Introduction to Optimal Control

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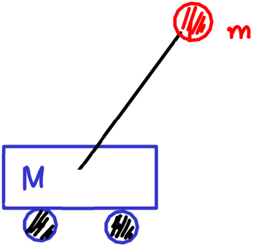
Inverted pendulum



Inverted pendulum

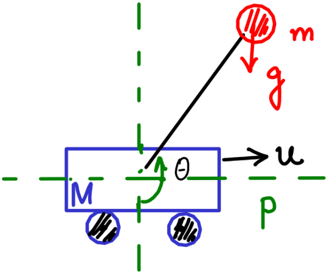
Degree of freedom: p and θ.

State vector:

$$x := \begin{bmatrix} p & \dot{p} & \theta & \dot{\theta} \end{bmatrix}^T$$

Dynamics: $\dot{x}(t) = f(x(t), u(t), t)$.

Desired states: $x_f = \begin{bmatrix} 0 & 0 & \pi & 0 \end{bmatrix}^T$.



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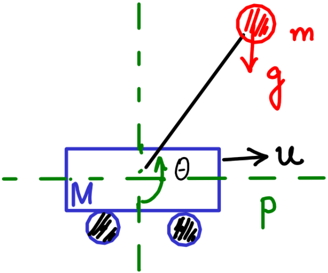

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Question: Is there an input $u(t)$ that helps us achieve this?

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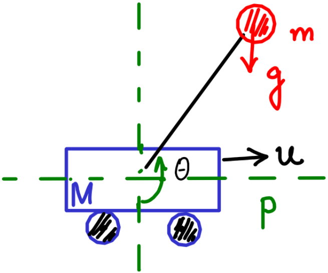

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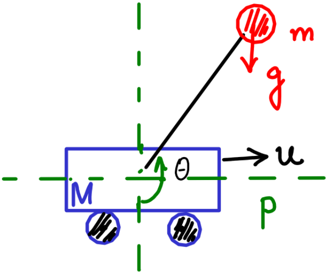

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Strategy:

- Linearize the model about $\theta = \pi$: $\dot{x}(t) = Ax(t) + Bu(t)$.
- Construct F if (A, B) controllable.
- Use the control law $u = -Fx$ in the actual system.

LQR problem

Problem

Given a system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

with initial condition x_0, find an input $u(t)$ such that

$$J(x(t), u(t), x_0) = \int_0^\infty (x^T Q x + u^T R u) dt,$$

where $Q \geqslant 0$ and $R > 0$ is minimized and $\lim_{t \to \infty} x(t) = 0$.

$x^T Q x$: penality on deviation from reference. $u^T R u$: penality on input.

Typical optimal control problem

Problem

Find an admissible control $u^*(t)$ which causes the system

$$\dot{x}(t) = a\left(x(t), u(t), t\right)$$

to follow an admissible trajectory $x^*(t)$ that minimizes the performance measure

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g\left(x(t), u(t), t\right) dt$$

$x^*(t)$: Optimal state trajectories, $u^*(t)$: Optimal control.

- We may not know $u^*(t)$ exists.
- If $u^*(t)$ exists, it may not be unique.

$$J^* = h(x^*(t_f), t_f) + \int_{t_0}^{t_f} g\left(x^*(t), u^*(t), t\right) dt \quad \leqslant h(x(t_f), t_f) + \int_{t_0}^{t_f} g\left(x(t), u(t), t\right) dt.$$

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Organization

Part - I
Calculus of Variation

Part - II
Dynamic Programming

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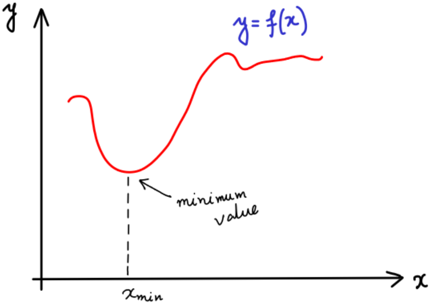
Part - I
Calculus of Variation

Part - II
Dynamic Programming

Calculus of variation approach

- Unconstrained optimization



$$\min_x y = \min_x f(x)$$

$$\text{Necessary:} \frac{dy}{dx} = 0$$

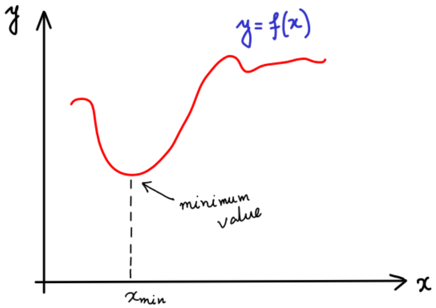
$$\text{Sufficiency:} \frac{d^2y}{dx^2} > 0.$$

- Problem: Find the stationary function of a functional.
- Functional: Function of functions. For example

$$J(x(t), u(t), t) = \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t)) dt.$$

Calculus of variation approach

- Unconstrained optimization



$$\min_{x \in \mathbb{R}^n} F(x)$$

$$\text{Necessary:} \nabla F(x) = 0$$

$$\text{Sufficiency:} \nabla^2 F(x) > 0.$$

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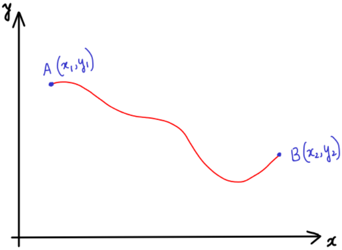

Find shortest path between A and B.

$$L = \int_A^B dS$$
$$= \int_A^B \sqrt{dx^2 + dy^2}$$
$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Find a function $y = f(x)$ between A and B such that L is minimized.

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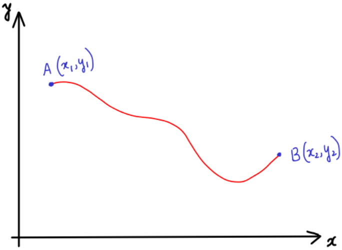

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Calculus of Variation

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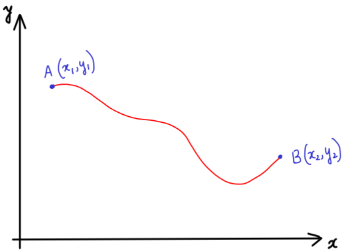

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Find a function $y = f(x)$ between A and B such that L is minimized.

Necessary conditions for stationary functions

Problem (COV based optimization problem)

Find $y = f(x)$ *such that the functional*

$$\int_{x_1}^{x_2} F\left(x, y, \frac{dy}{dx}\right) dx$$

is minimized/maximized.

Necessary condition for stationarity:

Euler-Lagrange Equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Example

Problem

Find a function $y = f(x)$ between A and B such that

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

is minimized.

E.L Equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0.$$

Here $F = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

$$\frac{d}{dx}\left[\frac{\partial}{\partial y'}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}\right] = 0 \Rightarrow \frac{d^2y}{dx^2} = 0 \Rightarrow y(x) = c_1 x + c_2.$$

On using boundary conditions: $y(x_1) = y_1$ and $y(x_2) = y_2$, we will get $c_1 = \frac{y_2 - y_1}{x_2 - x_1}$ and $c_2 = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$.

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Calculus of variation approach

- Constrained optimization problem: Minimize $F(x, y)$ subject to the constraint $g(x, y) = c$.
- Lagrange multipliers are introduced: $\nabla F(x, y) + \lambda \nabla (g(x, y) - c) = 0$.
- For example:

$$\text{Maximize} f(x, y) = x^2 y \text{ on the set } x^2 + y^2 = 1.$$

- Using the Lagrange multiplier idea (2 equations):

$$\nabla x^2 y + \lambda \nabla (x^2 + y^2 - 1) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} + \lambda \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = 0.$$

  The other equation: $x^2 + y^2 = 1$.
- Three equations and three unknowns (x, y, λ): Solvable.
- Langrangian: $\mathcal{L}(x, y, \lambda) := f(x, y) + \lambda(x^2 + y^2 - 1)$
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Constraint minimization of functionals

- Find the condition for $y^*(t)$ to be an extremal for a functional of the form

$$J(x) = \int_{t_0}^{t_f} g\left(y(x), \dot{y}(x), x\right) dt$$

where y is an $(n+m) \times 1$ vector of functions $n, m \geqslant 1$ that is required to satisfy n relations of the form

$$f_i(y(x), x) = 0, \ i = 1, 2, \ldots, n.$$

- We use the method of Lagrange multipliers –

$$g_a(y(x), \dot{y}(x), x) := g(y(x), \dot{y}(x), x) + p^T(x) f(y(x), x)$$

Necessary condition for $y^*(t)$ to be an extremal

$$\frac{\partial}{\partial y} g_a\Big(y^*(t), \dot{y}^*(x), p^*(x), x\Big) - \frac{d}{dx}\left[\frac{\partial}{\partial y'}\Big(y^*(x), {y'}^*(x), p^*(x), x\Big)\right] = 0.$$

Necessary condition for optimal control

Problem

Find an admissible control $u^*(t)$ that causes the system

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with initial condition x_0 to follow an admissible trajectory $x^*(t)$ that minimizes the performance measure

$$J(u(t)) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt$$

for t_0 specified.

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$$h(x(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}\left[h(x(t), t)\right] dt + h(x(t_0, t_0).$$

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for t_0 *specified.*

Necessary condition for optimal control

Define Hamiltonian

$$\mathscr{H} := g(x(t), u(t), t) + p^T(t)\left[a(x(t), u(t), t)\right]$$

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\begin{aligned}\frac{d}{dt}x^* &= \frac{\partial \mathscr{H}}{\partial p}\\ \frac{d}{dt}p^* &= -\frac{\partial \mathscr{H}}{\partial x}\\ 0 &= \frac{\partial \mathscr{H}}{\partial u}\end{aligned}$$

$$\left[\frac{\partial}{\partial x}h(x^*(t_f), t_f) - p^*(t_f)\right]^T \delta x_f + \Big[\mathscr{H}(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial}{\partial t}h(x^*(t_f), t_f)\Big]\delta t_f = 0.$$

LQR Problem

Problem

The plant is described by the linear state equations

$$\dot{x}(t) = Ax(t) + Bu(t),\ A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}.$$

The performance index to be minimized is

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

The final time t_f *is fixed,* $H, Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$, $H, Q \geqslant 0$ *and* $R > 0$.

Hamiltonian

$$\mathscr{H}(x(t), u(t), p(t), t) = \frac{1}{2}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] + p^T(t)\Big(Ax(t) + Bu(t)\Big)$$

LQR Problem

Hamiltonian

$$\mathscr{H}(x(t),u(t),p(t),t) = \frac{1}{2}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] + p^T(t)\Big(Ax(t) + Bu(t)\Big)$$

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\begin{aligned}
\frac{d}{dt}x^* &= \frac{\partial \mathscr{H}}{\partial p} \Rightarrow \frac{d}{dt}x^*(t) = Ax^*(t) + Bu^*(t) \\
\frac{d}{dt}p^* &= -\frac{\partial \mathscr{H}}{\partial x} \Rightarrow \frac{d}{dt}p^*(t) = -Qx^*(t) - A^T p^*(t) \\
0 &= \frac{\partial \mathscr{H}}{\partial u} \Rightarrow 0 = Ru^*(t) + B^T p^*(t).
\end{aligned}$$

$$\left[\frac{\partial}{\partial x}h(x^*(t_f),t_f) - p^*(t_f)\right]^T \delta x_f + \Big[\mathscr{H}(x^*(t_f),u^*(t_f),p^*(t_f),t_f) + \frac{\partial}{\partial t}h(x^*(t_f),t_f)\Big]\,\delta t_f = 0.$$

LQR Problem

Hamiltonian

$$\mathscr{H}(x(t), u(t), p(t), t) = \frac{1}{2}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] + p^T(t)\Big(Ax(t) + Bu(t)\Big)$$

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\frac{d}{dt}x^*(t) = Ax^*(t) + Bu^*(t)$$
$$\frac{d}{dt}p^*(t) = -Qx^*(t) - A^T p^*(t)$$
$$u^*(t) = -R^{-1}B^T p^*(t).$$

$$\frac{1}{2}\left[\frac{\partial}{\partial x}\Big(x^T(t_f)Hx(t_f)\Big) - p^*(t_f)\right]^T \delta x_f = 0 \Rightarrow p^*(t_f) = Hx^*(t_f).$$

LQR Problem

Hamiltonian

$$\mathscr{H}(x(t), u(t), p(t), t) = \frac{1}{2}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] + p^T(t)\Big(Ax(t) + Bu(t)\Big)$$

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\frac{d}{dt}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$p^*(t_f) = Hx^*(t_f).$$

Optimal control

$$u^*(t) = -R^{-1}B^Tp^*(t).$$

LQR Problem

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\frac{d}{dt}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} \Rightarrow \begin{bmatrix} x^*(t_f) \\ p^*(t_f) \end{bmatrix} = \Phi(t_f, t)\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\begin{bmatrix} x^*(t_f) \\ p^*(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f, t) & \Phi_{12}(t_f, t) \\ \Phi_{21}(t_f, t) & \Phi_{22}(t_f, t) \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^*(t_f) \\ Hx^*(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f, t) & \Phi_{12}(t_f, t) \\ \Phi_{21}(t_f, t) & \Phi_{22}(t_f, t) \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\Rightarrow p^*(t) = \underbrace{\left[\Phi_{22}(t_f, t) - H\Phi_{12}(t_f, t)\right]^{-1}\left[H\Phi_{11}(t_f, t) - \Phi_{21}(t_f, t)\right]}_{K(t)} x^*(t)$$

LQR Problem

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\frac{d}{dt}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\begin{bmatrix} x^*(t_f) \\ p^*(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f,t) & \Phi_{12}(t_f,t) \\ \Phi_{21}(t_f,t) & \Phi_{22}(t_f,t) \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^*(t_f) \\ Hx^*(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f,t) & \Phi_{12}(t_f,t) \\ \Phi_{21}(t_f,t) & \Phi_{22}(t_f,t) \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\Rightarrow p^*(t) = \underbrace{[\Phi_{22}(t_f,t) - H\Phi_{12}(t_f,t)]^{-1}[H\Phi_{11}(t_f,t) - \Phi_{21}(t_f,t)]}_{K(t)} x^*(t)$$

Optimal control

$$u^*(t) = -R^{-1}B^T K(t) x^*(t).$$

LQR Problem

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\frac{d}{dt}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\begin{aligned}
& p^*(t) = K(t)x^*(t) \\
\Rightarrow\ & \dot{p}^*(t) = \dot{K}(t)x^*(t) + K(t)\dot{x}^*(t) \\
\Rightarrow\ & -Qx^*(t) - A^T K(t)x^*(t) = \dot{K}(t)x^*(t) + K(t)(Ax^*(t) - BR^{-1}B^T p^*(t) \\
\Rightarrow\ & -Qx^*(t) - A^T K(t)x^*(t) = \dot{K}(t)x^*(t) + K(t)(Ax^*(t) - BR^{-1}B^T K x^*(t) \\
\Rightarrow\ & \dot{K}(t) = A^T K(t) + K(t)A + Q - BR^{-1}B^T
\end{aligned}$$

LQR Problem

At $(x^*(t), u^*(t), p^*(t))$ for all $t \in [t_0, t_f]$

$$\frac{d}{dt}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

$$\begin{aligned}
& p^*(t) = K(t)x^*(t) \\
\Rightarrow\ & \dot{p}^*(t) = \dot{K}(t)x^*(t) + K(t)\dot{x}^*(t) \\
\Rightarrow\ & -Qx^*(t) - A^T K(t)x^*(t) = \dot{K}(t)x^*(t) + K(t)(Ax^*(t) - BR^{-1}B^T p^*(t) \\
\Rightarrow\ & -Qx^*(t) - A^T K(t)x^*(t) = \dot{K}(t)x^*(t) + K(t)(Ax^*(t) - BR^{-1}B^T Kx^*(t) \\
\Rightarrow\ & \dot{K}(t) = A^T K(t) + K(t)A + Q - BR^{-1}B^T
\end{aligned}$$

Differential Riccati Equation

$$\dot{K}(t) = A^T K(t) + K(t)A + Q - BR^{-1}B^T$$

LQR Problem – finite horizon

Problem

The plant is described by the linear state equations

$$\dot{x}(t) = Ax(t) + Bu(t), \; A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}.$$

With fixed t_f, the performance index to be minimized is ($H, Q \geqslant 0$ and $R > 0$)

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

Optimal control

$$u^*(t) = -R^{-1}B^T K(t)x^*(t).$$

Differential Riccati Equation

$$\dot{K}(t) = A^T K(t) + K(t)A + Q - BR^{-1}B^T$$

LQR Problem – infinite horizon

Problem

The plant is described by the linear state equations (Given x_0 *and* $x_f = 0$*)*

$$\dot{x}(t) = Ax(t) + Bu(t),\ A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}.$$

The performance index to be minimized is ($H, Q \geqslant 0$ *and* $R > 0$*)*

$$J = \frac{1}{2}\int_{0}^{\infty} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

Optimal control

$$u^*(t) = -R^{-1}B^T K x^*(t) = -Fx^*(t),\ \text{where } F := -R^{-1}B^T K.$$

Algebraic Riccati Equation

$$0 = A^T K + KA + Q - BR^{-1}B^T$$

Organization

Part - I
Calculus of Variation

Part - II
Dynamic Programming

Dynamic Programming

Bellman's optimality criterion –

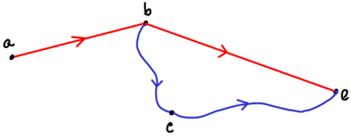
> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

Dynamic Programming

Bellman's optimality criterion –

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

If $a-b-e$ is the optimal path from a to e, then $b-e$ is the optimal path from b to e.

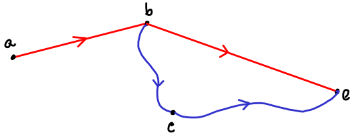


Dynamic Programming

Bellman's optimality criterion –

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

If $a - b - e$ is the optimal path from a to e, then $b - e$ is the optimal path from b to e.



$$J_{bce} < J_{be} \Rightarrow J_{ab} + J_{bce} < J_{ab} + J_{be} = J^*_{abe}.$$

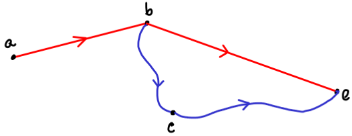
Contradiction!

Dynamic Programming

Bellman's optimality criterion –

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

If $a - b - e$ is the optimal path from a to e, then $b - e$ is the optimal path from b to e.



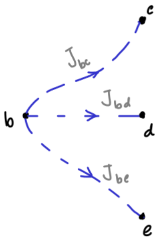
$$J_{bce} < J_{be} \Rightarrow J_{ab} + J_{bce} < J_{ab} + J_{be} = J^*_{abe}.$$

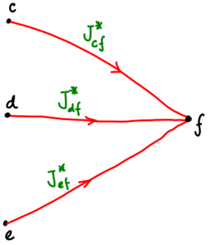
Contradiction!

Dynamic Programming - Example

Bellman's optimality criterion –

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

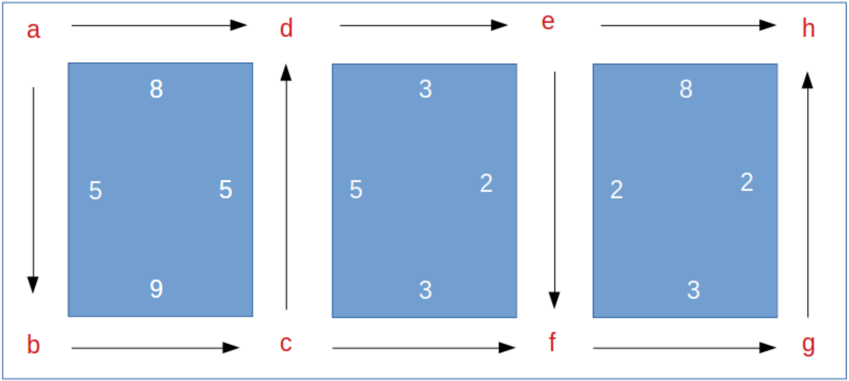




$$C^*_{bcf} = J_{bc} + J^*_{cf}$$
$$C^*_{bdf} = J_{bd} + J^*_{df}$$
$$C^*_{bef} = J_{be} + J^*_{ef}$$

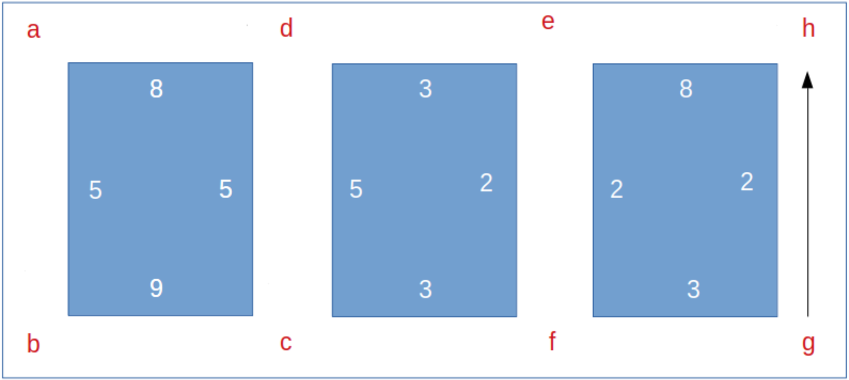
$\texttt{min}(C^*_{bcf}, C^*_{bdf}, C^*_{bef})$ is the optimal cost.

The travelling salesman



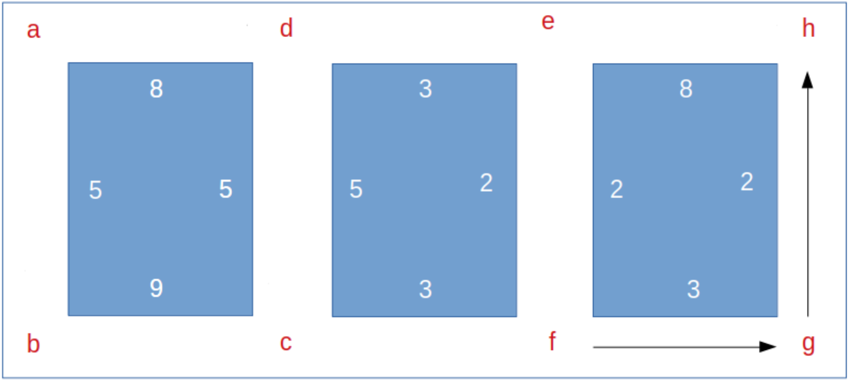
Problem: Travel from a to h in minimum time.

The travelling salesman



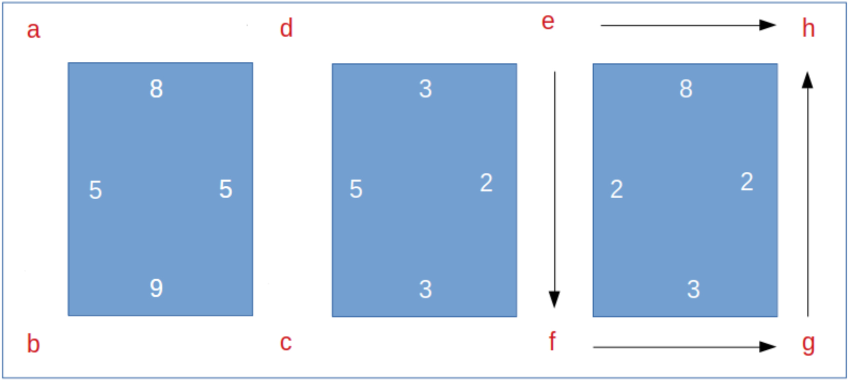
Cost: $C_{gh} = 2 = C^*_{gh}$.

The travelling salesman



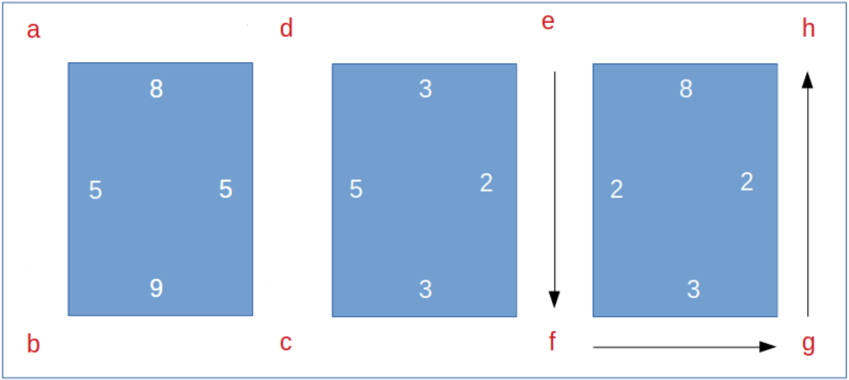
Cost: $C_{fh} = C_{fg} + C_{gh} = 5 = C^*_{fh}$.

The travelling salesman

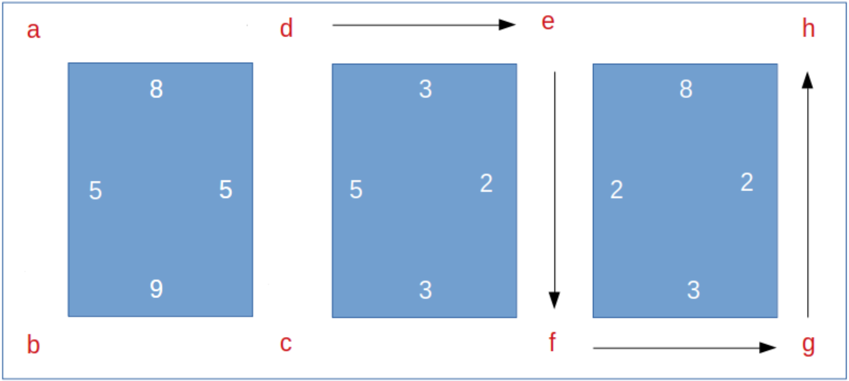


Cost: $C_{eh} = C_{ef} + C^{*}_{fh} = 7$ $\qquad$ $C_{eh} = 8$.

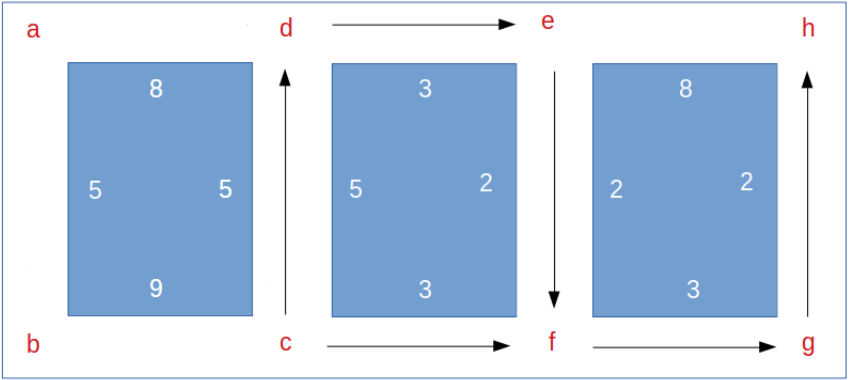
The travelling salesman



Cost: $\color{blue}{C_{eh} = C_{ef} + C^*_{fh} = 7 = C^*_{eh}}$ $\quad$ $\color{red}{C_{eh} = 8}$.

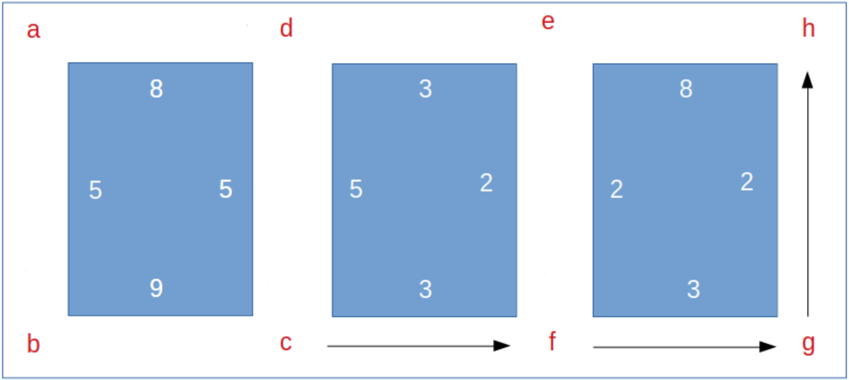


Cost: $C_{dh} = C_{de} + C^*_{eh} = 10 = C^*_{dh}$.



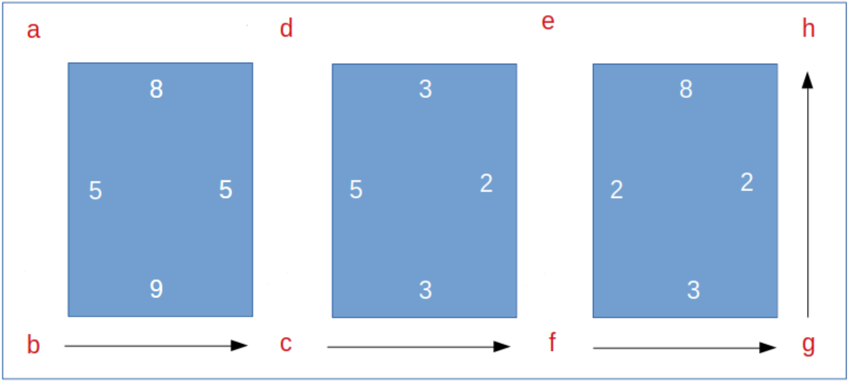
Cost: $C_{ch} = C_{cd} + C^*_{dh} = 15$ $\qquad$ $C_{ch} = C_{cf} + C^*_{fh} = 8$.

The travelling salesman



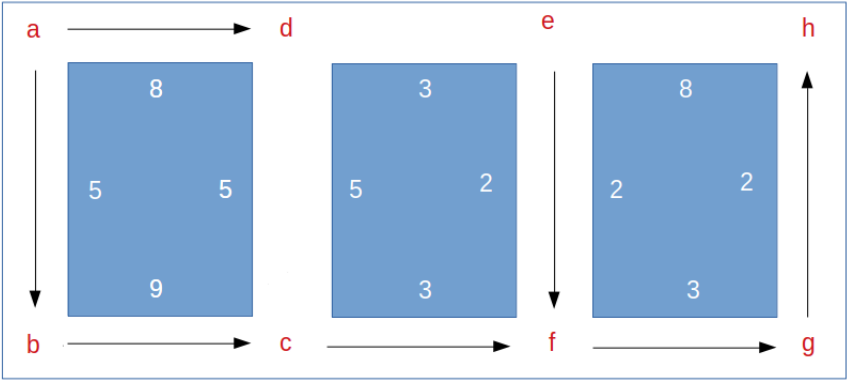
Cost: $C_{ch} = C_{cd} + C^*_{dh} = 15 = C^*_{ch}$ $\quad$ $C_{ch} = C_{cf} + C^*_{fh} = 8$.

The travelling salesman



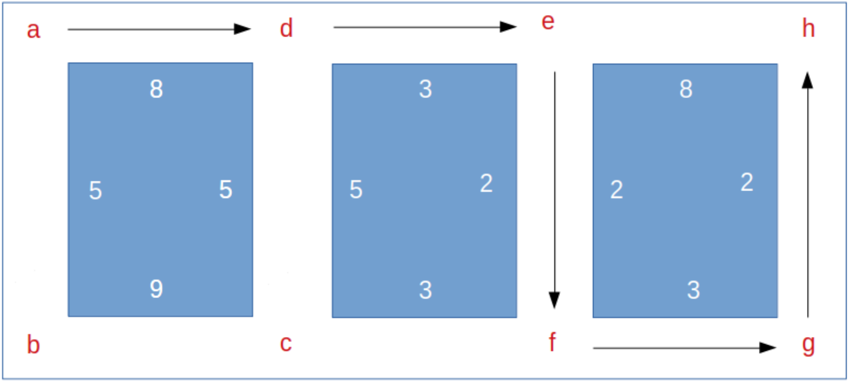
Cost: $C_{bh} = C_{bc} + C^*_{ch} = 17 = C^*_{bh}$.

The travelling salesman



Cost: $C_{ah} = C_{ab} + C^*_{bh} = 22$ $C_{ah} = C_{ad} + C^*_{dh} = 18$.

The travelling salesman



Cost: $C_{ah} = C_{ab} + C^*_{bh} = 22$ $\qquad C_{ah} = C_{ad} + C^*_{dh} = 18 = C^*_{ah}$.

The travelling salesman

Current Intersection α	Heading u_i	Next intersection x_i	Min cost α to h via x_i	Min cost to reach h from α	Optimal heading at α
g	N	h	$2 + 0 = 2$	2	N
f	E	g	$3 + 2 = 5$	5	E
e	E	h	$8 + 0 = 8$	7	S
	S	f	$2 + 5 = 7$		
d	E	e	$3 + 7 = 10$	10	E
c	N	d	$5 + 10 = 15$	8	E
	E	f	$3 + 5 = 8$		
b	E	c	$9 + 8 = 17$	17	E
a	E	d	$8 + 10 = 18$	18	E
	S	b	$5 + 17 = 22$		

Single-state single-input case

For a system with the following dynamics:

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

minimize the performance index:

$$J = x^2(T) + \alpha \int_0^T u^2(t)dt.$$

- System needs to be approximated by difference equation.
- Integral needs to be approximated by summation.

Divide 0 to T in N equal segments of size Δt, i.e, $N\Delta t = T$.

Single-input single-state

- Approximation of the system:

$$\frac{x(t+\Delta t)-x(t)}{\Delta t} \simeq ax(t)+bu(t) \Rightarrow x(t+\Delta t) \simeq (1+a\Delta t)\, x(t) + b\Delta t u(t).$$

- Assume Δt is small enough:
  $u(t)$ piecewise continuous with changes at $t = 0, \Delta t, \ldots, (N-1)\Delta t$.
- For $t = k\Delta t$:

$$x((k+1)\Delta t) = [1+a\Delta t]x(k\Delta t) + b\Delta t u(k\Delta t), \quad k = 0, 1, \ldots, N-1.$$

Discretized system

$$x[k+1] = [1+a\Delta t]x[k] + b\Delta t u[k]$$

Single-input single-state

Discretized system

$$x[k+1] = [1 + a\Delta t]x[k] + b\Delta t u[k]$$

- Discretization of the performance index:

$$J \simeq x^2(N\Delta t) + \alpha \left[\int_0^{\Delta t} u^2(0)dt + \int_{\Delta}^{2\Delta} u^2(\Delta t)dt + \cdots + \int_{(N-1)\Delta t}^{N\Delta t} u^2([N-1]\Delta t)dt \right]$$

$$\simeq x^2[N] + \alpha\Delta t \left[u[0] + u[1] + \cdots + u^2[N-1] \right]$$

Discretized performance index

$$J = x^2[N] + \alpha\Delta t \sum_{k=0}^{N-1} u^2[k].$$

Single-input single-state example

- $a = 0$, $b = 1$, $\alpha = 2$, and $T = 2$.

System	Cost	Constraint
$x(t) = u(t)$	$J = x^2(2) + 2\int_0^2 u^2(t)dt$	$0 \leqslant x(t) \leqslant 1.5, \quad -1 \leqslant u(t) \leqslant 1$

- Discretized problem: Assuming $\Delta t = 1$ ($N = 2$) Given the system

$$x[k+1] = x[k] + u[k]$$

minimize the performance index:

$$J = x^2[2] + 2u^2[0] + 2u^2[1].$$

subject to the constraints:

$$\begin{aligned} 0 \leqslant x[k] \leqslant 1.5, &\qquad \text{for } k = 0, 1, 2 \\ -1 \leqslant u[k] \leqslant 1, &\qquad \text{for } k = 0, 1 \end{aligned}$$

Problem from "Optimal Control Theory" authored by Donald E. Kirk

Single-input single-state example

- $a = 0$, $b = 1$, $\alpha = 2$, and $T = 2$.

System	Cost	Constraint
$\dot{x}(t) = u(t)$	$J = x^2(2) + 2\int_0^2 u^2(t)dt$	$0 \leqslant x(t) \leqslant 1.5, \quad -1 \leqslant u(t) \leqslant 1$

- Discretized problem: Assuming $\Delta t = 1$ ($N = 2$) Given the system

$$x[k+1] = x[k] + u[k]$$

minimize the performance index:

$$J = x^2[2] + 2u^2[0] + 2u^2[1].$$

subject to the constraints:

$$x[k] = \{0, 0.5, 1, 1.5\}$$
$$u[k] = \{-1, -0.5, 0, 0.5, 1\}.$$

Problem from "Optimal Control Theory" authored by Donald E. Kirk

Single-input single-state example

System: $x[k+1] = x[k] + u[k]$ Cost: $J = x^2[2] + 2u^2[0] + 2u^2[1]$.

$$x[k] = \{0, 0.5, 1, 1.5\}, u[k] = \{-1, -0.5, 0, 0.5, 1\}.$$

Current state $x(1)$	*Control* $u(1)$	*Next state* $x(2) = x(1) + u(1)$	*Cost* $x^2(2) + 2u^2(1) = J_{12}(x(1), u(1))$		*Minimum cost* $J^*_{12}(x(1))$	*Optimal control applied at* $k = 1$ $u^*(x(1), 1)$
1.5	0.0	1.5	$(1.5)^2 + 2(0.0)^2 =$	2.25		
	−0.5	1.0	$(1.0)^2 + 2(-0.5)^2 =$	1.50	$J^*_{12}(1.5) = 1.50$	$u^*(1.5, 1) = -0.5$
	−1.0	0.5	$(0.5)^2 + 2(-1.0)^2 =$	2.25		
1.0	0.5	1.5	$(1.5)^2 + 2(0.5)^2 =$	2.75		
	0.0	1.0	$(1.0)^2 + 2(0.0)^2 =$	1.00		
	−0.5	0.5	$(0.5)^2 + 2(-0.5)^2 =$	0.75	$J^*_{12}(1.0) = 0.75$	$u^*(1.0, 1) = -0.5$
	−1.0	0.0	$(0.0)^2 + 2(-1.0)^2 =$	2.00		
0.5	1.0	1.5	$(1.5)^2 + 2(1.0)^2 =$	4.25		
	0.5	1.0	$(1.0)^2 + 2(0.5)^2 =$	1.50		
	0.0	0.5	$(0.5)^2 + 2(0.0)^2 =$	0.25	$J^*_{12}(0.5) = 0.25$	$u^*(0.5, 1) = 0.0$
	−0.5	0.0	$(0.0)^2 + 2(-0.5)^2 =$	0.50		
0.0	1.0	1.0	$(1.0)^2 + 2(1.0)^2 =$	3.00		
	0.5	0.5	$(0.5)^2 + 2(0.5)^2 =$	0.75		
	0.0	0.0	$(0.0)^2 + 2(0.0)^2 =$	0.00	$J^*_{12}(0.0) = 0.00$	$u^*(0.0, 1) = 0.0$

Problem from "Optimal Control Theory" authored by Donald E. Kirk

Single-input single-state example

System: $x[k+1] = x[k] + u[k]$ Cost: $J = x^2[2] + 2u^2[0] + 2u^2[1]$.

$$x[k] = \{0, 0.5, 1, 1.5\}, u[k] = \{-1, -0.5, 0, 0.5, 1\}.$$

Current state $x(0)$	*Control* $u(0)$	*Next state* $x(1) = x(0) + u(0)$	*Minimum cost over last two stages for trial value* $u(0)$: $J_{01}(x(0), u(0)) + J^*_{12}(x(1)) =$ $2u^2(0) + J^*_{12}(x(1)) = C^*_{02}(x(0), u(0))$	*Minimum cost over last two stages* $J^*_{02}(x(0))$	*Optimal control applied at* $k = 0$: $u^*(x(0), 0)$
1.5	0.0	1.5	$2(0.0)^2 + 1.50 = 1.50$		
	-0.5	1.0	$2(-0.5)^2 + 0.75 = 1.25$	$J^*_{02}(1.5) = 1.25$	$u^*(1.5, 0) = -0.5$
	-1.0	0.5	$2(-1.0)^2 + 0.25 = 2.25$		
1.0	0.5	1.5	$2(0.5)^2 + 1.50 = 2.00$		
	0.0	1.0	$2(0.0)^2 + 0.75 = 0.75$	$J^*_{02}(1.0) = \begin{Bmatrix} 0.75 \\ 0.75 \end{Bmatrix}$	$u^*(1.0, 0) = \begin{Bmatrix} 0.0 \\ -0.5 \end{Bmatrix}$
	-0.5	0.5	$2(-0.5)^2 + 0.25 = 0.75$		
	-1.0	0.0	$2(-1.0)^2 + 0.00 = 2.00$		
0.5	1.0	1.5	$2(1.0)^2 + 1.50 = 3.50$		
	0.5	1.0	$2(0.5)^2 + 0.75 = 1.25$		
	0.0	0.5	$2(0.0)^2 + 0.25 = 0.25$	$J^*_{02}(0.5) = 0.25$	$u^*(0.5, 0) = 0.0$
	-0.5	0.0	$2(-0.5)^2 + 0.00 = 0.50$		
0.0	1.0	1.0	$2(1.0)^2 + 0.75 = 2.75$		
	0.5	0.5	$2(0.5)^2 + 0.25 = 0.75$		
	0.0	0.0	$2(0.0)^2 + 0.00 = 0.00$	$J^*_{02}(0.0) = 0.00$	$u^*(0.0, 0) = 0.0$

Problem from "Optimal Control Theory" authored by Donald E. Kirk

Hamiltonian-Jacobi-Bellman equation

- An alternate method without discretizing the system.
- Given the system

$$\frac{d}{dt}x(t) = a\left(x(t), u(t), t\right)$$

minimize the performance index

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g\left(x(\tau), u(\tau), \tau\right) d\tau$$

- Define

$$J^*(x(t), t) = \min_{\substack{u(\tau) \\ t \leqslant \tau \leqslant t_f}} \left\{ h(x(t_f), t_f) + \int_{t_0}^{t_f} g\left(x(\tau), u(\tau), \tau\right) d\tau \right\}$$

Hamiltonian-Jacobi-Bellman equation

- Define

$$J^*(x(t),t) = \min_{\substack{u(\tau)\\ t\leqslant\tau\leqslant t_f}} \left\{ h(x(t_f),t_f) + \int_{t_0}^{t_f} g\left(x(\tau),u(\tau),\tau\right) d\tau \right\}$$

- Subdividing the integral:

$$J^*(x(t),t) = \min_{\substack{u(\tau)\\ t\leqslant\tau\leqslant t_f}} \left\{ h(x(t_f),t_f) + \int_{t_0}^{t+\Delta t} g\left(x(\tau),u(\tau),\tau\right) d\tau \right.$$
$$\left. + \int_{t+\Delta t}^{t_f} g\left(x(\tau),u(\tau),\tau\right) d\tau \right\}$$

- By principle of optimality:

$$J^*(x(t),t) = \min_{\substack{u(\tau)\\ t\leqslant\tau\leqslant t_f}} \left\{ \int_{t_0}^{t+\Delta t} g d\tau + J^*(x(t+\Delta t), t+\Delta t) \right\}$$

HJB equations

- By principle of optimality:

$$J^*(x(t),t) = \min_{\substack{u(\tau) \\ t\leqslant\tau\leqslant t_f}} \left\{ \int_{t_0}^{t+\Delta t} g d\tau + J^*(x(t+\Delta t), t+\Delta t) \right\}$$

- Using Taylor series expansion and Δt small assumption:

$$\frac{\partial}{\partial t} J^*(x(t),t) + \min_{u(t)} \left\{ g(x(t),u(t),t) + \frac{\partial}{\partial x}\left(J^*(x(t),t)\right)^T [a(x(t),u(t),t] \right\} = 0.$$

- Define

$$\mathcal{H} := g(x(t),u(t),t) + \frac{\partial}{\partial x}\left(J^*(x(t),t)\right)^T [a(x(t),u(t),t]$$

HJB equation

$$J_t^*\left(x(t),u(t),t\right) + \min_{u(t)} \mathcal{H}(x(t),u(t),J_x^*,t) = 0$$

Boundary condition: $J^*\left(x(t_f),t_f\right) = h\left(x(t_f),t_f\right)$

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Thank you