2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

On the link between storage functions of allpass systems and Gramians.

Chayan Bhawal, Debasattam Pal and Madhu N. Belur

Abstract— **In this paper, we bring out a link between storage functions of allpass systems and observability/controllability Gramians. We show that under a particular transformation, the storage function of an allpass system is induced by an identity matrix. Interestingly, certain algebraic relations between the states and costates/dual states of an allpass system capture the information of the storage function of the system. Further, we also prove that certain difference dynamics between states and costates of an allpass system is always present in the orthogonal complement of its controllable subspace.**

Keywords: Observability Gramians, Controllability Gramians, Costates/Dual states, Adjoint systems, Algebraic Riccati inequality, Balanced realization.

1. Introduction

Allpass systems are a special class of bounded real systems that find applications in different areas of communication, control [5]. Like any other bounded real system, allpass systems therefore admit storage functions. Such functions can be computed using the solutions of the bounded real linear matrix inequality (LMI) (see [2, Chapter 7]). One of the most popular methods to compute solutions of such an LMI is by using algebraic Riccati equation (ARE). However, existence of an ARE requires certain *invertability conditions* to be satisfied: which may not be satisfied always [8]. Allpass systems are one such class of systems for which the ARE does not exist. In this paper, we show that the storage function of an allpass system has an interesting link to the observability/controllability Gramian of the system. We also show that under a particular linear transformation (called the *balanced transformation*) the storage function of such a system is induced by an identity matrix, i.e., if x is the state-vector of the system then, the storage function is given by $x^T x$. This fascinating result stems from the fact that the Hankel singular values of an allpass system are unity: see [6]. Note that allpass systems under a suitable transformation gets mapped to lossless systems. LC oscillators, spring-mass mechanical systems are examples of lossless systems. We show that the set of storage functions remains invariant under such a transformation. Hence, study of storage functions of allpass systems and that of lossless systems are the same. Further, we also show that for lossless systems certain algebraic relations between the states and its dual states capture the information of the storage function of the system. These relations can be used to compute the storage function of a lossless system (in general true for conservative systems): see algorithms in [4]. All these results put together will give us an understanding of the properties of storage functions of allpass/lossless systems.

The paper is structured as follows: In Section 2 we review the notation and preliminaries required for the paper. We establish the link between the storage function of an allpass system, its observability/controllability Gramian and balanced realization in Section 3. In Section 4, we present a link between the storage function of an allpass system and its lossless counterpart. We unveil the algebraic relations between the states and costates of a lossless/allpass system in Section 5. Finally we present the concluding remarks in Section 6.

The authors are in the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. Corresponding author email: chayanbhawal@ee.iitb.ac.in. This work was supported in part by IIT Bombay (15IRCCSG 012), SERB (DST), BRNS and DST-INSPIRE Faculty Grant, Department of Science and Technology (DST), Govt. of India (IFA14-ENG-99), India.

2. Notation and Preliminaries

We use the symbols $\mathbb{R}$ and $\mathbb{C}$ to denote the sets of real and complex numbers, respectively. We use $\mathbb{R}[\xi]$ to denote the ring of polynomials in one indeterminate ξ with real coefficients. The set $\mathbb{R}^{\mathtt{w}\times\mathtt{p}}[\xi]$ denotes all $\mathtt{w}\times\mathtt{p}$ matrices with entries from $\mathbb{R}[\xi]$. We use $\bullet$ when a dimension is not required to be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows. $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$ denotes the set of polynomial matrices in two indeterminates: ζ and η, having $\mathtt{n}$ rows and $\mathtt{m}$ columns. $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}})$ denotes the set of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{m}}$, and $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{m}})$ denotes the subspace of all compactly supported trajectories in $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}})$. A subspace spanned by the columns of a matrix B is denoted as $\langle B\rangle$.

A. Behavior

The behavioral approach to control theory is used throughout the paper. We give a brief review of the same in this section: for a detailed review see [10].

Definition 2.1. *A linear differential behavior $\mathfrak{B}$ is defined as the subspace of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}})$ functions that satisfy a system of linear ordinary differential equations with constant coefficients, i.e.,*

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}, \qquad (1)$$

where $R(\xi) \in \mathbb{R}^{\bullet\times\mathtt{m}}[\xi]$.

Variables w in equation (1) are called the *manifest variables* of the behavior $\mathfrak{B}$. We use $\mathfrak{L}^{\mathtt{m}}$ to denote linear differential behaviors with $\mathtt{m}$ manifest variables. The behavior $\mathfrak{B}$ can also be represented as $\ker R(\frac{d}{dt})$ and hence such a representation is called the *kernel representation* of the behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{m}}$. As described in [10, Theorem 2.5.23], $R(\xi)$ is assumed to be of full row rank throughout the paper, without loss of generality. Note that there are different ways of partitioning the manifest variables into inputs and outputs, however the number of inputs and outputs of the behavior remains invariant. We denote the number of inputs and outputs of a behavior $\mathfrak{B}$ with $\mathtt{i}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B})$, respectively. Further note that $\mathtt{i}(\mathfrak{B}) = \mathtt{m} - \mathtt{p}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B}) = \text{rank } R(\xi)$. In this paper, we use the notion of controllability in the behavioral sense: see [10, Definition 5.2.2]. A behavior $\mathfrak{B} = \ker R(\frac{d}{dt})$ is controllable if and only if $R(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. This is the generalization of the well known PBH rank test for state-controllability: see [10, Theorem 5.2.5]. Further, since state space systems play a central role in this paper, controllability/observability of behavior/system is same as state space controllability/observability. We use notation $\mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ to represent the set of all controllable behaviors with $\mathtt{m}$ manifest variables.

Another important concept that we require in this paper is that of Σ-orthogonal complement behavior $\mathfrak{B}^{\perp_\Sigma}$ of a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{m}}_{\text{cont}}$. We define this next.

Definition 2.2. *Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ and a nonsingular, symmetric matrix $\Sigma \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$. The Σ-orthogonal complement $\mathfrak{B}^{\perp_\Sigma}$ of $\mathfrak{B}$ is defined as*

$$\mathfrak{B}^{\perp_\Sigma} := \{v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}}) \mid \int_{-\infty}^{\infty} v^T \Sigma w \, dt = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}\}.$$

The behavior $\mathfrak{B}^{\perp_\Sigma}$ is also known in the literature as the *adjoint* system of $\mathfrak{B}$: see details in [11, Section 10]. If (A,B,C,D) is a minimal state-space representation of a system $\mathfrak{B}$ then, with respect to $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ the system $\mathfrak{B}^{\perp_\Sigma}$, with manifest variables (e,f), admits a minimal state-space representation of the form $\dot{z} = -A^T z + C^T e$, $f = B^T z - D^T e$ (see [12, Section VI-A]). The vector z represents the dual-states of the system $\mathfrak{B}$ and satisfies $\frac{d}{dt} x^T z = u^T f + y^T e$ for $(u,y) \in \mathfrak{B}$ and $(e,f) \in \mathfrak{B}^{\perp_\Sigma}$. We use the terms *system* and *behavior* interchangeably throughout the paper.

B. Quadratic differential forms and dissipativity

In this section, we give a brief introduction to the link between dissipativity, two-variable polynomial matrices and quadratic forms of manifest variables and their derivatives. A detailed exposition to this can be found in [11].

Consider a two-variable polynomial matrix $\phi(\zeta,\eta) := \sum_{j,k} \phi_{jk} \zeta^j \eta^k \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$, where $\phi_{jk} \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$. The QDF Q_ϕ induced by $\phi(\zeta,\eta)$ is a map $Q_\phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined as

$$Q_\phi(w) := \sum_{j,k} (\frac{d^j w}{dt^j})^T \, \phi_{jk} \, (\frac{d^k w}{dt^k}).$$

In this paper, we use $\phi = \Sigma \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ and hence $Q_\Sigma = w^T \Sigma w$. Next we define dissipative systems using QDFs.

Definition 2.3. *Let $\Sigma \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ be a symmetric matrix. A controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ is said to be Σ-dissipative if*

$$\int_{\mathbb{R}} Q_\Sigma(w)\, dt \geqslant 0 \quad \textit{for every} \quad w \in \mathfrak{B} \cap \mathfrak{D}. \tag{2}$$

$Q_\Sigma(w)$ is the rate of supply of energy to the system and is called *the supply rate* (see [9] for supply rate and its link to classical notion of gain/phase margin). For simplicity, we also call Σ the supply rate. In this paper, we use real, symmetric, constant, nonsingular Σ only. It is a known result from [11, Remark 5.11] that for a Σ-dissipative system

$$\mathtt{i}(\mathfrak{B}) = \sigma_+(\Sigma), \tag{3}$$

where $\sigma_+(\Sigma)$ represents the number of positive eigenvalues of Σ and $\mathtt{i}(\mathfrak{B})$ represents the number of inputs. This is called the *maximum input cardinality* condition.

Next we review an important property of controllable, dissipative systems. A controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ is Σ-dissipative if and only if there exists a QDF $Q_\psi(w)$ such that

$$\frac{d}{dt} Q_\psi(w) \leqslant Q_\Sigma(w) \text{ for all } w \in \mathfrak{B}. \tag{4}$$

The QDF Q_ψ is called a storage function for $\mathfrak{B}$ with respect to the supply rate Σ. Note that inequality (4) is a version of the law of conservation of energy. It means that for dissipative systems, the rate of change in stored energy can not exceed the supplied power. Storage function is used to capture the idea of stored energy in inequality (4).

C. Conservative systems

An important class of dissipative systems is that of conservative systems. We call a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ *conservative* with respect to a supply rate $\Sigma \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ if

$$\int_{\mathbb{R}} Q_\Sigma(w)\, dt = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Note that conservative systems satisfy inequality (2) with equality and hence such systems satisfy equation (4), too, with equality. Further, it is known that for controllable systems, the energy stored (i.e., the storage function) can be expressed in the form $x^T K x$, where $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is symmetric and x corresponds to states of the system. Hence, for conservative, controllable systems, inequality (4) becomes

$$\frac{d}{dt}(x^T K x) = Q_\Sigma(w) \qquad \text{for all } w \in \mathfrak{B}. \tag{5}$$

Systems that are conservative with respect to the supply rate $2u^T y$, i.e., $\Sigma = \begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{p}} & 0 \end{bmatrix}$, where u and y are the input and output of the system, respectively are called lossless systems. We call this supply rate *passivity supply rate*. LC circuits, spring-mass systems are a few examples of such systems. On the other hand, systems conservative with respect to $u^T u - y^T y$, i.e., $\Sigma = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}$ are called allpass systems. We call this supply rate the *bounded real supply rate*. In this paper, we deal with allpass and lossless systems only.

D. Dissipativity LMI

Consider a system with minimal input-state-output (i/s/o) representation

$$\dot{x} = Ax + Bu \quad y = Cx + Du,$$

where $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $B, C^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{p}}$ and $D \in \mathbb{R}^{\mathrm{p}\times\mathrm{p}}$. Using the dissipation inequality (4) and $Q_\psi = x^T K x$, it can be shown that a system is dissipative with respect to the *bounded real supply rate* if and only if

$$\begin{bmatrix} A^T K + KA + C^T C & KB + C^T D \\ B^T K + D^T C & -(I_{\mathrm{p}} - D^T D) \end{bmatrix} \leqslant 0. \tag{6}$$

On the other hand, a system is dissipative with respect to the *passivity supply rate* if and only if

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{7}$$

We call these LMIs the *dissipativity LMIs* and the solutions of these LMIs act as the storage functions of the corresponding system. Note that conservative systems admit their corresponding dissipativity LMIs with equality. Therefore, a behavior $\mathfrak{B}$ is conservative with respect to bounded real supply rate (allpass) if and only if there exists a symmetric $K \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ such that

$$A^T K + KA + C^T C = 0 \quad \text{and} \quad KB + C^T D = 0. \tag{8}$$

Further, $x^T K x$ is the unique storage function of the system. On the other hand, a behavior $\mathfrak{B}$ is conservative with respect to passivity supply rate if and only if there exists a symmetric $K \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ such that

$$A^T K + KA = 0 \quad \text{and} \quad B^T K - C = 0. \tag{9}$$

Further, $x^T K x$ is the unique storage function of the system.

E. Gramian and balancing

Consider a stable, controllable and observable system with a minimal i/s/o representation $\dot{x} = Ax + Bu$ and $y = Cx + Du$. Then the Lyapunov equations $AP + PA^T + BB^T = 0$ and $A^T Q + QA + C^T C = 0$ have unique solutions $P > 0$ and $Q > 0$, respectively: see [1, Section 4.3]. P and Q are called the (infinite) controllability and observability Gramian matrices, respectively. Interestingly, the storage function of an allpass system also satisfies one of these Lyapunov equations. We explore this link between Lyapunov equations, Gramian and storage function of an allpass system in Section 3. To unravel an interesting property of the storage function of an allpass system, we need the concept of balancing. A system is said to be represented in a *balanced state space basis* if P and Q are equal. From a linear algebraic viewpoint, it is the simultaneous diagonalization of two positive definite matrices P and Q. The proposition next gives a procedure to compute the balancing transformation of a system: see [1, Lemma 7.3].

Proposition 2.4. *Consider a controllable, observable and stable system with a minimal i/s/o representation $\dot{x} = Ax + Bu$ and $y = Cx + Du$ and let the corresponding Gramians be P and Q. Assume $P := UU^*$ and $U^* QU = KS^2K^*$ then a balancing transformation is given by $T = \sqrt{S}\ K^* U^{-1}$ (see footnote 1 for definition*[1] *of $\sqrt{S}$).*

3. Stored energy and balanced realization

In this section, we report the link between allpass systems, Gramians and balanced realization.

One of the main results of this section is that the storage function of an allpass system is the observability Gramian of the system. We report this result next.

Theorem 3.1. *Consider a stable, allpass system with a minimal i/s/o representation $\dot{x} = Ax + Bu$, $y = Cx + Du$, where $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $B, C^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{p}}$ Assume $Q \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ to be the observability Gramian of the system. Then, $x^T Q x$ is the unique storage function of the system.*

Proof. We know that for an allpass system there exists a $K \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ that satisfies equations (8). Since the system is stable, the equation $A^T K + KA + C^T C = 0$ must have a unique solution. Note that $A^T K + KA + C^T C = 0$ is the observability Gramian equation. Therefore, $K = Q$. Hence, we conclude using equation (8) that $x^T Q x$ is a unique storage function of the system. □

For an allpass system, the observability Gramian Q and controllability Gramian P are related as $PQ = I_{\mathrm{n}}$: see [6, Theorem 5.1]. Hence, the matrix P^{-1} also induces the storage function associated with an allpass system.

The next result uses the concept of balanced basis as stated in Section 2-E to infer a noteworthy property of the storage function of an allpass system.

Theorem 3.2. *Consider a stable, allpass system $G(s) \in \mathbb{R}(s)^{\mathrm{p}\times\mathrm{p}}$. Let (A, B, C, D) be a minimal state space realization of $G(s)$ in a balanced state space basis. Suppose the symmetric matrix $K \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ induces the storage function $x^T K x$ associated with the state space representation (A, B, C, D). Then $K = I_{\mathrm{n}}$.*

Proof. Let the observability and controllability Gramian in the balanced state space basis be W_o and W_r, respectively. By the definition of balanced state space basis as described in Section 2-E, we have $W_o = W_r = W$. Since $G(s)$ is allpass, using [6, Theorem 5.1], we have $W_o W_r = I_{\mathrm{n}} \implies W^2 = I_{\mathrm{n}}$. Further, since $G(s)$ is stable, we have $W > 0$. Therefore, $W = I_{\mathrm{n}}$. Thus, using Theorem 3.1, we conclude that I_{n} induces the storage function of the system in the balanced basis. This completes the proof of the theorem. □

Exactly the same set of arguments, used to prove Theorem 3.2, can be used to show that the Hankel singular values of allpass systems are unity: see [6, Section 5]. Hence, from Theorem 3.2 we infer that the unique storage function of the allpass system in balanced basis is induced by a diagonal matrix with Hankel singular values as its diagonal entries.

[1] A matrix $R = R^T \geqslant 0$ is said to be the square root of another matrix $S = S^T \geqslant 0$ if $R^2 = S$. We denote such a matrix as $\sqrt{S} := R$.

4. Link between storage functions of bounded real systems and passive systems

Next we establish the link between the storage function of an allpass and lossless system. We prove the result, in general, for bounded real and passive systems first. Note that a system $\mathfrak{B}_b$ with manifest variables (u,y) is bounded real if and only if the system $\mathfrak{B}_p$ with manifest variables $\left(\frac{u+y}{\sqrt{2}},\frac{u-y}{\sqrt{2}}\right)$ is passive: see [7, Chapter 5]. Further, if an i/s/o representation of $\mathfrak{B}_b$ is $\dot{x}=Ax+Bu$ and $y=Cx+Du$, then an i/s/o representation of the corresponding passive system $\mathfrak{B}_p$ is $\dot{x}=\left(A-B(I+D)^{-1}C\right)x+\frac{1}{\sqrt{2}}\left(B+B(I+D)^{-1}(I-D)\right)v$ and $r=-\sqrt{2}(I+D)^{-1}Cx+(I+D)^{-1}(I-D)v$, where $v:=\frac{u+y}{\sqrt{2}}$ and $r:=\frac{u-y}{\sqrt{2}}$, see [2]. In this paper, we call $\mathfrak{B}_p$ to be the *passive counterpart* of $\mathfrak{B}_b$ and $\mathfrak{B}_b$ to be the *bounded real counterpart* of $\mathfrak{B}_p$.

Next we show that the set of storage function of $\mathfrak{B}_b$ and $\mathfrak{B}_p$ are the same.

Theorem 4.1. *Consider a bounded real system $\mathfrak{B}_b \in \mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ with manifest variables (u,y). Suppose $\mathfrak{B}_p$ is the passive counterpart of $\mathfrak{B}_b$ with manifest variables (v,r). Then the set of storage functions of $\mathfrak{B}_b$ and $\mathfrak{B}_p$ remains invariant.*

Proof. Define $J:=\begin{bmatrix} I_{\mathtt{p}} & I_{\mathtt{p}} \\ I_{\mathtt{p}} & -I_{\mathtt{p}}\end{bmatrix}$. Since $\mathfrak{B}_p$ is the passive counterpart of $\mathfrak{B}_b$, therefore $v=\frac{u+y}{\sqrt{2}}$ and $r=\frac{u-y}{\sqrt{2}}$. Note that for controllable systems the storage function is x^TKx, where x are the states of the system $\mathfrak{B}_b$ and $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Hence, inequality (4) adapted to the bounded real system $\mathfrak{B}_b$ gives

$$\frac{d}{dt}(x^TKx)\leqslant \begin{bmatrix} u \\ y\end{bmatrix}^T\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -I_{\mathtt{p}}\end{bmatrix}\begin{bmatrix} u \\ y\end{bmatrix}=\frac{1}{2}\begin{bmatrix} u \\ y\end{bmatrix}^T J^T\begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{p}} & 0\end{bmatrix}J\begin{bmatrix} u \\ y\end{bmatrix}.$$

Therefore, $\frac{d}{dt}(x^TKx)\leqslant \begin{bmatrix} \frac{u+y}{\sqrt{2}} \\ \frac{u-y}{\sqrt{2}}\end{bmatrix}^T\begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{p}} & 0\end{bmatrix}\begin{bmatrix} \frac{u+y}{\sqrt{2}} \\ \frac{u-y}{\sqrt{2}}\end{bmatrix}$.

This means that any symmetric $K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ that induces a storage function for $\mathfrak{B}_b$ also induces a storage function for $\mathfrak{B}_p$. Along similar lines, it can be shown that any K that induces a storage function for $\mathfrak{B}_p$ does the same for $\mathfrak{B}_b$. This proves that the set of storage function for $\mathfrak{B}_b$ and $\mathfrak{B}_p$ remains invariant. □

Note that the passive counterpart of an allpass system $\mathfrak{B}_{all}$ with transfer function $G(s)$ is a lossless system $\mathfrak{B}_\ell$ with transfer function $[1-G(s)][1+G(s)]^{-1}$. Hence, we call $\mathfrak{B}_\ell$ the *lossless counterpart* of $\mathfrak{B}_{all}$ and $\mathfrak{B}_{all}$ the *allpass counterpart* of $\mathfrak{B}_\ell$. The next corollary relates the storage function of an allpass system and its lossless counterpart.

Corollary 4.2. *Consider a lossless behavior $\mathfrak{B}_\ell\in\mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ with a minimal i/s/o representation $\dot{x}=Ax+Bu$ and $y=Cx+Du$. Let the bounded real counterpart of $\mathfrak{B}_\ell$ be $\mathfrak{B}_{all}$. $\mathfrak{B}_{all}$ has an i/s/o representation $\dot{x}=\hat{A}x+\hat{B}v$ and $r=\hat{C}x+\hat{D}v$, where $v:=\frac{u+y}{\sqrt{2}}$, $r:=\frac{u-y}{\sqrt{2}}$, $\hat{A}:=\left(A-B(I+D)^{-1}C\right)$, $\hat{B}=\frac{1}{\sqrt{2}}\left(B+B(I+D)^{-1}(I-D)\right)$, $\hat{C}=-\sqrt{2}(I+D)^{-1}C$ and $\hat{D}:=(I+D)^{-1}(I-D)$. Assume Q is the observability Gramian of $\mathfrak{B}_{all}$. Then x^TQx is the unique storage function of $\mathfrak{B}_\ell$.*

Proof. From Theorem 3.1, we know that the observability Gramian Q induces the storage function of $\mathfrak{B}_{all}$. Since $\mathfrak{B}_\ell$ is the lossless counterpart of $\mathfrak{B}_{all}$, from Theorem 4.1 we know that the storage function of $\mathfrak{B}_{all}$ and $\mathfrak{B}_\ell$ are the same. Therefore, x^TQx is the unique storage function of $\mathfrak{B}_\ell$. □

Note that the poles of a lossless system are on the imaginary axis of the $\mathbb{C}$-plane. Therefore, we cannot compute the observability/controllability Gramian using the Lyapunov equations. However, from Corollary 4.2, it is clear that the observability Gramian of it's allpass counterpart induces the storage function of such a system. The next example illustrates Theorem 3.1, Theorem 4.1 and Corollary 4.2.

Example 4.3. Consider the lossless system $G(s)=\dfrac{0.7s}{s^2+9}$. An minimal i/s/o representation of the system is

$$\frac{d}{dt}x=\underbrace{\begin{bmatrix} 0 & 1 \\ -9 & 0\end{bmatrix}}_{A}x+\underbrace{\begin{bmatrix} 0 \\ 1\end{bmatrix}}_{B}u,\quad y=\underbrace{\begin{bmatrix} 0 & 0.7\end{bmatrix}}_{C}x. \tag{10}$$

The corresponding allpass system $\mathfrak{B}_{all}$ is

$$\frac{d}{dt}x=\underbrace{\begin{bmatrix} 0 & 1 \\ -9 & -0.7\end{bmatrix}}_{\hat{A}}x+\underbrace{\begin{bmatrix} 0 \\ \sqrt{2}\end{bmatrix}}_{\hat{B}}v,\quad r=\underbrace{\begin{bmatrix} 0 & -\frac{\sqrt{98}}{10}\end{bmatrix}}_{\hat{C}}x+\underbrace{\begin{bmatrix} 1\end{bmatrix}}_{\hat{D}}v.$$

The observability Gramian matrix of $\mathfrak{B}_{all}$ is $Q=\begin{bmatrix} 6.3 & 0 \\ 0 & 0.7\end{bmatrix}$.

It is easy to verify that $\hat{A}^TQ+Q\hat{A}+\hat{C}^T\hat{C}=0$ and $Q\hat{B}+\hat{C}^T\hat{D}=0$. Hence x^TQx is a storage function of the allpass system $\mathfrak{B}_{all}$ (Theorem 3.1).

Further, it is easy to verify that $A^TQ+QA=0$ and $QB-C^T=0$. Hence, x^TQx is also a storage function of the lossless counterpart of $\mathfrak{B}_{all}$, i.e., $\mathfrak{B}_\ell$ (Theorem 4.1 and Corollary 4.2).

5. Algebraic relation between states and costates

In the previous section, we have seen that the storage function of an allpass system and it's suitably transformed lossless counterpart is the same. Hence, for easy exposition all the results reported in this section are for lossless systems. Storage functions of allpass systems follow the same results.

In this section, we show that certain algebraic relations between the states and costates of a lossless system capture the information of the storage function of the system. To unveil the main result of this section, we use the concept of orthogonal behavior introduced in Section 2-A. An interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ such that $u=e$ and $y=f$ results in a new behavior that we represent as $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$. It is easy to see that $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ admits a first order kernel representation of the following form:

$$R\left(\frac{d}{dt}\right)\begin{bmatrix}x\\z\\y\end{bmatrix}=0,\ \text{where}\, R(\xi)=\begin{bmatrix}\xi I_{\mathtt{n}}-A & 0 & -B\\ 0 & \xi I_{\mathtt{n}}+A^T & -C^T\\ -C & B^T & D+D^T\end{bmatrix} \tag{11}$$

Further, for lossless systems, $D+D^T=0$.

Note that the McMillan degree[2] of $\mathfrak{B}$ is $\mathtt{n}$. For lossless systems, it is known that $\mathfrak{B}=\mathfrak{B}^{\perp_\Sigma}$ (a special case of [3, Lemma 11]). Hence, the McMillan degree of $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ is also $\mathtt{n}$. However the first order representation of $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ in equation (11) has $2\mathtt{n}$ states. Hence, the states and costates must be related by some algebraic relations. Theorem 5.2 below helps extract this algebraic relations and in the process provides an interesting relation between states, costates and storage function of a lossless behavior. We present a theorem next that is used to prove the main result of this section. This result states that the difference dynamics $x(t)-Kz(t)$ of a lossless system is orthogonal to the subspace $\langle B\rangle$ if and only if it is orthogonal to the controllable subspace spanned by columns of $\begin{bmatrix}B & AB & \cdots & A^{\mathtt{n}-1}B\end{bmatrix}$. We use the symbol $\langle A|B\rangle$ to represent the controllable subspace.

Theorem 5.1. *Consider a lossless behavior $\mathfrak{B}\in\mathfrak{L}^{\mathtt{m}}$. An i/s/o representation of $\mathfrak{B}$ is $\dot{x}=Ax+Bu,\ y=Cx+Du$, where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D\in\mathbb{R}^{\mathtt{p}\times\mathtt{p}}$. Let the corresponding orthogonal complement behavior $\mathfrak{B}^{\perp_\Sigma}$ have an i/s/o representation $\dot{z}=-A^Tz+C^Tu,\ y=B^Tz-D^Tu$. Let $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then the difference dynamics $z(t)-Kx(t)$ satisfies the following for each $t>0$*

$$\Big(z(t)-Kx(t)\Big)\perp\langle B\rangle \iff \Big(z(t)-Kx(t)\Big)\perp\langle A|B\rangle.$$

Proof. Let the controllability matrix be $\mathscr{C}:=\begin{bmatrix}B & AB & A^2B & \cdots & A^{\mathtt{n}-1}B\end{bmatrix}$. We write $x(t),z(t)$ as x,z for ease, respectively. All the arguments here are true for each $t>0$.
(⇐) Given $\mathscr{C}^T(z-Kx)=0$, i.e., $B^T(z-Kx)=0$. Hence, $z(t)-Kx(t)\perp\langle B\rangle$ for each $t>0$.
(⇒) Note that since the system is lossless, $D+D^T=0$ and therefore $Cx-B^Tz=0$. We use the principle of mathematical induction to prove the lemma.
Base step: Using the i/s/o representation of $\mathfrak{B}^{\perp_\Sigma}$, $Cx-B^Tz=0$ and equations (9), we have

$$\begin{aligned}B^TA^T(z-Kx)&=B^T(A^Tz-A^TKx)\\&=B^T(-\dot{z}+K\dot{x})=\frac{d}{dt}B^T(Kx-z)=0.\end{aligned}$$

Induction step: Assume $B^T(A^T)^i(z-Kx)=0$, where $i\in\mathbb{N}$ and $i>1$. Therefore,

$$\begin{aligned}B^T(A^T)^{i+1}(z-Kx)&=B^T(A^T)^i(-\dot{z}+K\dot{x})\\&=\frac{d}{dt}B^T(A^T)^i(Kx-z)=0.\end{aligned}$$

By the principle of mathematical induction, $B^T(A^T)^i(z-Kx)=0$ for all $i\in\mathbb{N}$. Hence, $\mathscr{C}^T(z-Kx)=0$. This proves that $\Big(z(t)-Kx(t)\Big)\perp\langle A|B\rangle$ for each $t>0$.

This completes the proof of the theorem. □

Note that a controllable subspace is the smallest A invariant subspace containing $\langle B\rangle$. Hence, the (⇐) direction of Theorem 5.1 is true for any behavior. However, the (⇒) direction of the theorem is true for lossless and in general, for conservative systems. Next we present one of the main results of this section.

Theorem 5.2. *Consider a lossless behavior $\mathfrak{B}\in\mathfrak{L}^{\mathtt{m}}_{\text{cont}}$ with a minimal i/s/o representation $\dot{x}=Ax+Bu,\ y=Cx+Du$, where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D\in\mathbb{R}^{\mathtt{p}\times\mathtt{p}}$. Consider the corresponding orthogonal complement behavior $\mathfrak{B}^{\perp_\Sigma}$ with an i/s/o representation $\dot{z}=-A^Tz+C^Tu,\ y=B^Tz-D^Tu$. The first order representation of the behavior $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ is given by equation (11). Then the following statements are true.*

1) *$\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ is not autonomous i.e $\det R(\xi)=0$.*
2) *there exists a unique $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that*

$$\frac{d}{dt}x^TKx=2u^Ty \ \text{ for all } \ (u,y)\in\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}=\mathfrak{B}. \tag{12}$$

3) *there exists a unique $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that*

$$\text{rank}\begin{bmatrix} & R(\xi) & \\ -K & I & 0\end{bmatrix}=\ \text{rank}\ R(\xi). \tag{13}$$

Further, for $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$,
K satisfies equation (12) $\iff$ K satisfies equation (13).

Proof. **1**: Here, $G(\xi)+G(-\xi)^T=0\implies D+D^T=0$. Now, using the Schur complement to find $\det(R(\xi))$, we have

$$\begin{aligned}&\det\left(\begin{array}{cc|c}\xi I_{\mathtt{n}}-A & 0 & -B\\ 0 & \xi I_{\mathtt{n}}+A^T & -C^T\\ \hline -C & B^T & 0\end{array}\right)\\&=\det\left\{-C(\xi I_{\mathtt{n}}-A)^{-1}B+B^T(\xi I_{\mathtt{n}}+A^T)^{-1}C^T\right\}\\&=\det\left\{-D-C(\xi I_{\mathtt{n}}-A)^{-1}B-D^T+B^T(\xi I_{\mathtt{n}}+A^T)^{-1}C^T\right\}\\&=\det\left\{-G(\xi)-G(-\xi)^T\right\}=0.\end{aligned}$$

Using [10, Section 3.2], we know that a system is autonomous if and only if it admits a kernel representation $P(\frac{d}{dt})w=0$, where $P(\xi)\in\mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{m}}$ and $\det(P(\xi))\neq0$. Note that for $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$, $\det(R(\xi))=0$ hence the system $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ is not autonomous. This proves Statement 1.
2: Assume $K_1\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $K_2\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ are symmetric matrices that satisfy the dissipation equation (5) adapted to the passivity supply rate. Note that the storage function of a lossless system is unique : see [11, Remark 5.13]. Hence, $x^T(t)K_1x(t)=x^T(t)K_2x(t)$ for all $t\in\mathbb{R}$. This is true if and only if $K_1=K_2$. This proves the uniqueness of K.

Consider $K_1=K_2=:K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. By dissipation equation (5) adapted to passivity supply rate, we have

$$\frac{d}{dt}x^TKx=2u^Ty\ \text{ for all } (u,y)\in\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}=\mathfrak{B}.$$

This proves Statement 2.

[2]The minimum number of states required for an i/s/o representation of the system is called McMillan degree of the system.

3: Using the i/s/o representation of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$, we have

$$\begin{aligned}\frac{d}{dt}x^T z &= (Ax+Bu)^T z + x^T(-A^T z + C^T u) = u^T B^T z + u^T Cx \\ &= u^T(B^T z - D^T u) + u^T(Cx+Du) = u^T y + u^T y \\ &= 2u^T y \ \text{ for all } (u,y)\in\mathfrak{B}.\end{aligned}$$

Equation (5) adapted to lossless systems gives $2u^T y = \frac{d}{dt}x^T Kx$ for all $(u,y)\in\mathfrak{B}$. Hence, from Statement 2, we have

$$\frac{d}{dt}x^T z = \frac{d}{dt}x^T Kx \implies \dot{x}^T z + x^T\dot{z} - \dot{x}^T Kx - x^T K\dot{x} = 0. \quad (14)$$

Using i/s/o representation of $\mathfrak{B}$, $\mathfrak{B}^{\perp_\Sigma}$ and the linear matrix equations (9), equation (14) becomes

$$\begin{aligned}&(Ax+Bu)^T z + x^T(-A^T z + C^T u) \\ &\qquad\qquad - (Ax+Bu)^T Kx - x^T K(Ax+Bu) = 0 \\ &\implies u^T B^T z - x^T(A^T K + KA)x - x^T(KB - C^T)u - x^T KBu = 0 \\ &\implies u^T B^T(z-Kx) = 0.\end{aligned}$$

This is true for all system trajectories $(u,y)\in\mathfrak{B}$. Hence, $B^T(z-Kx)=0$. Using Theorem 5.1, we have $z-Kx\in\ker\mathscr{C}^T$.

However (A,B) is a controllable system with minimal state representation, hence $z-Kx=0$ is true for *all* trajectories in $\mathfrak{B}$. This proves that trajectories that satisfy $\begin{bmatrix}-K & I & 0\end{bmatrix}\begin{bmatrix}x\\z\\y\end{bmatrix}=0$ are trajectories in $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ $(=\mathfrak{B})$ as well. Hence, rank $\begin{bmatrix} & R(\xi) & \\ -K & I & 0\end{bmatrix}=$ rank $R(\xi)$. This proves Statement 3.

Next we prove that Statement 2 and 3 are equivalent.

(⇒) Proof of Statement 3 crucially used Statement 2. Hence, K satisfies equation (12) $\Rightarrow$ K satisfies equation (13).

(⇐) Note that equation (13) means the behavior $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ has trajectories that satisfy $z=Kx$. Further from [11, Section 10], it is clear that the 'states' and its 'costates' (i.e., states of the dual/adjoint system) satisfy

$$\frac{d}{dt}x^T z = 2u^T y, \quad \text{i.e.,} \quad \frac{d}{dt}x^T Kx = 2u^T y.$$

Hence, $x^T Kx$ is the storage function of $\mathfrak{B}$. Hence, K satisfies equation (12) $\Leftarrow$ K satisfies equation (13).

This completes the proof of Theorem 5.2. □

Theorem 5.2 shows that the algebraic relations between the states and costates of a lossless behavior have information about the storage function of the system. To find the storage function of a lossless system, we just need to find out equations of the form $z=Kx$ from the equation module of $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$. The algorithm to extract these algebraic relations can be found in [4].

6. Conclusion

In this paper, we have investigated some new properties of the storage functions of allpass systems. We showed that for an allpass system, the observability Gramian turns out to be the unique storage function (Theorem 3.1). We have also shown that in the balanced basis, the observability Gramian of an allpass system is induced by an identity matrix. Hence, $x^T x$ is the storage function of an allpass system in the balanced basis (Theorem 3.2). We showed that the set of storage function is invariant under the map that takes bounded real systems to passive systems and vice versa (Theorem 4.1). Under this transformation, allpass systems are mapped to lossless systems and hence in the balanced basis, the identity matrix induces the storage function of lossless systems as well. Further, we showed that certain difference dynamics between the states and costates of an allpass system is always present in the orthogonal complement of its controllable subspace (Theorem 5.1). We also established that certain algebraic relations between the states and costates of a lossless system capture the storage functions of the system (Theorem 5.2). The same result is easily extendable to any conservative system provided the system satisfies the maximum input cardinality condition namely the input cardinality of the system is equal to the positive signature of the supply rate.

References

[1] A.C. Antoulas, *Approximation of Large-scale Dynamical Systems*, Advances in Design and Control, SIAM, Philadelphia, 2005.

[2] B.D.O. Anderson, S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*, Dover Publications, New York, 2006.

[3] M.N. Belur, H.K. Pillai and H.L. Trentelman, Dissipative systems synthesis: a linear algebraic approach, *Linear Algebra and its Applications*, vol. 425, no. 2-3, pages 739-756, 2007.

[4] C. Bhawal, D. Pal, S. Kumar and M.N. Belur, New results and techniques for computation of stored energy in lossless/all-pass systems, *Transactions on Circuits and Systems I: Regular Papers*, vol. 64, no. 1, pages 72-85, 2017.

[5] A. Ferrante, G. Picci and S. Pinzoni, Spectral factorization and stochastic realization with zeros on the unit circle, *Proceedings of the IEEE Conference on Decision and Control (CDC), Hawaii, USA*, vol. 2, pages 1398-1403, 2003.

[6] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds, *International Journal of Control*, vol. 39, no. 6, pages 1115-1193, 1984.

[7] W.M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: a Lyapunov-based Approach*, Princeton University Press, Princeton, 2008.

[8] S.C. Jugade, D. Pal, R.K. Kalaimani and M.N. Belur, Stationary trajectories, singular Hamiltonian systems and ill-posed interconnection, *Proceedings of the IEEE European Control Conference (ECC), Zurich, Switzerland*, pages 1740-1745, 2013.

[9] D. Pal and M.N. Belur, Finite gain and phase margins as dissipativity conditions, *Proceedings of the IEEE American Control Conference (ACC), Baltimore, USA*, pages 337-342, 2010.

[10] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, Springer-Verlag, New York, 1998.

[11] J.C. Willems and H.L. Trentelman, On quadratic differential forms, *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pages 1703-1749, 1998.

[12] J.C. Willems and H.L. Trentelman, Synthesis of dissipative systems using quadratic differential forms: Parts I, *IEEE Transactions on Automatic Control*, vol. 47, no. 1, pages 53-69, 2002.