PD controllers to solve single-input, index-1 DAE based LQR problems

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***Abstract*— In this paper, we present a method to compute the optimal trajectories of a linear quadratic regulator (LQR) problem corresponding to a single input, index-1 constant coefficient DAE system. Further, we also show that using a suitably designed proportional-derivative (PD) state-feedback controller one can force the trajectories of the corresponding DAE system to its optimal trajectories. Unlike the results present in the literature, the results in this paper are applicable for any arbitrary initial condition of the system.**

***Keywords:* DAE, Hamiltonian system, LQR Problem.**

1. Introduction

Differential algebraic equation (DAE) based systems naturally arise in different practical applications in the area of fluid dynamics, electrical systems, metabolic networks, chemical processes, etc. Hence, analysis of such systems have garnered high interest among mathematicians and control engineers: see [1] and the references therein. In particular, optimal control problems related to DAE systems have been studied by different authors [2], [3], [4]. In this paper we deal with the linear quadratic regulator (LQR) problem corresponding to semi-explicit, single-input, constant coefficient DAE systems of the form

$$\widehat{E}\dot{x} = \widehat{A}x + \widehat{b}u, \text{ where } \widehat{E}, \widehat{A} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \text{ and } \widehat{b} \in \mathbb{R}^{\mathtt{n}}, \tag{1}$$

and where we assume that the pencil $s\widehat{E} - \widehat{A}$ is regular[1] and of index 1. The corresponding LQR problem is:

Problem 1.1. *Consider a descriptor system with index-1 pencil having a minimal*[2] *input-state-output (i/s/o) representation as given in Eq.* (1). *For any initial condition $x^0 \in \mathbb{R}^{\mathtt{n}}$, find* admissible inputs[3] *u such that the following performance-index*

$$J(x^0,u) := \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} \widehat{Q} & \widehat{S} \\ \widehat{S}^T & \widehat{R} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt, \text{ where } \begin{bmatrix} \widehat{Q} & \widehat{S} \\ \widehat{S}^T & \widehat{R} \end{bmatrix} \geqslant 0 \tag{2}$$

is minimized.

LQR problems corresponding to DAE systems have been an active area of research: see [2], [7], [8], [9] and the references therein. In [2, Section 3] the authors find necessary optimality conditions for LQR problems corresponding to DAE systems and extend the conditions to non-linear systems as well. However, [2] does not provide a state-feedback controller that solves the LQR problem. The authors in [7] and [9] also extensively study the LQR problem for DAE systems. In particular, [9] establishes the link between the optimal control law that solves the LQR problem and the solutions of the corresponding Lur'e equation. Further, the authors also proposed implicit controllers that solve the LQR problem at hand. However, in order to ensure that the optimal trajectories that minimize the functional (2) are locally square-integrable functions ($\mathscr{L}^2_{\mathtt{loc}}$) such that the integral in Eq. (2) is well-defined, the authors in [7] and [9] impose a restriction on the initial conditions of the DAE system. In this paper, we do not impose any such restriction on the initial conditions of the system. We show that even without restricting the initial conditions of the system, we can ensure that the integral in (2) remains well-defined. We accomplish this using the theory developed in [6]. In order to highlight the contributions of this paper, we first provide an illustrative example.

Example 1.2. *Consider the following descriptor system with index-1 pencil:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \tag{3}$$

For any initial condition, find an admissible input that minimizes $J := \int_0^\infty x_1^2(t)dt$. Apparently, the minimum cost achievable here is zero which requires $x_1 = 0$. Hence, we look for an input that makes x_1 zero. Using [10, Eq. 1-4.10], the states of the system corresponding to an initial condition $x^0 = \mathtt{col}(x_1^0, x_2^0)$, where $x_1^0, x_2^0 \in \mathbb{R}$ is:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1^0 + \int_0^t u(\tau)d\tau \\ -u(t) \end{bmatrix}. \tag{4}$$

It is evident that J can be made arbitrarily small using an input $u \in \mathscr{L}^2_{\mathtt{loc}}(\mathbb{R},\mathbb{R})$. However, J can never be made zero unless we chose $u = -x_1^0\delta \notin \mathscr{L}^2_{\mathtt{loc}}(\mathbb{R},\mathbb{R})$, where δ is the Dirac-delta impulse function[4]*. Of course if the initial conditions are such that $x_1^0 = 0$, then the optimal input would be $u = 0$ and the optimal state $x = 0$. In such a scenario, the results in [9] are applicable. Thus, the results in [9] are applicable only to the initial conditions of the form $\begin{bmatrix} 0 \\ x_2^0 \end{bmatrix}$. These are precisely those initial conditions that satisfy $\widehat{E}x(0) = \widehat{E}x^0$ as given in [9, Section 7], where $\widehat{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.*

From Example 1.2 it is evident that the theory of [9] does not cater to all initial conditions. It is also clear that the function space of locally square integrable functions might not always provide us with optimal inputs for the corresponding LQR problem. Hence, we extend the search for optimal inputs in the set of $\mathscr{L}^2_{\mathtt{loc}}$-functions to the set of impulsive-smooth distributions $\mathfrak{C}_{\mathtt{imp}}$ as defined in [6, Def. 1].

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[1] A matrix pencil $sE - A$ is called regular if $\mathtt{det}(s\mathscr{E} - \mathscr{A})$ is not a zero polynomial. The index will be defined later.

[2] In this paper, we call a system *minimal* if the system is *behaviorally controllable*. For more on this see Section 2-C.

[3] Admissible inputs for an optimal control problem are those inputs that result in states such that the objective function is well-defined: see [5], [6] for more on such inputs.

[4] In this paper, we use the symbol $\delta^{(i)}$ to represent the i-th distributional-derivative of δ with respect to t.

Remark 1.3. In this framework, because of the inclusion of delta distributions δ and their derivatives, the obtained control may factually *reset* the value of the state x at $t = 0$ to any *admissible*[5] value. For that reason the notion of an initial value x^0 defining the value of $x(0)$ is not appropriate here. One may use $x(0_-) = x^0$ to express this technicality. We will simply refer to an initial value as the system's state that holds *before* any control action is applied.

Another important contribution of [9] is that the $\mathscr{L}^2_{\mathtt{loc}}$ optimal trajectories satisfy $Px + Lu = 0$, where P and L are related to the stabilizing solutions[6] of the corresponding Lur'e equation. Apart from being difficult to implement, this implicit control law does not divulge any information about the existence of an optimal input. In fact, in Example 1.2, $L = 0$ and the control law is void: $Px = 0$. Hence, our approach that provides an *implementable* explicit method to design a controller adds to the existing theory also from a practical point of view.

Now that we have shed light on the restrictions of [9], we list the primary contributions of this paper.

1) In Section 3, we first characterize the optimal trajectories of the Problem 1.1. Unlike other works in the literature, this characterization is achieved for arbitrary initial condition.

2) We further establish that a DAE system can be forced to these optimal trajectories using a proportional/proportional-derivative (P/PD) state-feedback controller (Thm. 3.6). Previous works [7], [9] provide implicit controllers with restrictions on the system's initial condition (see Example 1.2).

3) In Section 4, apart from a few supplementary results, we relate the results presented in this paper to the theory developed in [9].

In the next section, we present a brief review of the results in the literature required to present the results in this paper.

2. Preliminaries

A. Weierstrass canonical form

It is known that for a descriptor system of the form given in Eq. (1), there exist nonsingular $U_1, U_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $U_1 E U_2 = \mathtt{diag}(I_{\mathtt{n}_1}, N)$ and $U_1 A U_2 = \mathtt{diag}(A_1, I_{\mathtt{n}_2})$, where $A_1 \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ and $N \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is a nilpotent matrix [10]. A descriptor system is said to admit an index-1 pencil if the index[7] of the nilpotent matrix N corresponding to that system is 1, i.e, for such systems $N = 0$. Thus, without loss of generality, the system in Eq. (1) can be rewritten as

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}_1} & 0 \\ 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & I_{\mathtt{n}_2} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b} u, \tag{5}$$

where $U_1\widehat{b} =: b$, $b_1 \in \mathbb{R}^{\mathtt{n}_1}$, $b_2 \in \mathbb{R}^{\mathtt{n}_2}$, and $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] := U_2^{-1} x$. Eq. (5) is called the *Weierstrass canonical form* of the system in Eq. (1). For the sake of simplicity, we solve the LQR Problem 1.1 for the system in Eq. (5). Hence, on rewriting the LQR Problem 1.1 for a descriptor system in the Weierstrass canonical form we have the following:

Problem 2.1. *Consider a descriptor system with index-1 pencil having a minimal input-state-output (i/s/o) representation as in Eq.* (5). *For any initial condition $x^0 \in \mathbb{R}^{\mathtt{n}}$, find* admissible inputs *u such that the performance-index*

$$\int_0^\infty \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 & S_1 \\ Q_2^T & Q_3 & S_2 \\ S_1^T & S_2^T & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} dt, \text{ where } \begin{bmatrix} Q_1 & Q_2 & S_1 \\ Q_2^T & Q_3 & S_2 \\ S_1^T & S_2^T & R \end{bmatrix} \geqslant 0, \tag{6}$$

with $Q := \left[\begin{smallmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{smallmatrix}\right] \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $S := \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right] \in \mathbb{R}^{\mathtt{n}}$ partitioned complying with the partitions in A and b, respectively, is minimized.

B. Extended Hamiltonian matrix pencil

The optimal trajectories of the LQR Problem 2.1 are intimately linked with certain special subspaces associated with the following matrix pencil:

$$s\underbrace{\begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathscr{E}} - \underbrace{\begin{bmatrix} A & 0 & b \\ -Q & -A^T & -S \\ S^T & b^T & R \end{bmatrix}}_{\mathscr{A}}. \tag{7}$$

Using the terminology used in [11], we call Eq. (7) the *Extended Hamiltonian matrix pencil (EHP)* and $(\mathscr{E}, \mathscr{A})$ the *Hamiltonian matrix pair*. Observe that the pencil in Eq. (7) can be converted to the even matrix pencil $s\mathscr{E} - \mathscr{A}$ used in [9] using pre- and post-multiplication by suitable permutation matrices.

C. Behavioral controllability

It is important to note that throughout the paper we consider the DAE system to be *behaviorally controllable* (minimal). For the ease of exposition, we present the definition of behavioral controllability using its algebraic characterization next. For a trajectory-level definition refer to [9, Def. 2.2].

Definition 2.2. *The DAE system in Eq.* (1) *is called* behaviorally controllable *if* $\mathtt{rank}\begin{bmatrix} \lambda\widehat{E} - \widehat{A} & \widehat{B} \end{bmatrix} = \mathtt{n}$ *for all* $\lambda \in \mathbb{C}$.

For the system in Eq. (5) to be behaviorally controllable, we must have for all $\lambda \in \mathbb{C}$, $\mathtt{rank}\begin{bmatrix} \lambda E - A & b \end{bmatrix} = \mathtt{n}$, i.e.,

$$\mathtt{rank}\begin{bmatrix} \lambda I_{\mathtt{n}_1} - A_1 & 0 & b_1 \\ 0 & -I_{\mathtt{n}_2} & b_2 \end{bmatrix} = \mathtt{n} \Leftrightarrow \begin{bmatrix} \lambda I_{\mathtt{n}_1} - A_1 & b_1 \end{bmatrix} = \mathtt{n}_1.$$

Thus, the system in Eq. (5) is behaviorally controllable if and only if (A_1, b_1) is controllable. Hence, we assume (A_1, b_1) to be controllable throughout the paper.

3. Main results

The primary idea used to derive the results of this paper is the well-known fact that a descriptor system with index-1 pencil can be converted to a standard state-space system. Using this fact we first convert the Problem 2.1 to its state-space counterpart Problem 3.3 in Section 3-A. Then, we show that using the optimal trajectories of the equivalent state-space system based LQR problem, we can construct the optimal trajectories of the LQR Problem 2.1 (Thm. 3.5). We present this in Section 3-B. Finally, in Section 3-C

[5] The word *admissible* in this context indicates those states that are reachable by an impulsive control.

[6] In [9], a triplet (K, P, L) is called a stabilizing solution if K is a solution of the LMI $\mathscr{L}(K) := \left[\begin{smallmatrix} A^T KE + E^T KA + Q & E^T KB + S \\ B^T KE + S^T & R \end{smallmatrix}\right] \geqslant 0$ such that $\mathscr{L}(K) = \left[\begin{smallmatrix} P & L \end{smallmatrix}\right]^T \left[\begin{smallmatrix} P & L \end{smallmatrix}\right]$ and $\mathtt{rank}_{\mathbb{R}(s)} \left[\begin{smallmatrix} -\lambda E + A & B \\ P & L \end{smallmatrix}\right]$ for all $\lambda \in \mathbb{C}_-$. For the sake of brevity, in this paper we call K to be a stabilizing solution if it leads to the triplet (K, P, L).

[7] Nilpotency index of a nilpotent matrix N is the smallest positive integer $\mathtt{h}$ for which $N^{\mathtt{h}-1} \neq 0$ and $N^{\mathtt{h}} = 0$. Further, in this paper, the symbol $\sigma(E, A)$ is used to represent the multiset of the roots of $\mathtt{det}(sE - A)$. The roots are also referred as the eigenvalues of the matrix pair (E, A). We also use the symbol $\mathtt{degdet}(sE - A)$ to represent the degree of the polynomial $\mathtt{det}(sE - A)$.

we show that the corresponding descriptor system can be forced to these optimal trajectories using a PD state-feedback controller designed in Thm. 3.6.

A. Transformation of LQR Problem 2.1 to a state-space system based LQR problem

Observe that for the system in Eq. (5), we have $x_2 = -b_2 u$. On substituting x_2 with $-b_2 u$ in Problem 2.1, it can be restated as follows:

Problem 3.1. *For any initial condition $x^0 \in \mathbb{R}^{\mathtt{n}}$, find an admissible input u such that the following performance index*

$$\int_0^\infty \begin{bmatrix} x_1 \\ u \end{bmatrix}^T \begin{bmatrix} Q_1 & S_1 - Q_2 b_2 \\ (S_1 - Q_2 b_2)^T & R + b_2^T Q_3 b_2 - b_2^T S_2 - S_2^T b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix} dt \quad (8)$$

is minimized subject to $\dot{x}_1 = A_1 x_1 + b_1 u$.

Problem 3.1 is similar to an LQR problem corresponding to a state-space system except for the fact that the initial conditions are from $\mathbb{R}^{\mathtt{n}}$ instead of $\mathbb{R}^{\mathtt{n}_1}$. Hence, in the next step we decompose the initial conditions from $\mathbb{R}^{\mathtt{n}}$ as components of two unique subspaces which helps us to convert the LQR Problem 3.1 to a state-space system based LQR problem. This is done via the decomposition $\mathbb{R}^{\mathtt{n}} = \mathtt{im}\, E \oplus \ker E$ that holds for descriptor systems with index-1 pencils. With that, any initial condition of the system (5) can be decomposed as:

$$x^0 = \begin{bmatrix} x_1^0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^0 \end{bmatrix}, \text{ where } x_1^0 \in \mathbb{R}^{\mathtt{n}_1} \text{ and } x_2^0 \in \mathbb{R}^{\mathtt{n}_2}. \quad (9)$$

Note that all initial conditions of the form $\begin{bmatrix} 0 \\ x_2^0 \end{bmatrix}$ belong to the subspace $\ker E$ and the ones of the form $\begin{bmatrix} x_1^0 \\ 0 \end{bmatrix}$ belong to $\mathtt{im}\, E$. Next we write down the trajectories of the system (5) using [10, Eq. 1-4.10] to understand their role in the cost function of Eq. (8):

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{A_1 t} x_1^0 + \int_0^\infty e^{A_1(t-\tau)} b_1 u(\tau) d\tau \\ -b_2 u(t) \end{bmatrix}. \quad (10)$$

Thus, the trajectories of the system do not depend on x_2^0. Hence, for initial conditions in $\ker E$, taking the input to be $u = 0$, the cost turns out to be zero. Note that this is the minimum cost attainable, since the cost matrix in Eq. (8) is positive semi-definite as we confirm in the next lemma:

Lemma 3.2. *Let $Q_1 \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_1}, Q_2 \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}, Q_3 \in \mathbb{R}^{\mathtt{n}_2 \times \mathtt{n}_2}, S_1 \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{p}}, S_2 \in \mathbb{R}^{\mathtt{n}_2 \times \mathtt{p}}$ and $R \in \mathbb{R}^{\mathtt{p} \times \mathtt{p}}$ be such that $\begin{bmatrix} Q_1 & Q_2 & S_1 \\ Q_2^T & Q_3 & S_2 \\ S_1^T & S_2^T & R \end{bmatrix} \geqslant 0$. Define $S_{\mathtt{r}} := S_1 - Q_2 B_2$ and $R_{\mathtt{r}} := R + B_2^T Q_3 B_2 - B_2^T S_2 - S_2^T B_2$, where $B_2 \in \mathbb{R}^{\mathtt{n}_2 \times \mathtt{p}}$. Then, $\begin{bmatrix} Q_1 & S_{\mathtt{r}} \\ S_{\mathtt{r}}^T & R_{\mathtt{r}} \end{bmatrix} \geqslant 0$.*

Proof. Define $U_1 := \begin{bmatrix} I_{\mathtt{n}_1} & 0 & 0 \\ 0 & -B_2^T & I_{\mathtt{p}} \\ 0 & I_{\mathtt{n}_2} & 0 \end{bmatrix}$ and $L := \begin{bmatrix} Q_1 & Q_2 & S_1 \\ Q_2^T & Q_3 & S_2 \\ S_1^T & S_2^T & R \end{bmatrix}$. Since U_1 is nonsingular, $L \geqslant 0 \Rightarrow U_1^T L U_1 \geqslant 0$. Since all the principal minors of a positive semi-definite matrix are nonnegative and one of the principal minors of $U_1 L U_1^T$ is $\begin{bmatrix} Q_1 & S_{\mathtt{r}} \\ S_{\mathtt{r}}^T & R_{\mathtt{r}} \end{bmatrix}$, the result follows. $\square$

Now that we know the optimal input for initial conditions in $\ker E$, we solve Problem 3.1 for $x^0 \in \mathtt{im}\, E$, i.e., the ones of the form $\begin{bmatrix} x_1^0 \\ 0 \end{bmatrix}$, where $x_1^0 \in \mathbb{R}^{\mathtt{n}_1}$. This combined with the fact that $x_2 = -b_2 u$ implies that we need to first solve the following LQR problem to solve Problem 3.1.

Problem 3.3. *For any initial condition $x_1^0 \in \mathbb{R}^{\mathtt{n}_1}$, find admissible inputs u such that the performance index in Eq. (8) is minimized subject to $\dot{x}_1 = A_1 x_1 + b_1 u$.*

This is an LQR problem corresponding to a state-space system. In the next section, we show that using the optimal trajectories of the LQR Problem 3.3 we can construct the optimal trajectories of the LQR Problem 2.1.

B. Optimal trajectories of the LQR Problem 2.1

Note that the optimal trajectories of a standard state-space system based LQR problem have already been characterized in [6]. For the ease of exposition we review a few results from [6], in terms of the matrices used in this paper. Observe that the EHP corresponding to Problem 3.3 is as follows:

$$s \underbrace{\begin{bmatrix} I_{\mathtt{n}_1} & 0 & 0 \\ 0 & I_{\mathtt{n}_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathscr{E}_{\mathtt{r}}} - \underbrace{\begin{bmatrix} A_1 & 0 & b_1 \\ -Q_1 & -A_1^T & -S_{\mathtt{r}} \\ S_{\mathtt{r}} & b_1^T & R_{\mathtt{r}} \end{bmatrix}}_{\mathscr{A}_{\mathtt{r}}}, \quad (11)$$

where $S_{\mathtt{r}} := S_1 - Q_2 b_2$ and $R_{\mathtt{r}} := R + b_2^T Q_3 b_2 - b_2^T S_2 - S_2^T b_2$. Corresponding to a regular EHP, we define the following: $\mathtt{degdet}(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}}) =: 2\mathtt{s}$ and $\mathtt{f} := \mathtt{n}_1 - \mathtt{s}$. It has been established in [6] that for suitable $V_1 \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{s}}$ the matrix

$$X_1 := \begin{bmatrix} V_1 & b_1 & A_1 b_1 & \cdots & A_1^{\mathtt{f}-1} b_1 \end{bmatrix} \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_1} \quad (12)$$

is a nonsingular matrix (see [6, Prop. 1] for the properties of V_1). Hence, any arbitrary initial condition $x_1^0 \in \mathbb{R}^{\mathtt{n}_1}$ can be uniquely decomposed in the following subspaces:

$$\mathbb{R}^{\mathtt{n}_1} = \mathtt{im} V_1 \oplus \mathtt{im} b_1 \oplus \cdots \oplus \mathtt{im} A_1^{\mathtt{f}-1} b_1. \quad (13)$$

Such a decomposition of the initial condition helps in characterizing the exponential and impulsive optimal trajectories of the LQR problem. Based on the results in [6], we tabulate the optimal trajectories of the LQR Problem 3.3 corresponding to all the initial conditions of the system $\dot{x}_1 = A_1 x_1 + b_1 u$ in the next proposition.

Proposition 3.4. *[6, Thm. 1] Consider the LQR Problem 3.3. Let the EHP be as defined in Eq. (11). Assuming the EHP to be regular and $\sigma(\mathscr{E}_{\mathtt{r}}, \mathscr{A}_{\mathtt{r}}) \cap j\mathbb{R} = \emptyset$, define $\mathtt{degdet}(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}}) =: 2\mathtt{s}$ and $\mathtt{f} := \mathtt{n}_1 - \mathtt{s}$. Let $V \in \mathbb{R}^{(2\mathtt{n}_1+1) \times \mathtt{s}}$ be a full-column rank matrix such that $\mathscr{A}_{\mathtt{r}} V = \mathscr{E}_{\mathtt{r}} V \Gamma$, where $\sigma(\Gamma) = \sigma(\mathscr{E}_{\mathtt{r}}, \mathscr{A}_{\mathtt{r}}) \cap \mathbb{C}_-$. Conforming to the partition in $\mathscr{A}_{\mathtt{r}}$, V is partitioned as $V =: \mathtt{col}(V_1, V_2, V_3)$. Let X_1 be as defined in Eq. (12). Further, let $\beta, \alpha_i \in \mathbb{R}$, where $i \in \{0, 1, \cdots, \mathtt{f}-1\}$. Then, corresponding to initial condition $x_1^0 \in \mathbb{R}^{\mathtt{n}_1}$, the optimal state $x_{\mathtt{opt}}$ and the optimal input $u_{\mathtt{opt}}$ are as tabulated below:*

x_1^0	$x_{\mathtt{opt}}$	$u_{\mathtt{opt}}$
$V_1\beta$	$V_1 e^{\Gamma t}\beta$	$V_3 e^{\Gamma t}\beta$
$b_1\alpha_0$	0	$-\delta\alpha_0$
$A_1 b_1 \alpha_1$	$-b_1\alpha_1\delta$	$-\delta^{(1)}\alpha_1$
$A_1^2 b_1 \alpha_2$	$-(b_1\delta^{(1)} + A_1 b_1 \delta)\alpha_2$	$-\delta^{(2)}\alpha_2$
$\vdots$	$\vdots$	$\vdots$
$A_1^{\mathtt{f}-1} b_1 \alpha_{\mathtt{f}-1}$	$-\left(\sum_{i=0}^{\mathtt{f}-2} A_1^{\mathtt{f}-2-i} b_1 \delta^{(i)}\right)\alpha_{\mathtt{f}-1}$	$-\delta^{(\mathtt{f}-1)}\alpha_{\mathtt{f}-1}$

It has already been established in [6, Thm. 1] that the inputs $u_{\mathtt{opt}}$ are admissible. Now we use Prop. 3.4 to characterize the optimal trajectories of the LQR Problem 2.1.

Observe that using Eq. (9) and Eq. (13), it is evident that any initial condition $x^0 \in \mathbb{R}^{\mathtt{n}}$ can be uniquely decomposed in the following subspaces:

$$\mathbb{R}^{\mathtt{n}} = \mathtt{im}\begin{bmatrix} V_1 \\ 0 \end{bmatrix} \oplus \mathtt{im}\begin{bmatrix} b_1 \\ 0 \end{bmatrix} \oplus \cdots \oplus \mathtt{im}\begin{bmatrix} A_1^{\mathtt{f}-1} b_1 \\ 0 \end{bmatrix} \oplus \mathtt{im}\begin{bmatrix} 0 \\ I_{\mathtt{n}_2} \end{bmatrix}. \quad (14)$$

We use this fact in the next theorem to characterize the optimal trajectories of the LQR Problem 2.1.

Theorem 3.5. *Consider the LQR Problem 2.1. Let $V_1, V_2, V_3, \mathtt{s}, \mathtt{f}, x_{\mathtt{opt}}, u_{\mathtt{opt}}$ and x_1^0 be as defined in Prop. 3.4. Then, the following statements are true:*
1) Corresponding to an initial condition $\mathtt{col}(x_1^0, 0)$, the optimal input and the optimal state are $u_{\mathtt{opt}}$ and $\mathtt{col}(x_{\mathtt{opt}}, -b_2 u_{\mathtt{opt}})$, respectively.
2) Corresponding to an initial condition $\mathtt{col}(0, x_2^0)$, where $x_2^0 \in \mathbb{R}^{\mathtt{n}_2}$, the optimal input and the optimal state are 0 and $0_{\mathtt{n},1}$, respectively.

Proof: *(1):* Observe that for a given x_1 and u the cost function in Eq. (8) evaluates to the same value as that of Eq. (6), if we choose $x_2 = -b_2 u$ in Eq. (6). Further, from Prop. 3.4 it is evident that $x_1 = x_{\mathtt{opt}}$ and $u = u_{\mathtt{opt}}$ minimizes the cost function in Eq. (8). Therefore, with $x_2 = -b_2 u$ the optimal state and the optimal input must be given by $\begin{bmatrix} x_{\mathtt{opt}} \\ -b_2 u_{\mathtt{opt}} \end{bmatrix}$ and $u = u_{\mathtt{opt}}$, respectively.
(2): Using Lemma 3.2 it is evident that the minimum cost achievable is zero. An initial condition $\mathtt{col}(0, x_2^0)$ implies that $x_1^0 = 0$ in Eq. (10). Thus, corresponding to zero input, we have $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0$ for all $t \in \mathbb{R}$. On evaluation of Eq. (6), it is evident that the minimum cost of zero is achieved by zero input and corresponding zero states. ☐

C. Design of the PD feedback controllers

In this section, we show that a descriptor system with index-1 pencil can be forced to its optimal trajectories (characterized in Thm. 3.5) using a PD state-feedback controller.

Theorem 3.6. *Consider the LQR Problem 2.1. Let X_1 and V_3 be as defined in Prop. 3.4. Let $g_0, g_1, \ldots, g_{\mathtt{f}-1} \in \mathbb{R}$. Define*

$$F_{\mathtt{p}} := \begin{bmatrix} V_3 & g_0 & g_1 & \cdots & g_{\mathtt{f}-1} & 0_{1,\mathtt{n}_2} \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & I_{\mathtt{n}_2} \end{bmatrix}^{-1}, \quad (15)$$

$$F_{\mathtt{d}} := \begin{bmatrix} 0_{1,\mathtt{s}} & 1 & -g_0 & \cdots & -g_{\mathtt{f}-2} & 0_{1,\mathtt{n}_2} \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & I_{\mathtt{n}_2} \end{bmatrix}^{-1} \quad (16)$$

with $\mathtt{det}\left(s(E - bF_{\mathtt{d}}) - (A + bF_{\mathtt{p}})\right) \neq 0$. Then, application of the PD state-feedback law $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \dot{x}$ forces the system into its optimal trajectories characterized in Theorem 3.5.

Proof: Define

$$\widehat{F}_{\mathtt{p}} := \begin{bmatrix} V_3 & g_0 & g_1 & \cdots & g_{\mathtt{f}-1} \end{bmatrix} X_1^{-1},$$
$$\widehat{F}_{\mathtt{d}} := \begin{bmatrix} 0_{1,\mathtt{s}} & 1 & -g_0 & \cdots & -g_{\mathtt{f}-2} \end{bmatrix} X_1^{-1}.$$

Define $E_{\mathtt{c}} := E - bF_{\mathtt{d}}$ and $A_{\mathtt{c}} := A + bF_{\mathtt{p}}$. Observe that

$$E_{\mathtt{c}} = \begin{bmatrix} I_{\mathtt{n}_1} - b_1 \widehat{F}_{\mathtt{d}} & 0 \\ -b_2 \widehat{F}_{\mathtt{d}} & 0 \end{bmatrix}, A_{\mathtt{c}} = \begin{bmatrix} A + b_1 \widehat{F}_{\mathtt{p}} & 0 \\ b_2 \widehat{F}_{\mathtt{p}} & I_{\mathtt{n}_2} \end{bmatrix}.$$

Thus, we have

$$sE_{\mathtt{c}} - A_{\mathtt{c}} = \begin{bmatrix} s(I_{\mathtt{n}_1} - b_1 \widehat{F}_{\mathtt{d}}) - (A_1 + b_1 \widehat{F}_{\mathtt{p}}) & 0 \\ -sb_2 \widehat{F}_{\mathtt{d}} - b_2 \widehat{F}_{\mathtt{p}} & -I_{\mathtt{n}_2} \end{bmatrix}.$$

Therefore, we have

$$\mathtt{det}(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0 \Leftrightarrow \mathtt{det}\left(s(I_{\mathtt{n}_1} - b_1 \widehat{F}_{\mathtt{d}}) - (A_1 + b_1 F_{\mathtt{p}})\right) \neq 0.$$

It has already been established in [6, Lemma 3] that such choices of $g_0, g_1, \cdots, g_{\mathtt{f}-1}$ always exist that would make $\mathtt{det}\left(s(I_{\mathtt{n}_1} - b_1 \widehat{F}_{\mathtt{d}}) - (A_1 + b_1 F_{\mathtt{p}})\right) \neq 0$.

On application of the control law $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \dot{x}$ on the system in Eq. (5), we get the following closed loop system:

$$\begin{bmatrix} I_{\mathtt{n}_1} - b_1 \widehat{F}_{\mathtt{d}} & 0 \\ -b_2 \widehat{F}_{\mathtt{d}} & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 + b_1 \widehat{F}_{\mathtt{p}} & 0 \\ b_2 \widehat{F}_{\mathtt{p}} & I_{\mathtt{n}_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (17)$$

Thus we have a system that satisfies

$$(I_{\mathtt{n}_1} - b_1 \widehat{F}_{\mathtt{d}}) \dot{x}_1 = (A_1 + b_1 \widehat{F}_{\mathtt{p}}) x_1. \quad (18)$$
$$-b_2 \widehat{F}_{\mathtt{d}} \dot{x}_1 = b_2 \widehat{F}_{\mathtt{p}} x_1 + x_2 \Rightarrow x_2 = -b_2(\widehat{F}_{\mathtt{p}} x_1 + \widehat{F}_{\mathtt{d}} \dot{x}_1). \quad (19)$$

For initial conditions in $\ker E$, we have $x_1^0 = 0$. Then, from the table in Proposition (3.4) we infer that $x_1 = 0$ for all $t > 0$. Further, from Eq. (19) it is clear that $x_2 = 0$ for all $t > 0$. Thus, for an initial condition in $\ker E$ the closed loop system (17) has the optimal state-trajectory 0.

Next we look into the case when the initial conditions are in $\mathtt{im}\, E$, i.e., $x_2^0 = 0$. It has been established in [6, Thm. 2] that the trajectories of the system in Eq. (18) are $x_{\mathtt{opt}}$ (as defined in Prop. 3.4). Hence, from Eq. (19) we have $x_2 = -b_2(\widehat{F}_{\mathtt{p}} x_{\mathtt{opt}} + \widehat{F}_{\mathtt{d}} \dot{x}_{\mathtt{opt}}) = -b_2 u_{\mathtt{opt}}$. Thus, for an initial condition in $\mathtt{im}\, E$, the closed-loop system (17) has the optimal state-trajectory $\begin{bmatrix} x_{\mathtt{opt}} \\ -b_2 u_{\mathtt{opt}} \end{bmatrix}$.

Further, since with a suitable choice of $g_0, g_1, \cdots, g_{\mathtt{f}-1}$ we can ensure that $\mathtt{det}(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$, we can get an autonomous closed loop DAE system with *unique* trajectories; the trajectories being the optimal ones. ☐

Thus, we have established that a PD state-feedback controller can force the trajectories of the system to the optimal states provided the EHP is regular and does not have eigenvalues on the imaginary axis. We shed light on these two conditions next.

1) $\mathtt{det}(s\mathscr{E} - \mathscr{A}) \not\equiv 0$: In order to discuss this condition we first establish that $\mathtt{det}(s\mathscr{E} - \mathscr{A}) \not\equiv 0 \Leftrightarrow \mathtt{det}(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}}) \not\equiv 0$.

Lemma 3.7. *Consider the EHPs $(s\mathscr{E} - \mathscr{A})$ and $(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}})$ defined in Eq. (7) and Eq. (11), respectively. Then, $\mathtt{det}(s\mathscr{E} - \mathscr{A}) = \pm\mathtt{det}(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}})$.*

Proof: This follows from using Schur complement with respect to $\begin{bmatrix} sE - A & 0 \\ Q & sE + A^T \end{bmatrix}$ while computing $\mathtt{det}(s\mathscr{E} - \mathscr{A})$. ☐

From Lemma 3.7 it is evident that $(s\mathscr{E} - \mathscr{A})$ is a regular pencil if and only if $(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}})$ is regular. Observe that for $R_{\mathtt{r}}$ nonsingular, $(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}})$ is always regular. Further, it has been shown in [12, Thm. 3.5] that for a single-input controllable system with zero cost, i.e., $R_{\mathtt{r}} = 0$, $\mathtt{det}(s\mathscr{E}_{\mathtt{r}} - \mathscr{A}_{\mathtt{r}})$ is never a zero polynomial. Thus, $\mathtt{det}(s\mathscr{E} - \mathscr{A}) \not\equiv 0$ for a single-input (A_1, b_1) controllable system. In other words, $\mathtt{det}(s\mathscr{E} - \mathscr{A}) \not\equiv 0$ for a behaviorally controllable system.

2) $\sigma(\mathscr{E}, \mathscr{A}) \cap j\mathbb{R} = \emptyset$: This is a condition to ensure that the closed-loop system is asymptotically stable. If the EHP admits imaginary axis eigenvalues, then we have optimal trajectories that are periodic in nature. Such trajectories are not zero in the limit. Note that the LQR framework is in general used in tracking problems, where the states

correspond to the error in tracking the reference signal. Hence, the objective in such problems is not only to ensure that the quadratic cost function is minimized but also that the error is zero in the limit. Hence, it is standard to assume that the states are zero in the limit for an LQR problem. Thus, the assumption $\sigma(\mathscr{E},\mathscr{A})\cap j\mathbb{R}=\emptyset$ is reasonable.

4. Supplementary results

Now that we have established that an LQR problem corresponding to a single-input descriptor system with index-1 pencil can be solved using PD feedback, we present a few supplementary results on the computation of eigenspace of the Hamiltonian system and the optimal cost attained by the optimal trajectories in this section. We also relate the results in this paper to [9].

A. Relation between eigenspace-basis of $(\mathscr{E},\mathscr{A})$ and $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$

It has been shown in Thm. 3.5 and Thm. 3.6 that a basis of the eigenspace corresponding to the stable eigenvalues of the matrix pair $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$ is essential to design the PD controllers. In the next theorem we establish the relation between a basis of the eigenspace of $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$ and that of $(\mathscr{E},\mathscr{A})$. From Lemma 3.7 it is evident that the eigenvalues of the matrix pair $(\mathscr{E},\mathscr{A})$ and $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$ are the same.

Theorem 4.1. *Let $V:=\mathtt{col}\left(V_1,\widetilde{V}_1,V_2,\widetilde{V}_2,V_3\right)$ be a basis of the eigenspace corresponding to the stable eigenvalues of $(\mathscr{E},\mathscr{A})$. Then, $\mathtt{col}(V_1,V_2,V_3)$ is a basis of the eigenspace corresponding to the stable eigenvalues of $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$.*

Proof: Note that since $\mathscr{A}V=\mathscr{E}V\Gamma$, where $\Gamma\in\mathbb{R}^{\mathtt{s}\times\mathtt{s}}$ and $\mathtt{s}:=\mathtt{degdet}(s\mathscr{E}-\mathscr{A})/2$ with $\sigma(\Gamma)\subseteq\mathbb{C}_-$, we have

$$\begin{bmatrix} A_1 & 0 & 0 & 0 & b_1\\ 0 & I_{\mathtt{n}_2} & 0 & 0 & b_2\\ -Q_1 & -Q_2 & -A_1^T & 0 & -S_1\\ -Q_2^T & -Q_3 & 0 & -I_{\mathtt{n}_2} & -S_2\\ S_1^T & S_2^T & b_1^T & b_2^T & R\end{bmatrix}\begin{bmatrix}V_1\\ \widetilde{V}_1\\ V_2\\ \widetilde{V}_2\\ V_3\end{bmatrix}=\begin{bmatrix}V_1\Gamma\\ 0\\ V_2\Gamma\\ 0\\ 0\end{bmatrix}. \tag{20}$$

From Eq. (20) it is clear that

$$A_1V_1+B_1V_3=V_1\Gamma. \tag{21}$$

Observe that from Eq. (20) we have

$$\widetilde{V}_1+b_2V_3=0\Rightarrow\widetilde{V}_1=-b_2V_3, \tag{22}$$

$$\widetilde{V}_2=-Q_2^TV_1-Q_3\widetilde{V}_1-S_2V_3. \tag{23}$$

Further, we also have from Eq. (20)

$$-Q_1V_1-Q_2\widetilde{V}_1-A_1^TV_2-S_1V_3=V_2\Gamma, \tag{24}$$

$$S_1^TV_1+S_2^T\widetilde{V}_1+b_1^TV_2+b_2^T\widetilde{V}_2+RV_3=0. \tag{25}$$

Using Eq. (22) in Eq. (24), we have

$$-Q_1V_1-A_1^TV_2-S_{\mathtt{r}}V_3=V_2\Gamma. \tag{26}$$

Further, using Eq. (23) in Eq. (25), we further have

$$S_{\mathtt{r}}V_1+b_1^TV_2+R_{\mathtt{r}}V_3=0. \tag{27}$$

Using Eq. (21), Eq. (26) and Eq. (27), we have

$$\begin{bmatrix} A_1 & 0 & b_1\\ -Q_1 & -A_1^T & -S_{\mathtt{r}}\\ S_{\mathtt{r}}^T & b_1^T & R_{\mathtt{r}}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\\ V_3\end{bmatrix}=\begin{bmatrix}I_{\mathtt{n}_1} & 0 & 0\\ 0 & I_{\mathtt{n}_1} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}V_1\\ V_2\\ V_3\end{bmatrix}\Gamma.$$

Thus, $\mathtt{col}(V_1,V_2,V_3)$ is a basis of the eigenspace corresponding to the stable eigenvalues of $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$. ☐

From Thm. 4.1 it is evident that while designing a controller for the DAE system as given in Thm. 3.6, instead of computing an eigenbasis of the reduced EHP $(\mathscr{E}_{\mathtt{r}},\mathscr{A}_{\mathtt{r}})$ we can also compute an eigenbasis of the EHP $(\mathscr{E},\mathscr{A})$ and construct the controller matrices as given in Thm. 3.6.

B. Cost attained by optimal trajectories of Problem 2.1

In the next theorem we compute the cost attained by the optimal trajectories characterized in Thm. 3.5.

Theorem 4.2. *Consider the LQR Problem 2.1. Let $V_1,V_2,\mathtt{s},\mathtt{f}$ be as defined in Thm. 3.5. Assume $x^0\in\mathbb{R}^{\mathtt{n}}$ to be an arbitrary initial condition such that*

$$x^0=\begin{bmatrix}V_1\\ 0\end{bmatrix}\beta+\begin{bmatrix}W_1\\ 0\end{bmatrix}\alpha+\begin{bmatrix}0\\ I_{\mathtt{n}_2}\end{bmatrix}\gamma,\ \text{where } \alpha\in\mathbb{R}^{\mathtt{s}},\beta\in\mathbb{R}^{\mathtt{f}},\gamma\in\mathbb{R}^{\mathtt{n}_2}$$

with $W_1=\begin{bmatrix}b_1 & A_1b_1 & \cdots & A_1^{\mathtt{f}-1}b_1\end{bmatrix}$. Then, the cost attained by the corresponding optimal trajectories is $\beta^TV_1^TV_2\beta$.

Proof: Define $x_1^0:=V_1\beta+W_1\alpha$. Let $K_{\mathtt{r}}$ be the maximal rank minimizing solution of the LMI

$$\mathscr{L}(K):=\begin{bmatrix}A_1^TK+KA_1+Q_1 & Kb_1+S_1-Q_2b_2\\ b_1^TK+S_1^T-b_2^TQ_2 & R-S_2^Tb_2-b_2^TS_2+b_2^TQ_3b_2\end{bmatrix}\geqslant 0. \tag{28}$$

Observe that the LMI in Eq. (28) is the LMI corresponding to the LQR Problem 3.3. Since the optimal cost only depends on x_1 and u, from [6, Thm. 1] it is evident that for the case when $\gamma=0$, the optimal cost is $(x_1^0)^TK_{\mathtt{r}}x_1^0$. For initial conditions with $\alpha,\beta=0$ and $\gamma\neq 0$, the optimal state trajectories from Thm. 3.5 are $0_{\mathtt{n},1}$ and the optimal input is zero. Hence, the cost in such a case is zero. Therefore, it is evident that the optimal cost for any arbitrary initial condition is $(x_1^0)^TK_{\mathtt{r}}(x_1^0)$.

From [6, Prop. 1] and Thm. 4.1 the maximal rank-minimizing solution of the LMI (28) is:

$$K_{\mathtt{r}}=\begin{bmatrix}V_2 & 0_{\mathtt{n}_1,\mathtt{f}}\end{bmatrix}\begin{bmatrix}V_1 & W_1\end{bmatrix}^{-1}. \tag{29}$$

Therefore, we must have $K_{\mathtt{r}}W_1\alpha=0$. This means that

$$(x_1^0)^TK_{\mathtt{r}}x_1^0=\left([V_1\beta+W_1\alpha]^TK_{\mathtt{r}}[V_1\beta+W_1\alpha]\right)=\beta^TV_1^TV_2\beta.$$

This completes the proof of the theorem. ☐

C. Relation with deflating subspace method in the literature

As mentioned in Section 1, the authors in [9] have extensively studied the LQR problem for DAE systems. However, the problem has been solved assuming that the initial conditions satisfy $Ex(0)=Ex^0$, where E is as defined in Eq. (5). This condition restricts the states from admitting jumps. We have relaxed this condition in Thm. 3.5. A natural question that arises in this context is: how are the results in [9] related to our results. We discuss this next.

Observe that for the case when $R_{\mathtt{r}}$ is nonsingular in Problem 3.3, we must have $\mathtt{s}=\mathtt{n}_1$ and $\mathtt{f}=0$. In such a case, it is clear from Thm. 3.6 that the state-feedback controller that will force the system to its optimal trajectories is given by $u=\begin{bmatrix}V_3 & 0_{1,\mathtt{n}_2}\end{bmatrix}\begin{bmatrix}X_1 & 0\\ 0 & I_{\mathtt{n}_2}\end{bmatrix}^{-1}x$. In this case, the optimal input and optimal state are given by $V_3e^{\Gamma t}\beta$ and

$\begin{bmatrix} V_1 e^{\Gamma t}\beta \\ -b_2 V_3 e^{\Gamma t}\beta \end{bmatrix}$, respectively (Thm. 3.5). Therefore, for the case when $R_{\mathtt{r}}$ is nonsingular, all the initial conditions of the system satisfy $Ex(0) = Ex^0$ and $\lim_{t\to\infty} Ex(t) = 0$, since $\sigma(\Gamma) \subseteq \mathbb{C}_-$. Thus, [9] caters to all initial conditions if $R_{\mathtt{r}}$ is nonsingular. However, if $R_{\mathtt{r}}$ is singular, then [9] does not cater to the initial conditions in $\mathtt{im}\begin{bmatrix} W_1 \\ 0 \end{bmatrix}$.

Remark 4.3. For the optimal trajectories corresponding to the initial conditions of the form $x^0 = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}\beta + \begin{bmatrix} 0 \\ I_{\mathtt{n}_2} \end{bmatrix}\gamma$, i.e. the initial conditions that satisfy $Ex(0) = Ex^0$, the cost achieved by the optimal trajectories is $(x^0)^T E^T K E x^0$, where K is a stabilizing solution of the LQR LMI [9, Section 7]. Conforming to the partitions in A, we partition the stabilizing solution of the LQR LMI as $K =: \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_3 \end{bmatrix}$. Then, we have $(x^0)^T E^T K E x^0 = \beta^T V_1^T K_1 V_1 \beta$. Therefore, if we choose K_1 such that $K_1 V_1 = V_2$, then we get the optimal cost $\beta^T V_1^T V_2 \beta$ as claimed in Thm. 4.2. This is possible if we chose K_1 to be equal to the maximal rank-minimizing solution of the LQR LMI (28), i.e., $K_{\mathtt{r}}$ as defined in Eq. (29). Hence, by [9, Lemma 5.4], $K_{\mathtt{r}}$ has to be a stabilizing solution of LMI (28).

D. Illustrative example

In this section we present examples to demonstrate the results presented in this paper. The first example is the one we used to motivate the paper in Section 1.

Example 4.4. *Note that for the problem in Example 1.2, we have* $\mathtt{n}_1 = \mathtt{f} = 1$*,* $\mathtt{s} = 0$*, and* $\mathtt{n}_2 = 1$*, since* $\det(s\mathscr{E} - \mathscr{A}) = -1$*, where* $(\mathscr{E}, \mathscr{A})$ *is the corresponding Hamiltonian matrix pair. Here* $b_1 = 1$ *and* $b_2 = 1$*. Further,* $X_1 = b_1 = 1$*. Therefore, from Thm. 3.6 we have* $F_{\mathtt{p}} = \begin{bmatrix} g_0 & 0 \end{bmatrix}$ *and* $F_{\mathtt{d}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$*. On application of the control law* $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \dot{x}$*, we get the following closed loop system*

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} g_0 & 0 \\ 1+g_0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus, for all $g_0 \in \mathbb{R}\setminus\{0\}$*, we have PD controllers that will force the system to its optimal trajectories.*

Recall that we had started the paper with the LQR Problem 1.1 and to make the exposition in the paper easier we considered the transformed LQR Problem 2.1. In the next example we start with semi-explicit descriptor system (not in its Weierstrass canonical form) and solve Problem 1.1.

Example 4.5. *Consider the DAE system in Eq.* (1) *with*

$$\widehat{E} := \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \widehat{A} := \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \widehat{b} := \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Let the cost-functional to be minimized be as in Eq. (2) *with*

$$\widehat{Q} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & 16 \end{bmatrix}, \widehat{S} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \widehat{R} = 1.$$

Define the matrices

$$U_1 := \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, U_2 := \frac{1}{9}\begin{bmatrix} 4 & -3 & 2 \\ -3 & 0 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Define $E := U_1 \widehat{E} U_2$*,* $b := U_1 \widehat{b}$*,* $A := U_1 \widehat{A} U_2$*. Thus, we have*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $U_2^{-1}x =: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$*. Then, the Weierstrass canonical form is:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u. \tag{30}$$

The new cost functional will be as given in Eq. (6)*, with*

$$Q = U_2^T \widehat{Q} U_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, S = U_2^T \widehat{S} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, R = 1.$$

It can be verified that $\det(s\mathscr{E} - \mathscr{A}) = 1 - 4s^2$ *and hence,* $\mathtt{s} = 1$*. Thus, for this problem we have,* $\mathtt{n}_1 = 2$*,* $\mathtt{n}_2 = 1$*,* $\mathtt{s} = 1$ *and* $\mathtt{f} = 1$*. An eigenvector corresponding to the stable eigenvalue* $-\frac{1}{2}$ *of the EHP pair is* $V = \begin{bmatrix} 1 & 3 & 1.5 & -8 & 8 & -4 & -1.5 \end{bmatrix}^T$*. Hence,* $V_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$*,* $V_2 = \begin{bmatrix} -8 & 8 \end{bmatrix}^T$ *and* $V_3 = -1.5$*. Using Thm. 3.6 and choosing* $g_0 = 0$*, the controller matrices are:* $F_{\mathtt{p}} = \begin{bmatrix} 0.75 & 0.75 & 0 \end{bmatrix}$*, and* $F_{\mathtt{d}} = \begin{bmatrix} 1.5 & -0.5 & 0 \end{bmatrix}$*. Hence, a PD controller that forces the trajectories of the DAE system to its optimal trajectories is*

$$\begin{aligned} u &= \begin{bmatrix} 0.75 & 0.75 & 0 \end{bmatrix} U_2^{-1}x + \begin{bmatrix} 1.5 & -0.5 & 0 \end{bmatrix} U_2^{-1}\dot{x} \\ \Rightarrow u &= \begin{bmatrix} 1.5 & -0.75 & -0.75 \end{bmatrix} x + \begin{bmatrix} 2 & -1.5 & 0.5 \end{bmatrix}\dot{x} \end{aligned}$$

Observe that for standard state-space systems, $\mathtt{f} = 0$ if and only if the input cost matrix R is singular [6]. However, from Example 4.5 it is evident that even if R is nonsingular, we might have $\mathtt{f} \neq 0$.

5. Concluding remarks

In this paper, we presented a method to design PD state-feedback controllers that force the trajectories of a single-input descriptor system with index-1 pencil to its optimal trajectories corresponding to an LQR problem. The method presented in this paper uses the results in [6] that crucially used the idea of weak-unobservability and strong-controllability of a standard state-space system. Using the same notions we plan to extend the results of this paper to the multi-input case in our forthcoming paper.

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