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A 2D-DFT based method to compute the Bezoutian and a link to Lyapunov equations

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***Abstract*— This paper deals with the computation of the Bezoutian using two dimensional discrete Fourier transform techniques. The Bezoutian is an important tool in the study of stability of systems, especially in localization of roots of a polynomial. This paper gives an alternate method to compute the Bezoutian. Results exploring the link between the Bezoutian and Lyapunov operator under suitable assumption are also presented in the paper. It is shown in this paper that solving the Bezoutian is equivalent to solving a Lyapunov equation. Hence the 2D-DFT technique used to compute the Bezoutian is applicable to computing solutions of the Lyapunov equation as well. The results in this paper follow from a new link that we establish between two variable polynomials, 2D-DFT and Lyapunov operators.**
Keywords: Bezoutian, Lyapunov operator, Lyapunov equation, Two dimensional discrete Fourier transform.

1. Introduction

Problems of localization of roots of a polynomial is important in the study of stability of systems. One of the most useful tools used in such problems is the Bezoutian of two polynomials. The Bezoutian is a special structured quadratic form which finds applications not only in stability analysis but also in the field of elimination theory. Depending on the application, different forms of the Bezoutian are present in the literature: for details see [4], [9, Section 13.3]. In this paper, we deal with the Bezoutian of the form shown in equation (5) that finds application in areas like stability analysis (see [4, Page 40, Section 6]) and computation of storage function of conservative systems (see [6, Section 3A]). One of the methods used to solve the Bezoutian is the Euclidean long division method: see [9, Section 13.3]. However, error accumulation is a major disadvantage of this method. Another method used to compute the Bezoutian is based on multiplication of triangular Hankel matrices of the coefficients of the polynomials: for a detailed analysis see [9, Section 13.3 and Proposition 1]. Such a method results in $\mathscr{O}(\mathtt{n}^3)$ complexity. An iterative method to find the Bezoutian matrix has also been presented in [1]. In this paper we propose a new method to compute the Bezoutian using 2D-discrete Fourier transform (2D-DFT). Since the method proposed in this paper uses structured and sparse matrices in the computation of the Bezoutian, we believe the method proposed will have a complexity of at most $\mathscr{O}(\mathtt{n}^2)$. Further, 2D-DFT inherently uses orthogonal matrices and hence we believe the method will be less prone to errors as well.

Development of the new technique to solve for the Bezoutian using 2D-DFT results in an interesting insight into solving Lyapunov equations of a special structure using 2D-DFT technique. The polynomial interpretation of the FFT algorithms is well known. In this paper we show how two variable polynomials are linked to Lyapunov operators under suitable assumptions. Further, we give one variable analogous results as well to understand the intrinsic features of its two variable counterpart.

The rest of the paper is organized as follows. Section 2 contains the notation and preliminaries required for the paper. The main results of the paper that links Lyapunov operator, 2D-DFT and two variable polynomials are presented in Section 3. Section 4 has results for the 1D case. This section has results analogous to Section 3. In Section 5 we present the link between the Bezoutian and Lyapunov operators. The section contains an algorithm to compute the Bezoutian of two polynomials based on 2D-DFT. The same algorithm can also be used to solve Lyapunov equation of a special kind. Concluding remarks are presented in Section 7.

2. Notation and Preliminaries

A. Notation

We follow standard notation in this paper: $\mathbb{R}$ and $\mathbb{C}$ denote fields of real and complex numbers respectively. $\mathbb{Z}$ and $\mathbb{N}$ denotes the set of integers and natural numbers respectively. The ring of polynomials in $\mathtt{n}$ variables $x_1, x_2, \ldots, x_\mathtt{n}$ with complex coefficients is denoted by $\mathbb{C}[x_1, x_2, \ldots, x_\mathtt{n}]$. We use the notation $\langle p, q\rangle$ to denote the ideal generated by p, q: the ring usually being clear from the context. The vector corresponding to an $(\mathtt{N}-1)$ degree polynomial $p(s) = a_0 + a_1 s + \cdots + a_{\mathtt{N}-1}s^{\mathtt{N}-1}$ is represented as $p = \begin{bmatrix} a_0 & a_1 & \cdots & a_{\mathtt{N}-1}\end{bmatrix} \in \mathbb{C}^\mathtt{N}$. $A = [a_{ij}]$ represents a matrix A with elements a_{ij} where i and j are the row and column indices of the matrix. The symbol $A \otimes B$ represents Kronecker product of matrices A and B. A matrix of the form $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ is represented as $\text{col}(B_1, B_2)$. The notation $\mathbf{0}_r$ (or $\mathbf{0}_c$) is used to represent a row (or column) vector of zero elements. $\mathbf{1}_\mathtt{n}$ represents a column vector of length $\mathtt{n}$ with all elements 1. $I_\mathtt{N}$ represents an $\mathtt{N}\times\mathtt{N}$ identity matrix. We use the notation $\mathbf{X}$ to represent column vectors of the form $\begin{bmatrix} 1 & x & x^2 & \cdots & x^{\mathtt{N}-1}\end{bmatrix}^T \in \mathbb{R}^\mathtt{N}[x]$. The notation $A * B$, $A \odot B$ and $A./B$ is used to imply 2D-convolution, Hadamard product (i.e. entry-wise product) and entry-wise division of matrices A and B respectively. We use the symbol $\mathfrak{f}(\cdot)$ and $\mathscr{F}(\cdot)$ to represent 1D-DFT and 2D-DFT respectively. $\mathscr{F}^{-1}(\cdot)$ represents the inverse 2D-DFT.

B. Discrete Fourier transform

This section contains a quick review of the one and two dimensional discrete Fourier transform (DFT). See [7] for

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an elaboration. Given a vector $x \in \mathbb{R}^{\mathtt{N}}$, from a linear algebra perspective, the operation of N-point 1D-DFT is written as a matrix-vector product $X = \Omega x$, where $x \in \mathbb{R}^{\mathtt{N}}$ is the N-point signal and $\Omega \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the Vandermonde matrix constructed from the roots of unity as follows:

$$\Omega = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{\mathtt{N}-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(\mathtt{N}-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{\mathtt{N}-1} & \omega^{2(\mathtt{N}-1)} & \cdots & \omega^{(\mathtt{N}-1)^2} \end{bmatrix} \text{ with } \omega := e^{-j\frac{2\pi}{\mathtt{N}}}$$

In 2D-DFT operation, the argument is a matrix. We represent the 2D-DFT of $Y \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ as $\mathscr{F}(Y) = [F(k,\ell)] \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ where $k,\ell = 0,1,2,\ldots,\mathtt{N}-1$ and $F(k,\ell)$ is given by

$$F(k,\ell) = \sum_{p=0}^{\mathtt{N}-1} \left\{ \sum_{q=0}^{\mathtt{N}-1} Y(p,q)\omega^{qk} \right\} \omega^{p\ell} \qquad \text{where } \omega = e^{-j\frac{2\pi}{\mathtt{N}}}.$$

C. Controller canonical form

In this section, we recall the controller canonical form: see [5, Section 5.1] for an elaboration. Consider a system with transfer function

$$G(s) = \frac{b_{\mathtt{n}-1}s^{\mathtt{n}-1} + b_{\mathtt{n}-2}s^{\mathtt{n}-2} + \cdots + b_0}{s^{\mathtt{n}} + a_{\mathtt{n}-1}s^{\mathtt{n}-1} + \cdots + a_1 s + a_0} = \frac{n(s)}{d(s)}$$

Define the controller canonical form state space representation of the system: $\dot{x} = Ax + Bu$ and $y = Cx + Du$ (1)
where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T \in \mathbb{R}^{\mathtt{n}}$ and $D = 0$ with A, B, C as follows

$$A := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{\mathtt{n}-2} & -a_{\mathtt{n}-1} \end{bmatrix},\ B := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C := \begin{bmatrix} b_0 \\ \vdots \\ b_{\mathtt{n}-2} \\ b_{\mathtt{n}-1} \end{bmatrix}^T.$$

Matrices of the form $E := \begin{bmatrix} 0 & I_{\mathtt{N}-1} \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ (2)
are used extensively in this paper. For easy reference, we call such a matrix *the unit cyclic matrix*. Note that the matrix E is the controller canonical form of a system with a strictly proper transfer function $G(s) = \frac{n(s)}{s^{\mathtt{N}}-1}$. Further, the eigenvalues of the unit cyclic matrix are the N roots of unity.

D. Lyapunov operator

Given a matrix $A \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$, the continuous-time Lyapunov operator $\mathscr{L}_A : \mathbb{C}^{\mathtt{N}\times\mathtt{N}} \to \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is defined as

$$\mathscr{L}_A(P) := A^T P + PA. \tag{3}$$

The role of Lyapunov operators in stability theory is well-known. Note that the $\mathscr{L}_A(\cdot)$ is a linear operator and can be written in the form of a linear equation: see details in [8, Section 5.3]. We use these linear equations to prove the results in this paper. Hence a brief description of the procedure to construct them is presented next. In equation (3), consider the columns of P are $P_{*1}, P_{*2}, \ldots, P_{*\mathtt{N}}$. The linear equations corresponding to $\mathscr{L}_A(P)$ is given by

$$(I_{\mathtt{N}} \otimes A^T + A^T \otimes I_{\mathtt{N}})\tilde{P} = \mathrm{col}\left(\mathscr{L}_A(P)\right) \tag{4}$$

where $\tilde{P} := \mathrm{col}(P_{*1}, P_{*2}, \ldots, P_{*\mathtt{N}})$. The eigenvalues and 'eigenmatrices' of the Lyapunov operator have been extensively used in this paper. Below is a standard result on the eigenvalues and eigenmatrices of a Lyapunov operator stated as a proposition for easy reference. (We use indices 0 to $\mathtt{N}-1$ due to the proposition's use in the context of DFT.)

Proposition 2.1. *Consider the Lyapunov operator $\mathscr{L}_A(P) := A^T P + PA$ where $A \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ and $P \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$. Let the eigenvalues of A^T be $\{\lambda_0, \lambda_1, \ldots, \lambda_{\mathtt{N}-1}\}$ corresponding to eigenvectors $\{v_0, v_1, \ldots, v_{\mathtt{N}-1}\}$. Then $V = v_i v_j^* \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the eigenmatrix of $\mathscr{L}_A(\cdot)$ corresponding to eigenvalue $\lambda_i + \bar{\lambda}_j$ for $i,j = 0,1,\ldots,\mathtt{N}-1$.*

E. Quotient ring and Gröbner basis

In order to establish the link between Lyapunov operators and two variable polynomials, we need the notion of equivalence classes and quotient ring. Consider a commutative ring $\mathfrak{R}$ and an ideal $\mathbb{I} \subseteq \mathfrak{R}$. The quotient ring associated with $\mathfrak{R}$ is the set of all equivalence classes obtained from the equivalence relation: two elements $p_1, p_2 \in \mathfrak{R}$ are related if $p_1 - p_2 \in \mathbb{I}$. The quotient ring, i.e. the collection of equivalence classes, is denoted by $\mathfrak{R}/\mathbb{I}$. The equivalence class of an element $p \in \mathfrak{R}$ is represented as $[p]$. In this paper, we deal with the two variable polynomial ring $\mathbb{C}[x_1,x_2]$ and the ideal $\mathbb{I} \subseteq \mathbb{C}[x_1,x_2]$ generated by a given set of polynomials $\{f_1, f_2, \ldots, f_{\mathtt{t}}\} \in \mathbb{C}[x_1,x_2]$. Given a polynomial $p \in \mathbb{C}[x_1,x_2]$ we want to uniquely represent it in the quotient ring $\mathbb{C}[x_1,x_2]/\mathbb{I}$. This is a standard problem in commutative algebra and Gröbner basis helps here: see [2, Chapter 2].

Given an ideal $\mathbb{I} \subseteq \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$, the Gröbner basis is a special set of polynomials which generate $\mathbb{I}$ and possess useful properties for computational analysis. The Gröbner basis, in general, is not unique. For uniqueness of Gröbner basis we need ordering of the monomials in the polynomial ring. Such an ordering is called *term ordering*[1]. Among the different types of term ordering present in the literature, we use the lexicographic ordering for this paper: see [2, Chapter 2]. Once the ordering is fixed, we get a unique Gröbner basis[2] for an ideal $\mathbb{I} \subseteq \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$. Next we state a property of Gröbner basis that we use in this paper.

Proposition 2.2. *Let $G = \{g_1, g_2, \ldots, g_{\mathtt{t}}\}$ be a Gröbner basis (generators) for an ideal $\mathbb{I} \subseteq \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$ and let $p \in \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$. Then there exists polynomial $q_1, q_2, \ldots, q_{\mathtt{t}} \in \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$ such that $p = q_1 g_1 + q_2 g_2 + \cdots + q_{\mathtt{t}} g_{\mathtt{t}} + r$ where r is the remainder with respect to G and leading*[3] *term (LT) of $r \leqslant LT(g_i)$ for every $i = 1,2,\ldots,\mathtt{t}$. Moreover r is unique.*

Thus given an ideal $\mathbb{I} \in \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$ with Gröbner basis (generators) $G = \{g_1, g_2, \ldots, g_{\mathtt{t}}\}$, the map $\mathscr{M} : \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}] \to \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]/\mathbb{I}$ takes any element $p \in$

[1]For two monomials x and y, if $x \prec y$ and z is any other monomial, then $xz \prec yz$ and if x is any monomial in the given polynomial ring then $1 \preceq x$. There are different types of term ordering viz. lexicographic, graded lexicographic order, graded reverse lexicographic order.

[2]In general, even for a given term ordering, the Gröbner basis is not unique unless the basis is minimal and reduced. For this paper, we assume the Gröbner basis to be minimal and reduced.

[3]Let $f = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$ and $\succ$ be a term ordering. Then leading term of f is $LT(f) = a_{d(f)}x^{d(f)}$ where multidegree of f i.e. $d(f) = \max(\alpha \in \mathbb{Z}^{\mathtt{n}}_{\geq 0} : a_\alpha \neq 0)$: see [2, Chapter 2].

$\mathbb{C}[x_1,x_2,\ldots,x_{\mathtt{n}}]$ to its unique remainder r obtained by multivariate division of p by G (irrespective of the order of division). The equivalence class $[p]$ is therefore uniquely represented by the remainder $r \in \mathbb{C}[x_1,x_2,\cdots,x_{\mathtt{n}}]$.

F. Bezoutian

The Bezoutian of two polynomials is a well known concept and is widely used in stability theory. Note that based on the application, different forms of the Bezoutian are known in the literature. Given two polynomials $p(x)$ and $q(x)$, the Bezoutian of the two polynomials is defined as

$$b(x,y) := \frac{p(x)q(y)+p(y)q(x)}{x+y}. \tag{5}$$

Such a form of the Bezoutian arises in stability analysis of real polynomials (see Liénard-Chipart criterion [9, Section 13.7]), computation of stored energy in conservative systems (see [6, Section 3A]) etc. Another form of the Bezoutian well known in the literature (see [4, Page 35], for example) is

$$\tilde{b}(x,z) = \frac{p(x)q(z)-p(z)q(x)}{x-z} \tag{6}$$

Interestingly, if $p(x)$ is an odd polynomial and $q(z)$ an even polynomial, then replacing z by $-y$ transforms $\tilde{b}(x,z)$ to $b(x,y)$. For the rest of the paper, we use equation (5) as the definition of Bezoutian[4] of polynomials $p(x)$ and $q(x)$.

3. Two variable polynomial interpretation of Lyapunov operator and its link to 2D-DFT

In this section we present results that explore the relation between Lyapunov operators, two variable polynomials and 2D-DFT. The first theorem gives an insight into the link between two-variable polynomials and the eigenmatrices of Lyapunov operator corresponding to the unit cyclic matrix.

Consider the polynomial ring $\mathbb{C}[x,y]$ and the ideal $\mathbb{A} := \langle x^{\mathtt{N}}-1, y^{\mathtt{N}}-1\rangle \subset \mathbb{C}[x,y]$, $\mathtt{N} \in \mathbb{N}$. Define the map

$$\Pi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y]/\mathbb{A} \tag{7}$$

Under the action of Π, an element $p \in \mathbb{C}[x,y]$ goes to $[p]$. Note that the Gröbner basis corresponding to the ideal $\mathbb{A}$ is the set $\{x^{\mathtt{N}}-1, y^{\mathtt{N}}-1\}$ itself. Hence as mentioned in Section 2-E, $[p]$ is uniquely represented by the remainder obtained on division of p by $x^{\mathtt{N}}-1$ and $y^{\mathtt{N}}-1$ (irrespective of the order of division).

Theorem 3.1. *Consider the Lyapunov operator $\mathscr{L}_E(P) := E^TP+PE$ where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix (as defined in Equation* (2)*). Suppose $V \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the eigenmatrix of $\mathscr{L}_E(\cdot)$ corresponding to eigenvalue $\mu \in \mathbb{C}$. The two-variable polynomial associated with V is $v(x,y) := \mathbf{X}^TV\mathbf{Y}$. Consider the map $\Pi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y]/\mathbb{A}$ as defined in* (7)*. Then the following statements are equivalent*

1) $\mathscr{L}_E(V) = \mu V$.
2) $\Pi\Big((x+y)v(x,y)\Big) = \mu v(x,y)$.

[4]The 2D-DFT method reported in this paper to compute the Bezoutian in equation (5) can also be used to compute the Bezoutian of the form in equation (6) by changing variable z to $-y$ and careful manipulation of the sign in the numerator. We do not dwell on this further in the paper.

Proof. $(\mathbf{1} \Rightarrow \mathbf{2})$: From Statement 1, we have

$$\mathscr{L}_E(V) = \mu V \Rightarrow E^TV+VE = \mu V. \tag{8}$$

The two variable polynomial multiplication $(x+y)v(x,y) =$

$$(x+y)\mathbf{X}^TV\mathbf{Y} = \begin{bmatrix}\mathbf{X}\\ x^{\mathtt{N}}\end{bmatrix}^T \left\{\begin{bmatrix}\mathbf{0}_r & 0\\ V & \mathbf{0}_c\end{bmatrix} + \begin{bmatrix}\mathbf{0}_c & V\\ 0 & \mathbf{0}_r\end{bmatrix}\right\} \begin{bmatrix}\mathbf{Y}\\ y^{\mathtt{N}}\end{bmatrix}.$$

Under the action of the map Π defined in (7) the polynomial $(x+y)v(x,y)$ goes to its equivalence class represented by the unique remainder obtained when $(x+y)v(x,y)$ is divided by $x^{\mathtt{N}}-1$ and $y^{\mathtt{N}}-1$. The remainder of this multivariate polynomial division operation can be represented as

$$\begin{aligned}&\begin{bmatrix}\mathbf{X}\\ x^{\mathtt{N}}\end{bmatrix}^T \left\{\left[\begin{array}{c|c}\mathbf{0}_c & E^T\\ \hline 0 & \mathbf{0}_r\end{array}\right]\left[\begin{array}{c|c}\mathbf{0}_r & 0\\ \hline V & \mathbf{0}_c\end{array}\right] + \left[\begin{array}{c|c}\mathbf{0}_c & V\\ \hline 0 & \mathbf{0}_r\end{array}\right]\left[\begin{array}{c|c}\mathbf{0}_r & 0\\ \hline E & \mathbf{0}_c\end{array}\right]\right\} \begin{bmatrix}\mathbf{Y}\\ y^{\mathtt{N}}\end{bmatrix}\\ &= \mathbf{X}^T\{E^TV+VE\}\mathbf{Y} \qquad (9)\\ &= \mathbf{X}^T\{\mu V\}\mathbf{Y} = \mu v(x,y) \quad \text{(Using equation (8))}.\end{aligned}$$

This proves Statement 2 of Theorem 3.1.
$(\mathbf{2} \Rightarrow \mathbf{1})$: From Statement 2 and equation (9), we have

$$\begin{aligned}&\Pi\Big((x+y)v(x,y)\Big) = \mu v(x,y)\\ &\Longrightarrow \mathbf{X}^T\{E^TV+VE\}\mathbf{Y} = \mu\mathbf{X}^TV\mathbf{Y}\\ &\Longrightarrow \mathbf{X}^T\{E^TV+VE-\mu V\}\mathbf{Y} = 0\\ &\Longrightarrow E^TV+VE-\mu V = 0 \ \text{ i.e. } \mathscr{L}_E(V) = \mu V.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Next we present a theorem which relates the eigenvalues of the Lyapunov operator $\mathscr{L}_E(\cdot)$ to the 2D-DFT of a special structured matrix.

Theorem 3.2. *Consider the two-variable polynomial $r(x,y) = (x+y)$. The matrix $R \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ associated with the polynomial $r(x,y)$ is of the form $\begin{bmatrix}\tilde{I} & 0\\ 0 & 0\end{bmatrix}$ where $\tilde{I} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \in \mathbb{C}^{2\times 2}$. Suppose the 2D-DFT matrix corresponding to R is $\mathscr{F}(R) \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$. Let $\mathscr{L}_E(P) := E^TP+PE$ be a Lyapunov operator where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix. The eigenvalues of $\mathscr{L}_E(\cdot)$ are $\{\mu_1,\mu_2,\ldots,\mu_{\mathtt{M}}\}$ where $\mathtt{M} := \frac{\mathtt{N}(\mathtt{N}+1)}{2}$. Then the elements of $\mathscr{F}(R)$ are equal to the corresponding eigenvalues of $\mathscr{L}_E(\cdot)$.*

Proof. Let the eigenvalues of E^T be $\{\lambda_0,\lambda_1,\ldots,\lambda_{\mathtt{N}-1}\}$. Note that the eigenvalues of E^T are the $\mathtt{N}$ roots of unity i.e. $\lambda_i = e^{-j\frac{2\pi}{\mathtt{N}}i}$ where $i = 0,1,\ldots,\mathtt{N}-1$. Hence the eigenvalues of the Lyapunov operator $\mathscr{L}_E(\cdot)$ are $\{\bar{\lambda}_k+\lambda_\ell | k,\ell = 0,1,\ldots,\mathtt{N}-1\}$ i.e. pairwise sums of the $\mathtt{N}$ roots of unity (Proposition 2.1).
The formula for 2D-DFT is given as

$$X[k,\ell] = \sum_{p=0}^{\mathtt{N}-1}\left[\sum_{q=0}^{\mathtt{N}-1} R(p,q)e^{-j\frac{2\pi}{\mathtt{N}}qk}\right]e^{-j\frac{2\pi}{\mathtt{N}}p\ell}.$$

In this case $R(0,1) = 1$ and $R(1,0) = 1$ and all other entries of R are zero. Hence we specialize $X[k,\ell]$ for the matrix R.

$$X[k,\ell] = e^{-j\frac{2\pi}{\mathtt{N}}k}+e^{-j\frac{2\pi}{\mathtt{N}}\ell} \tag{10}$$

This shows that the elements of $\mathscr{F}(R)$ are also pairwise sum of the $\mathtt{N}$ roots of unity. Therefore the elements of $\mathscr{F}(R)$ are equal to corresponding eigenvalues of $\mathscr{L}_E(\cdot)$. □

The eigenvalues of $\mathscr{L}_E(\cdot)$ are the pairwise sums of the eigenvalues of E and the eigenvalues of E lie on the unit circle in the $\mathbb{C}$-plane hence, $\mathscr{L}_E(\cdot)$ will have a nullspace if and only if some pair of eigenvalues add to zero. This fact in addition to the symmetry in the locations of the eigenvalues of E on the unit circle results in the following theorem.

Theorem 3.3. *Consider the Lyapunov operator* $\mathscr{L}_E(P) := E^T P + PE$ *where* $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *is the unit cyclic matrix. Assume* $\mathtt{N}$ *is odd and let* $P \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *satisfy* $\mathscr{L}_E(P) = 0$. *Then* $P = 0$.

Proof. Since E is the unit cyclic matrix, the eigenvalues λ_i (where $i = 0, 1, \ldots, \mathtt{N}-1$) are $\mathtt{N}$ roots of unity i.e. $e^{-j\frac{2\pi}{\mathtt{N}}i}$. Note that when $\mathtt{N}$ is odd, no two eigenvalues of E adds to zero. As mentioned in subsection 2-D, the eigenvalues of $\mathscr{L}_E(\cdot)$ is the set $\{\lambda_i + \lambda_j | i = 0, 1, \ldots, \mathtt{N}-1 \text{ and } j = 0, 1, \ldots, \mathtt{N}-1\}$. Hence the Lyapunov operator is nonsingular if and only if $\mathtt{N}$ is odd. This proves that when $\mathtt{N}$ is odd $\mathscr{L}_E(P) = 0 \Rightarrow P = 0$. □

Using Theorem 3.3 and Statement 2 of Theorem 3.1 we infer the following corollary.

Corollary 3.4. *Consider the Lyapunov operator* $\mathscr{L}_E(P) := E^T P + PE$ *where* $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *is the unit cyclic matrix and* $P \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$. *Assume* $\mathtt{N}$ *is even. Then there exists a two variable polynomial* $v(x,y) \in \mathbb{C}[x,y]$ *such that* $(x+y)v(x,y) \in \langle x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1\rangle$.

Proof. From Theorem 3.3, we know that for $\mathtt{N}$ even there exists $V := v_i v_j^*$ such that $\mathscr{L}_E(V) = 0$. The vectors v_i and v_j are eigenvectors of E^T corresponding to λ_i and λ_j such that $\mu := \lambda_i + \bar{\lambda}_j = 0$. Let the two variable polynomial corresponding to $v_i v_j^*$ be $v(x,y)$. By Statement 2 of Theorem 3.1, we have $\Pi\Big((x+y)v(x,y)\Big) = 0$ i.e. the remainder is zero. Further, since $\langle x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1\rangle$ is not a prime ideal[5] for $\mathtt{N} > 1$, $v(x,y) \in \langle x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1\rangle$ is *not* guaranteed. This proves that the polynomial $(x+y)v(x,y) \in \langle x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1\rangle$. □

From Theorem 3.2 we have the eigenvalues of $\mathscr{L}_E(\cdot)$ and the elements of the 2D-DFT matrix $\mathscr{F}(R)$ are equal. Further, the proof from Theorem 3.3 shows that $\mathscr{L}_E(\cdot)$ is nonsingular if $\mathtt{N}$ is odd. Hence all $\mathtt{N}^2$ elements of $\mathscr{F}(R)$ are nonzero if $\mathtt{N}$ is odd. We present this result as a theorem next.

Theorem 3.5. *Consider a matrix* $R := \begin{bmatrix} \tilde{I} & 0 \\ 0 & \mathbf{0} \end{bmatrix} \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *where* $\tilde{I} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $\mathbf{0}$ *is a* $(\mathtt{N}-2)\times(\mathtt{N}-2)$ *zero matrix. Suppose* $\mathtt{N}$ *is odd. Let the two dimensional discrete Fourier transform of* R *be* $\mathscr{F}(R) \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$. *Then* $\mathscr{F}(R)$ *has all* $\mathtt{N}^2$ *elements nonzero.*

Theorem 3.5 is crucially used to find the Bezoutian of two polynomials using 2D-DFT (Algorithm 5.1).

All the results till now dealt with Lyapunov operator $\mathscr{L}_A(\cdot)$ with A as the unit cyclic matrix E. The next result gives a characterization of matrices in the nullspace of a Lyapunov operator $\mathscr{L}_A(\cdot)$ when A is in the controller canonical form.

Theorem 3.6. *Consider the Lyapunov operator* $\mathscr{L}_A(P) := A^T P + PA$ *where* $A \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ *is in the controller canonical form. Let* $Q \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *be a matrix with its last row and last column both zero and suppose* $\mathscr{L}_A(Q) = 0$. *Then* $Q = 0$.

We skip a detailed proof of the theorem due to paucity of space. Let $\tilde{q} := \text{col}\{q_1, q_2, \ldots q_{\mathtt{N}}\}$ where q_i are the columns of Q. Using the fact that the first $\mathtt{N}(\mathtt{N}-1)$ columns of the matrix in the equation $(I_{\mathtt{N}} \otimes A^T + A^T \otimes I_{\mathtt{N}})\tilde{q} = 0$ is of full column rank, it can be easily shown that $Q = 0$.

[5] An ideal $\mathfrak{p}$ in a ring $\mathfrak{R}$ is prime if $\mathfrak{p} \neq \mathfrak{R}$ and if $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

4. Comparison with the 1D case

This section contains results analogous to Section 3. Of course, the 1D results are straightforward; we present them to compare how certain features are intrinsic to the 2D case.

Consider the polynomial ring $\mathbb{C}[s]$ and for $\mathtt{N} \in \mathbb{N}$, the ideal $\mathfrak{a} := \langle x^{\mathtt{N}} - 1\rangle \subset \mathbb{C}[s]$. Define the map

$$\pi : \mathbb{C}[s] \longrightarrow \mathbb{C}[s]/\mathfrak{a} \ . \tag{11}$$

Under the action of π, an element $p \in \mathbb{C}[s]$ goes to $[p]$. As mentioned in Section 2-E, $[p]$ is represented by the remainder obtained by division of p by $s^{\mathtt{N}} - 1$.

Theorem 4.1. *Consider the operator* $\ell_E(p) := Ep$ *where* $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *is the unit cyclic matrix (as defined in equation* (2)*). Suppose* $q \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ *is the eigenvector of* $\ell_E(\cdot)$ *corresponding to the eigenvalue* λ. *Let the polynomial associated with* q *be* $v(s)$. *Consider the map* $\pi : \mathbb{C}[s] \longrightarrow \mathbb{R}[s]/\mathfrak{a}$ *as defined in* (11). *Then,*

1) $\ell_E(q) = \lambda q$.
2) $\pi\Big(sv(s)\Big) = \lambda^{(\mathtt{N}-1)} v(s)$.

Proof. Statement 1 is trivial and follows from the definition of eigenvalues and eigenvectors. We prove Statement 2 next. Let $\omega = e^{-j\frac{2\pi}{\mathtt{N}}}$. The general form of the eigenvectors of E is $q = \Big(1, \omega^k, \omega^{2k}, \ldots, \omega^{(\mathtt{N}-1)k}\Big)$ corresponding to eigenvalue $\lambda = \omega^k$ where $k = 0, 1, \ldots, \mathtt{N}-1$. Therefore

$$v(s) = 1 + \omega^k s + \omega^{2k} s^2 + \cdots + \omega^{(\mathtt{N}-1)k} s^{\mathtt{N}-1}.$$

Further, $\lambda^{\mathtt{N}-1} v(s) = \omega^{(\mathtt{N}-1)k} v(s) =$

$$\omega^{(\mathtt{N}-1)k} + \omega^{\mathtt{N}k} s + \omega^{(\mathtt{N}+1)k} s^2 + \cdots + \omega^{2(\mathtt{N}-1)k} s^{\mathtt{N}-1} \tag{12}$$

Note that $\omega^{\mathtt{N}} = 1, \omega^{\mathtt{N}+1} = \omega, \ldots, \omega^{2(\mathtt{N}-1)} = \omega^{\mathtt{N}-2}$. Hence

$$\lambda^{(\mathtt{N}-1)} v(s) = \omega^{(\mathtt{N}-1)k} + s + \omega^k s^2 + \cdots + \omega^{(\mathtt{N}-2)k} s^{\mathtt{N}-1}. \tag{13}$$

Using Euclidean long division it can be shown that

$$\begin{aligned} \pi\Big(sv(s)\Big) &= \omega^{(\mathtt{N}-1)k} + s + \omega^k s^2 + \cdots + \omega^{(\mathtt{N}-2)k} s^{\mathtt{N}-1} \\ &= \omega^{(\mathtt{N}-1)k} v(s) = \lambda^{(\mathtt{N}-1)} v(s) \quad \text{(From equation (13))}. \end{aligned}$$

This proves Statement 2 of Theorem 4.1. □

Unlike the 2-variable counterpart Theorem 3.1, where the two statements were *equivalent*, in Theorem 4.1 above, Statement 1 is only a sufficient condition for Statement 2. Consider $\lambda = e^{-j\frac{2\pi}{6}3}$ and the vector $v = (1,1,1)$. Here $\mathtt{N} = 3$. The polynomial corresponding to v is $v(s) = 1 + s + s^2$. Since $\lambda^{\mathtt{N}-1} = \lambda^2 = 1$, it is easy to see that vector v follows Statement 2 of Theorem 4.1.

$$\pi\Big(sv(s)\Big) = 1 + s + s^2 = \lambda^{\mathtt{N}-1} v(s).$$

However, λ is not an eigenvalue of E and hence does not satisfy Statement 1 of Theorem 4.1.

The next theorem is the one variable counterpart of Theorem 3.2 and Theorem 3.5.

Theorem 4.2. *Consider the one-variable polynomial $m(s) = s$. The vector $w \in \mathbb{C}^{\mathtt{N}}$ associated with the polynomial $m(s)$ is of the form $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$. Consider the 1D-DFT vector corresponding to w be $\mathfrak{f}(w) \in \mathbb{C}^{\mathtt{N}}$. Further, let $\ell_E(w) := Ew$ be an operator where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix. The eigenvalues of $\ell_E(\cdot)$ are $\{\lambda_0, \lambda_1, \ldots, \lambda_{\mathtt{N}-1}\}$. Then,*

1) *Every element of $\mathfrak{f}(w)$ is nonzero.*
2) *Elements of the $\mathfrak{f}(w)$ are the eigenvalues of $\ell_E(\cdot)$.*

Proof. **1:** As discussed in Section 2-B, 1D-DFT is given by $X = \Omega x$. Hence, $\mathfrak{f}(w) = \left(1, \omega, \omega^2, \ldots, \omega^{\mathtt{N}-1}\right)$ where $\omega = e^{-j\frac{2\pi}{\mathtt{N}}}$. The elements of $\mathfrak{f}(w)$ are the N roots of unity and hence are nonzero. This proves statement 1.
2: The characteristic equation of E is $s^{\mathtt{N}} - 1 = 0$. Hence the eigenvalues of E are the $\mathtt{N}$ roots of unity. We have seen above that the elements of $\mathfrak{f}(w)$ are also the $\mathtt{N}$ roots of unity. This proves statement 2. □

Note that Theorem 3.5 and Statement 1 of Theorem 4.2 are analogous. The vector $\mathfrak{f}(\cdot)$ or matrix $\mathscr{F}(\cdot)$ having all elements nonzero indicates presence of all frequencies in the argument (say signal in 1D-DFT case and image in 2D-DFT case). All elements of $\mathfrak{f}(w)$ being nonzero is expected since the vector w represents an impulse signal (possibly shifted) and hence contains all the frequencies. Interestingly however its two dimensional counterpart R, has all frequency components if $\mathtt{N}$ is odd. Hence the order of the matrix plays an important role in deciding the presence of certain frequencies. This is an intrinsic property in the 2D-DFT case.

The next lemma is an analogy to Theorem 3.3.

Lemma 4.3. *Consider the operator $\ell_E(p) := Ep$ where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix. Suppose vector $v \in \mathbb{C}^{\mathtt{N}}$ satisfies $\ell_E(v) = 0$. Then $v = 0$.*

The proof is skipped due to paucity of space.

Note that the order of the system plays an important role in case of the Lyapunov operator result (Theorem 3.3).

The comparison of the results related to the Lyapunov operator $\mathscr{L}_E(\cdot)$ in Section 3 and that of the operator $\ell_E(\cdot)$ in Section 4 shows that for $\mathscr{L}_E(\cdot)$ the order of the operator plays a crucial role. We use this intrinsic property of $\mathscr{L}_E(\cdot)$ and properties of 2D-DFT to compute the Bezoutian of two polynomials in the next section.

5. Bezoutian and the Lyapunov equation

In this section we describe a procedure to compute the Bezoutian of two polynomials using 2D-DFT. However, before we present the procedure we discuss the link between $\mathscr{L}_E(\cdot)$ and the Bezoutian $b(x,y)$. The Bezoutian $b(x,y)$ in equation (5) is rewritten as

$$(x+y)b(x,y) = p(x)q(y) + p(y)q(x) \qquad (14)$$

Consider the deg $p(x) = \mathtt{N}-1$ and deg $q(x) = \mathtt{M}-1$ where $\mathtt{N} \geqslant \mathtt{M}$. Rewrite the right hand side of equation (14) as

$$p(x)q(y) + p(y)q(x) = \mathbf{X}^T \Phi \mathbf{Y} \text{ where } \Phi \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}. \qquad (15)$$

Note that the degree of $b(x,y)$ is $\mathtt{N}-2$. Therefore let

$$b(x,y) := \sum_{i,j=0}^{\mathtt{N}-2} B_{ij} x^i y^j \text{ where } B = [B_{i,j}] \in \mathbb{C}^{(\mathtt{N}-1)\times(\mathtt{N}-1)}.$$

Appending zeros in B we rewrite $b(x,y)$ as

$$b(x,y) = \mathbf{X}^T \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Y} =: \mathbf{X}^T \tilde{B} \mathbf{Y},\ \tilde{B} \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}} \qquad (16)$$

Interestingly the operation $(x+y)b(x,y)$ in equation (14) can be related to the Lyapunov operator $\mathscr{L}_E(\cdot)$ where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix. Using Equation (15) and (16), we rewrite Equation (14) as

$$\mathbf{X}^T \left(E^T \tilde{B} + \tilde{B} E\right) \mathbf{Y} = \mathbf{X}^T \Phi \mathbf{Y} \qquad (17)$$

Hence solution of the Lyapunov equation $E^T \tilde{B} + \tilde{B} E = \Phi$ with the additional constraint that the solution $\tilde{B}$ has the structure shown in equation (16) is equivalent to finding the Bezoutian of the polynomials $p(x)$ and $q(x)$.

Now that we have established the link between $\mathscr{L}_E(\cdot)$ and $b(x,y)$ we present a new method to solve the Bezoutian using 2D-DFT. Consider the matrix $R \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ as given in Theorem 3.5. Rewrite the term $x+z$ in equation (14) as

$$x + z = \mathbf{X}^T R \mathbf{Y} \qquad (18)$$

Note that two variable polynomial multiplication is equivalent to 2D-convolution. Further, 2D-convolution is equivalent to entry-wise multiplication in the 2D-DFT domain (see [7, Section 5.2.8]). Hence from equation (14), (15), (16) and (18), we have

$$R * \tilde{B} = \Phi \Rightarrow \mathscr{F}(R) \odot \mathscr{F}(\tilde{B}) = \mathscr{F}(\Phi) \qquad (19)$$

Hence, we compute the Bezoutian matrix using the formula

$$B = \tilde{B}(1:\mathtt{N}-1, 1:\mathtt{N}-1) \text{ where } \tilde{B} = \mathscr{F}^{-1}[\mathscr{F}(\Phi)./\mathscr{F}(R)] \qquad (20)$$

Note that for the entry-wise division to be possible, $\mathscr{F}(R)$ must have *every* element nonzero. From Theorem 3.5, we conclude that $\mathscr{F}(R)$ has all entries nonzero if $\mathtt{N}$ is odd. For the case when $\mathtt{N}$ is even i.e. when the polynomial $p(x)q(y) + p(y)q(x)$ has an even degree (in the lex ordering), we append the matrices Φ and R with zeros. For even $\mathtt{N}$, construct

$$\tilde{\Phi} := \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{(\mathtt{N}+1)\times(\mathtt{N}+1)} \text{ and } \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{(\mathtt{N}+1)\times(\mathtt{N}+1)}$$

Then the Bezoutian matrix is given by

$$B := \tilde{B}(1:\mathtt{N}-1, 1:\mathtt{N}-1) \text{ where } \tilde{B} := \mathscr{F}^{-1}[\mathscr{F}(\tilde{\Phi})./\mathscr{F}(\tilde{R})] \qquad (21)$$

Thus we have a procedure to compute the Bezoutian of two polynomials using 2D-DFT. Further, since the solution to the Lyapunov equation $E^T \tilde{B} + \tilde{B} E = \Phi$ is the same as the Bezoutian matrix corresponding to $b(x,y)$, we have a procedure to solve the Lyapunov operator corresponding to the unit cyclic matrix. An algorithm to compute the Bezoutian matrix B is presented next.

Algorithm 5.1 Bezoutian computation algorithm

Input: $p(x), q(x) \in \mathbb{C}[x]$ where deg $p(x) \geqslant$ deg $q(x)$.
Output: Bezoutian matrix $B \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$.

1: Extract coefficients of polynomials $p(x)$ and $q(x)$ into arrays $N \in \mathbb{C}^{1\times\mathtt{N}}$ and $D \in \mathbb{C}^{1\times\mathtt{M}}$. (Here $\mathtt{N} \geqslant \mathtt{M}$).
2: Equate the lengths of array of N and D by appending zeros to D i.e. $D(\mathtt{M}+1:\mathtt{N}-\mathtt{M}) = 0$. Result: $D \in \mathbb{C}^{1\times\mathtt{N}}$.

Construct $\Phi := N^T D + D^T N \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$.
Construct $R = \begin{bmatrix} \tilde{I} & 0 \\ 0 & \mathbf{0} \end{bmatrix} \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ where $\tilde{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
if $\mathtt{N}$ is odd **then**
  $\tilde{\Phi} = \Phi$ and $\tilde{R} = R \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$.
else (i.e $\mathtt{N}$ is even)
  $\tilde{\Phi} = \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix}$ and $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{(\mathtt{N}+1)\times(\mathtt{N}+1)}$.
end if
Compute 2D-DFT of $\tilde{\Phi}$ and $\tilde{R}$. **Result:** $\mathscr{F}(\tilde{\Phi})$ and $\mathscr{F}(\tilde{R})$.
Using entry-wise division compute $T := \mathscr{F}(\tilde{\Phi})./\mathscr{F}(\tilde{R})$.
Use inverse 2D-DFT to find $\tilde{B} = \mathscr{F}^{-1}(T)$.
The Bezoutian matrix is given by

$$B = \tilde{B}(1:\mathtt{N}-1, 1:\mathtt{N}-1).$$

6. Example

In this section we use a few examples to demonstrate the results reported in Section 3 and 5. We start the section with an example to demonstrate Theorem 3.1.

Example 6.1. *Consider the Lyapunov operator for* $\mathtt{N} = 4$ *i.e.* $E = \begin{bmatrix} 0 & I_3 \\ 1 & 0 \end{bmatrix}$. *The eigenvalues of* E^T *are* $\{1, -i, -1, i\}$.

Corresponding to $\lambda_0 = 1$ *and* $\lambda_2 = -1$ *the eigenmatrix is* $P_2 = \begin{bmatrix} \mathbf{1}_4 & -\mathbf{1}_4 & \mathbf{1}_4 & -\mathbf{1}_4 \end{bmatrix}$. *Hence,* $\mathscr{L}_E(P_2) = E^T P_2 + P_2 E = 0$. *(Statement 1 Theorem 3.1).*

$$(x+y)p_2(x,y) = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \end{bmatrix}$$

It is easy to verify now that $\Pi\Big((x+y)p_2(x,y)\Big) = 0$ *because* $(x+y)p_2(x,y) = (x^4-1)(-y^3+y^2-y+1) - (y^4-1)(x^3+x^2+x+1)$ *(as given in Statement 2 Theorem 3.1). Thus for* $\mathtt{N} = 4$, *there exist polynomial* $(x+y)p_2(x,y) \in \langle x^4-1, y^4-1 \rangle$ *as stated in Corollary 3.4.*

The next example demonstrates the method to compute the Bezoutian matrix using 2D-DFT as described in Algorithm 5.1.

Example 6.2. *Consider the two variable polynomial* $p(x) = 8x^2+1$ *and* $q(x) = 6x^3+x$. *From equation* (5), *we have*

$$b(x,y) = \frac{48x^3y^2 + 6x^3 + 48x^2y^3 + 8x^2y + 8xy^2 + x + 6y^3 + y}{x+y}$$

Hence

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 6 \\ 1 & 0 & 8 & 0 \\ 0 & 8 & 0 & 48 \\ 6 & 0 & 48 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\mathtt{N}$ *is even, we append a zero row and column after the last row and column of* Φ *and* R *to get* $\tilde{\Phi}$ *and* $\tilde{R}$. *Using equation* (21) *we have*

$$B = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & 0 \\ 6 & 0 & 48 \end{bmatrix} \quad i.e.\ b(x,y) = 48x^2y^2 + 6x^2 + 2xy + 6y^2 + 1.$$

7. Conclusion

In this paper we presented a two variable polynomial interpretation of Lyapunov operator corresponding to the unit cyclic matrix and showed its link to the Bezoutian of two polynomials. In Theorem 3.1, we showed the interplay between eigenmatrices of the Lyapunov operator, two variable polynomials and multivariate polynomial division. The link between 2D-DFT values of a special structured matrix R and eigenvalues of Lyapunov operator was explored in Theorem 3.2. Interestingly, this special structured matrix is associated with the Bezoutian of two polynomials as shown in equation (18) and (19). In order to get an idea about the intrinsic features of the Lyapunov operator, we presented some 1D results in Section 4. A few of this results were analogous to their 2D-counterpart whereas the few others captured the fact that order of the Lyapunov operator plays an important role in characterizing its nullspace and image (Theorem 3.3). In Section 5, we showed that solution of the Lyapunov equation in (17) is equivalent to solving the Bezoutian of two polynomials.

The link between 2D-DFT and Bezoutian brought out by Theorem 3.5 and Sections 5 was used to formulate an algorithm to compute the Bezoutian using 2D-DFT technique (Algorithm 5.1). This algorithm can also be used to solve Lyapunov equations corresponding to unit cyclic matrix. The relation between Lyapunov operator corresponding to unit cyclic matrix and 2D-DFT explored in this paper opens up possibilities to compute solutions of Lyapunov equation in general using 2D-DFT technique. This is an area which needs further research. Note that the use of 2D-DFT technique to compute the Bezoutian (Algorithm 5.1) possibly leads to lower computation (due to use of structured and sparse matrices) and lower error (due to use of orthogonal matrices) compared to the techniques present in the literature. The complexity and error analysis of this method is another aspect that needs exploration.

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