Generalized eigenvalue problem

Chayan Bhawal

Students' Reading Group

October 26, 2017

Standard eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists nonzero $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda x, \text{ where } A \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}.$$

x is the **eigenvector** of A corresponding to **eigenvalue** λ.

- For example $A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Eigenvalues of A are 1 and 2.

Then

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cc|c|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

$$A\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} J$$

Standard eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists nonzero $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda x, \text{ where } A \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}.$$

x is the **eigenvector** of A corresponding to **eigenvalue** λ.

- For example $A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Eigenvalues of A are 1 and 2.

  Then

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cc|c|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

$$A \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} J$$

Standard Eigenvalue problem: Choice of Algorithm

Properties to be considered[1]:

- Is the matrix real or complex?
- What special properties does the matrix have?
  - symmetric, Hermitian, skew symmetric, unitary.
- Structure?
  - band, sparse, structured sparseness, Toeplitz.
- Eigenvalues required?
  - largest, smallest in magnitude, real part of eigenvalues negative, sums of intermediate eigenvalues.

[1] G.H. Golub and H.A. van der Vorst. *Eigenvalue computation in the 20th century*, Journal of Computational and Applied Mathematics 123, no. 1 (2000).

Standard Eigenvalue problem: Choice of Algorithm

Properties to be considered[1]:

- Is the matrix real or complex?
- What special properties does the matrix have?
  - symmetric, Hermitian, skew symmetric, unitary.
- Structure?
  - band, sparse, structured sparseness, Toeplitz.
- Eigenvalues required?
  - largest, smallest in magnitude, real part of eigenvalues negative, sums of intermediate eigenvalues.

[1] G.H. Golub and H.A. van der Vorst. *Eigenvalue computation in the 20th century*, Journal of Computational and Applied Mathematics 123, no. 1 (2000).

Properties to be considered[1]:

- Is the matrix real or complex?
- What special properties does the matrix have?
  - symmetric, Hermitian, skew symmetric, unitary.
- Structure?
  - band, sparse, structured sparseness, Toeplitz.
- Eigenvalues required?
  - largest, smallest in magnitude, real part of eigenvalues negative, sums of intermediate eigenvalues.

[1] G.H. Golub and H.A. van der Vorst. *Eigenvalue computation in the 20th century*, Journal of Computational and Applied Mathematics 123, no. 1 (2000).

Standard Eigenvalue problem: Choice of Algorithm

Properties to be considered[1]:

- Is the matrix real or complex?
- What special properties does the matrix have?
  - symmetric, Hermitian, skew symmetric, unitary.
- Structure?
  - band, sparse, structured sparseness, Toeplitz.
- Eigenvalues required?
  - largest, smallest in magnitude, real part of eigenvalues negative, sums of intermediate eigenvalues.

[1] G.H. Golub and H.A. van der Vorst. *Eigenvalue computation in the 20th century*, Journal of Computational and Applied Mathematics 123, no. 1 (2000).

Standard Eigenvalue problem: Choice of Algorithm

Properties to be considered[1]:

- Is the matrix real or complex?
- What special properties does the matrix have?
  - symmetric, Hermitian, skew symmetric, unitary.
- Structure?
  - band, sparse, structured sparseness, Toeplitz.
- Eigenvalues required?
  - largest, smallest in magnitude, real part of eigenvalues negative, sums of intermediate eigenvalues.

[1] G.H. Golub and H.A. van der Vorst. *Eigenvalue computation in the 20th century*, Journal of Computational and Applied Mathematics 123, no. 1 (2000).

Standard Eigenvalue problem: Choice of Algorithm

$$\begin{bmatrix} 2 & 6 & 1 & 5 & 7 & 9 & 8 \\ 3 & 1 & 2 & 4 & 7 & 8 & 10 \\ 0 & 4 & 1 & 7 & 4 & 6 & 3 \\ 0 & 0 & 7 & 6 & 5 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Upper Hessenberg form

$$\begin{bmatrix} 2 & 6 & 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 7 & 0 & 0 & 0 \\ 0 & 0 & 8 & 6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Tridiagonal form

Standard Eigenvalue problem: Choice of Algorithm

Method	Applies to	Produces	Description
Householder	General	Hessenberg	Reflect each column through a subspace to zero out its lower entries.
Givens rotations	General	Hessenberg	Apply planar rotations to zero out each entry.
Arnoldi iteration	General	Hessenberg	Perform Gram-Schmidt orthogonalization on Krylov subspaces
Lanczos algorithm	Hermitian	Tridiagonal	Arnoldi iteration for Hermetian matrices, with shortcuts

Standard Eigenvalue problem: Choice of Algorithm

Method	Applies to	Produces Eigenvalue	Description
Power iteration	General	Largest	Repeatedly applies matrix to an arbitrary initial vector.
Inverse iteration	General	closest to μ	Power iteration $(A - \mu I)^{-1}$
Rayleigh quotient iteration	Hermitian	closest to μ	Power iteration using Rayleigh quotient.
QR algorithm	Hessenberg	All eigenvalues	Factors $A = QR$, Q: orthogonal R: triangular.

QR Algorithm: Schur decomposition

Schur decomposition:
Given a matrix $A \in \mathbb{R}^{\texttt{n}\times\texttt{n}}$ there exists a unitary matrix $Q \in \mathbb{C}^{\texttt{n}\times\texttt{n}}$ such that

$$Q^* A Q = U$$

where $U \in \mathbb{C}^{\texttt{n}\times\texttt{n}}$ is a upper triangular matrix.

Real-Schur decomposition:
Given a matrix $A \in \mathbb{R}^{\texttt{n}\times\texttt{n}}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{\texttt{n}\times\texttt{n}}$ such that

$$Q^T A Q = U$$

where $U \in \mathbb{R}^{\texttt{n}\times\texttt{n}}$ is a quasi-upper triangular matrix.

QR Algorithm: Schur decomposition

Schur decomposition:
Given a matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists a unitary matrix $Q \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^* A Q = U$$

where $U \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ is a upper triangular matrix.

Real-Schur decomposition:
Given a matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^T A Q = U$$

where $U \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a quasi-upper triangular matrix.

For example:

$$A = \begin{bmatrix} -2 & -2 & -1 \\ -2 & -4 & -3 \\ 1 & 3 & 1 \end{bmatrix}$$

Use Scilab: [Q,U] = schur(A);

$$Q = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & 1 & \sqrt{3} \\ -\sqrt{2} & 2 & 0 \end{bmatrix}, \quad U = \frac{1}{\sqrt{12}} \begin{bmatrix} -3\sqrt{12} & -8\sqrt{6} & -4\sqrt{2} \\ 0 & -\sqrt{12} & 2 \\ 0 & -6 & -\sqrt{12} \end{bmatrix}$$

Eigenvalues of A: $\{-3, 1 \pm j1\}$

QR Algorithm: Real-Schur decomposition

For example:

$$A = \begin{bmatrix} -2 & -2 & -1 \\ -2 & -4 & -3 \\ 1 & 3 & 1 \end{bmatrix}$$

Use Scilab: [Q,U] = schur(A);

$$Q = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & 1 & \sqrt{3} \\ -\sqrt{2} & 2 & 0 \end{bmatrix}, \quad U = \frac{1}{\sqrt{12}} \begin{bmatrix} -3\sqrt{12} & -8\sqrt{6} & -4\sqrt{2} \\ 0 & -\sqrt{12} & 2 \\ 0 & -6 & -\sqrt{12} \end{bmatrix}$$

Eigenvalues of A: $\{-3, 1 \pm j1\}$

For example:

$$A = \begin{bmatrix} -2 & -2 & -1 \\ -2 & -4 & -3 \\ 1 & 3 & 1 \end{bmatrix}$$

Use Scilab: [Q,U] = schur(A);

$$Q = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & 1 & \sqrt{3} \\ -\sqrt{2} & 2 & 0 \end{bmatrix}, \quad U = \frac{1}{\sqrt{12}} \begin{bmatrix} -3\sqrt{12} & -8\sqrt{6} & -4\sqrt{2} \\ 0 & -\sqrt{12} & 2 \\ 0 & -6 & -\sqrt{12} \end{bmatrix}$$

Eigenvalues of A: $\{-3, 1 \pm j1\}$

Standard Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda x, \text{ where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

  x is the **eigenvector** of the matrix A corresponding to **eigenvalue** λ.

- Algorithm: Schur decomposition.
- Number of eigenvalues = $\mathtt{n}$.
- Number of eigenvectors depends on the algebraic multiplicity and geometric multiplicity of eigenvalues.

Standard Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda I x, \text{ where } A \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}.$$

  x is the **eigenvector** of the matrix pair (I, A) corresponding to **eigenvalue** λ.

- Algorithm: Schur decomposition.
- Number of eigenvalues = $\mathtt{n}$.
- Number of eigenvectors depends on the algebraic multiplicity and geometric multiplicity of eigenvalues.

Generalized Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda Bx, \text{ where } A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

x is the **generalized eigenvector** of the matrix (B, A) corresponding to **generalized eigenvalue** λ.

- How to find the *generalized* eigenvalues?

$$(A - \lambda B)x = 0.$$

Will Real-Schur decomposition work?

Generalized Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda B x, \text{ where } A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

  x is the **generalized eigenvector** of the matrix (B, A) corresponding to **generalized eigenvalue** λ.

- How to find the *generalized* eigenvalues?

$$(A - \lambda B)x = 0.$$

  Will Real-Schur decomposition work?

Generalized Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda B x, \text{ where } A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

  x is the **generalized principal eigenvector** of the matrix (B, A) corresponding to **generalized eigenvalue** λ.

- How to find the *generalized* eigenvalues?

$$(A - \lambda B)x = 0.$$

  Will Real-Schur decomposition work?

Generalized Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda Bx, \text{ where } A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

  x is the **generalized principal eigenvector** of the matrix (B, A) corresponding to **generalized eigenvalue** λ.

- How to find the *generalized* eigenvalues?

$$(A - \lambda B)x = 0.$$

  Will Real-Schur decomposition work?

Generalized Eigenvalue problem

- Find $\lambda \in \mathbb{C}$ such that there exists $x \in \mathbb{C}^{\mathtt{n}}$ such that

$$Ax = \lambda {\color{red}B}x, \text{ where } A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

  x is the **generalized principal eigenvector** of the matrix (B, A) corresponding to **generalized eigenvalue** λ.

- How to find the *generalized* eigenvalues?

$$(A - \lambda B)x = 0.$$

  Will Real-Schur decomposition work? 

Will Real-Schur decomposition work?

$$Q^T A Q = U \Rightarrow A = Q U Q^T$$

$$Q^T B Q = \tilde{B} \Rightarrow B = Q \tilde{B} Q^T$$

$$\begin{aligned} Ax &= \lambda B x \\ QUQ^T x &= \lambda Q \tilde{B} Q^T x \\ {\color{blue}U} Q^T x &= \lambda {\color{blue}\tilde{B}} Q^T x \end{aligned}$$

For example:

$$\underbrace{\begin{bmatrix} -2 & -2 & -1 \\ -2 & -4 & -3 \\ 1 & 3 & 1 \end{bmatrix}}_{A} x = \lambda \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B} x$$

Will Real-Schur decomposition work?

$$Q^T A Q = U \Rightarrow A = Q U Q^T$$

$$Q^T B Q = \tilde{B} \Rightarrow B = Q \tilde{B} Q^T$$

$$\begin{aligned} Ax &= \lambda Bx \\ QUQ^T x &= \lambda Q\tilde{B}Q^T x \\ UQ^T x &= \lambda \tilde{B} Q^T x \end{aligned}$$

For example:

$$\underbrace{\frac{1}{\sqrt{12}}\begin{bmatrix} -3\sqrt{12} & -8\sqrt{6} & -4\sqrt{2} \\ 0 & -\sqrt{12} & 2 \\ 0 & -6 & -\sqrt{12} \end{bmatrix}}_{U} Q^T x = \lambda \underbrace{\frac{1}{6}\begin{bmatrix} 2 & \sqrt{2} & \sqrt{6} \\ \sqrt{2} & 1 & \sqrt{3} \\ -\sqrt{6} & -\sqrt{3} & 3 \end{bmatrix}}_{\tilde{B}} Q^T x$$

QZ Algorithm: Generalized Schur decomposition

Generalized Schur decomposition:

Given a matrix $A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists a unitary matrix $Q, Z \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^*AZ = U \quad Q^*BZ = T$$

where $U, T \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ are upper triangular matrix.

Generalized Real-Schur decomposition:

Given a matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists an orthogonal matrix $Q, Z \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^TAZ = U \quad Q^TBZ = T$$

where $U \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a quasi-upper triangular matrix and T is a triangular matrix.

QZ Algorithm: Generalized Schur decomposition

Generalized Schur decomposition:
Given a matrix $A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists a unitary matrix $Q, Z \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^*AZ = U \quad Q^*BZ = T$$

where $U, T \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ are upper triangular matrix.

Generalized Real-Schur decomposition:
Given a matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists an orthogonal matrix $Q, Z \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^TAZ = U \quad Q^TBZ = T$$

where $U \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a quasi-upper triangular matrix and T is a triangular matrix.

QZ Algorithm: Generalized Schur decomposition

Generalized Schur decomposition:
Given a matrix $A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists a unitary matrix $Q, Z \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^* AZ = U \quad Q^* BZ = T$$

where $U, T \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ are upper triangular matrix.

Generalized Real-Schur decomposition:
Given a matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ there exists an orthogonal matrix $Q, Z \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$Q^T AZ = U \quad Q^T BZ = T$$

where $U \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a quasi-upper triangular matrix and T is a triangular matrix.

Generalized Schur decomposition

$$Q^T A Q = U \Rightarrow A = Q U Q^T$$

$$Q^T B Q = \tilde{B} \Rightarrow B = Q \tilde{B} Q^T$$

$$Ax = \lambda Bx$$
$$QUQ^T x = \lambda Q\tilde{B}Q^T x$$
$$UQ^T x = \lambda \tilde{B} Q^T x$$

$$\underbrace{\frac{1}{\sqrt{12}}\begin{bmatrix} -3\sqrt{12} & -8\sqrt{6} & -4\sqrt{2} \\ 0 & -\sqrt{12} & 2 \\ 0 & -6 & -\sqrt{12} \end{bmatrix}}_{U} Q^T x = \lambda \underbrace{\frac{1}{6}\begin{bmatrix} 2 & \sqrt{2} & \sqrt{6} \\ \sqrt{2} & 1 & \sqrt{3} \\ -\sqrt{6} & -\sqrt{3} & 3 \end{bmatrix}}_{\tilde{B}} Q^T x$$

Generalized Schur decomposition

$$Q^T A Z = U \Rightarrow A = QUZ^T$$

$$Q^T B Z = T \Rightarrow B = QTZ^T$$

$$Ax = \lambda Bx$$
$$QUZ^T x = \lambda QTZ^T x$$
$$UZ^T x = \lambda TZ^T x$$

Use Scilab: [U,T,Q,Z] = schur(A,B);

$$Q = \frac{1}{3\sqrt{2}} \begin{bmatrix} -3 & 2\sqrt{2} & 1 \\ 3 & 2\sqrt{2} & 1 \\ 0 & -\sqrt{2} & 2 \end{bmatrix}, \quad Z = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{bmatrix}$$

Generalized Schur decomposition

$$Q^T AZ = U \Rightarrow A = QUZ^T$$

$$Q^T BZ = T \Rightarrow B = QTZ^T$$

$$Ax = \lambda Bx$$
$$QUZ^T x = \lambda QTZ^T x$$
$$UZ^T x = \lambda TZ^T x$$

Use Scilab: [U,T,Q,Z] = schur(A,B);

$$\begin{bmatrix} -1 & 1 & -\sqrt{2} \\ 0 & 3\sqrt{2} & -5 \\ 0 & 0 & \sqrt{2} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} Z^T x$$

Generalized Schur decomposition

$$Q^T A Z = U \Rightarrow A = QUZ^T$$

$$Q^T B Z = T \Rightarrow B = QTZ^T$$

$$Ax = \lambda Bx$$
$$QUZ^T x = \lambda QTZ^T x$$
$$UZ^T x = \lambda TZ^T x$$

Use Scilab: [U,T,Q,Z] = schur(A,B);

$$\begin{bmatrix} -1-\lambda & 1-\lambda & -\sqrt{2}-\frac{\lambda}{\sqrt{2}} \\ 0 & 3\sqrt{2} & -5-\frac{2\lambda}{\sqrt{3}} \\ 0 & 0 & \sqrt{2}-\frac{\lambda}{3\sqrt{2}} \end{bmatrix} Z^T x = 0$$

Values of λ at which we have a kernel: $\{-1, 6\}$. These are the finite eigenvalues of (A, B). Where is the other eigenvalue?

Generalized eigenvalue problem

$$Ax = \lambda Bx$$

$$\begin{bmatrix} a_{11} & \star & \cdots & \star \\ 0 & a_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{\mathrm{nn}} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} b_{11} & \star & \cdots & \star \\ 0 & b_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & b_{\mathrm{nn}} \end{bmatrix} Z^T x.$$

- Eigenvalues: $\{(a_{11}, b_{11}), (a_{22}, b_{22}), \cdots, (a_{\mathrm{nn}}, b_{\mathrm{nn}})\}$
- Finite eigenvalue: $\lambda_i = \frac{a_{ii}}{b_{ii}}$.
- For our example:

$$\begin{bmatrix} -1 & 1 & -\sqrt{2} \\ 0 & 3\sqrt{2} & -5 \\ 0 & 0 & \sqrt{2} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} Z^T x$$

- Eigenvalues: $\{(-1, 1), (3\sqrt{2}, 0), (\sqrt{2}, \frac{1}{3\sqrt{2}})\}$.
  Finite eigenvalues: $\{-1, 6\}$.

Generalized eigenvalue problem

$$Ax = \lambda Bx$$

$$\begin{bmatrix} a_{11} & \star & \cdots & \star \\ 0 & a_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{\mathtt{nn}} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} b_{11} & \star & \cdots & \star \\ 0 & b_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & b_{\mathtt{nn}} \end{bmatrix} Z^T x.$$

- Eigenvalues: $\{(a_{11}, b_{11}), (a_{22}, b_{22}), \cdots, (a_{\mathtt{nn}}, b_{\mathtt{nn}})\}$
- Finite eigenvalue: $\lambda_i = \frac{a_{ii}}{b_{ii}}$.
- For our example:

$$\begin{bmatrix} -1 & 1 & -\sqrt{2} \\ 0 & 3\sqrt{2} & -5 \\ 0 & 0 & \sqrt{2} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} Z^T x$$

- Eigenvalues: $\{(-1, 1), (3\sqrt{2}, 0), (\sqrt{2}, \frac{1}{3\sqrt{2}})\}$.
  Finite eigenvalues: $\{-1, 6\}$.

Generalized eigenvalue problem

$$Ax = \lambda Bx$$

$$\begin{bmatrix} a_{11} & \star & \cdots & \star \\ 0 & a_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{\mathtt{nn}} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} b_{11} & \star & \cdots & \star \\ 0 & b_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & b_{\mathtt{nn}} \end{bmatrix} Z^T x.$$

- Eigenvalues: $\{(a_{11}, b_{11}), (a_{22}, b_{22}), \cdots, (a_{\mathtt{nn}}, b_{\mathtt{nn}})\}$
- Finite eigenvalue: $\lambda_i = \frac{a_{ii}}{b_{ii}}$.
- For our example:

$$\begin{bmatrix} -1 & 1 & -\sqrt{2} \\ 0 & 3\sqrt{2} & -5 \\ 0 & 0 & \sqrt{2} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} Z^T x$$

- Eigenvalues: $\{(-1, 1), (3\sqrt{2}, 0), (\sqrt{2}, \frac{1}{3\sqrt{2}})\}$.

  Finite eigenvalues: $\{-1, 6\}$.

Generalized eigenvalue problem

$$Ax = \lambda Bx$$

$$\begin{bmatrix} a_{11} & \star & \cdots & \star \\ 0 & a_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{\mathtt{nn}} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} b_{11} & \star & \cdots & \star \\ 0 & b_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & b_{\mathtt{nn}} \end{bmatrix} Z^T x.$$

- Eigenvalues: $\{(a_{11}, b_{11}), (a_{22}, b_{22}), \cdots, (a_{\mathtt{nn}}, b_{\mathtt{nn}})\}$
- Finite eigenvalue: $\lambda_i = \frac{a_{ii}}{b_{ii}}$.
- For our example:

$$\begin{bmatrix} -1 & 1 & -\sqrt{2} \\ 0 & 3\sqrt{2} & -5 \\ 0 & 0 & \sqrt{2} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} Z^T x$$

- Eigenvalues: $\{(-1, 1), (3\sqrt{2}, 0), (\sqrt{2}, \frac{1}{3\sqrt{2}})\}$.

  Finite eigenvalues: $\{-1, 6\}$.

Generalized eigenvalue problem

$$Ax = \lambda Bx$$

$$\begin{bmatrix} a_{11} & \star & \cdots & \star \\ 0 & a_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{\mathtt{nn}} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} b_{11} & \star & \cdots & \star \\ 0 & b_{22} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & b_{\mathtt{nn}} \end{bmatrix} Z^T x.$$

- Eigenvalues: $\{(a_{11}, b_{11}), (a_{22}, b_{22}), \cdots, (a_{\mathtt{nn}}, b_{\mathtt{nn}})\}$
- Finite eigenvalue: $\lambda_i = \frac{a_{ii}}{b_{ii}}$.
- For our example:

$$\begin{bmatrix} -1 & 1 & -\sqrt{2} \\ 0 & 3\sqrt{2} & -5 \\ 0 & 0 & \sqrt{2} \end{bmatrix} Z^T x = \lambda \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} Z^T x$$

- Eigenvalues: $\{(-1, 1), (3\sqrt{2}, 0), (\sqrt{2}, \frac{1}{3\sqrt{2}})\}$.
  Finite eigenvalues: $\{-1, 6\}$.

Generalized eigenvalue problem: A few properties

- $\texttt{deg}\,\texttt{det}(sI - A) = \texttt{n}$. Hence $\texttt{n}$ finite eigenvalues.
- $\texttt{deg}\,\texttt{det}(sB - A) \leqslant \texttt{n}$. Let $\texttt{deg}\,\texttt{det}(sB - A) = \texttt{n}_f$.

  Number of finite eigenvalues $= \texttt{n}_f$.

  Number of inifite eigenvalue $= \texttt{n}_\infty$.
- For our example:

  $\texttt{det}(sB - A) = s^2 - 5s + 6$.
  $\texttt{n}_f = 2$ and $\texttt{n}_\infty = 3 - 2 = 1$.

  Roots of $\texttt{det}(sB - A) = \{-1, 6\}$ = Finite eigenvalues of (A, B).

Generalized eigenvalue problem: A few properties

- $\mathtt{deg}\,\mathtt{det}(sI - A) = \mathtt{n}$. Hence $\mathtt{n}$ finite eigenvalues.
- $\mathtt{deg}\,\mathtt{det}(sB - A) \leqslant \mathtt{n}$. Let $\mathtt{deg}\,\mathtt{det}(sB - A) = \mathtt{n}_f$.

  Number of finite eigenvalues $= \mathtt{n}_f$.

  Number of inifite eigenvalue $= \mathtt{n}_\infty$.
- For our example:

  $\mathtt{det}(sB - A) = s^2 - 5s + 6$.
  $\mathtt{n}_f = 2$ and $\mathtt{n}_\infty = 3 - 2 = 1$.

  Roots of $\mathtt{det}(sB - A) = \{-1, 6\}$ = Finite eigenvalues of (A, B).

Generalized eigenvalue problem: A few properties

- $\texttt{deg}\,\texttt{det}(sI - A) = \texttt{n}$. Hence $\texttt{n}$ finite eigenvalues.
- $\texttt{deg}\,\texttt{det}(sB - A) \leqslant \texttt{n}$. Let $\texttt{deg}\,\texttt{det}(sB - A) = \texttt{n}_f$.

  Number of finite eigenvalues $= \texttt{n}_f$.

  Number of inifite eigenvalue $= \texttt{n}_\infty$.
- For our example:

  $\texttt{det}(sB - A) = s^2 - 5s + 6$.
  $\texttt{n}_f = 2$ and $\texttt{n}_\infty = 3 - 2 = 1$.

  Roots of $\texttt{det}(sB - A) = \{-1, 6\} =$ Finite eigenvalues of (A, B).

Generalized eigenvalue problem: A few properties

- $\texttt{deg}\,\texttt{det}(sB - A) = \texttt{n}$ if and only if B is nonsingular.

$$Ax = \lambda Bx$$
$$B^{-1}Ax = \lambda x$$

Never ever ever ... Too costly

- $\texttt{det}(sB - A) = 0 \Rightarrow \texttt{det}(\lambda B - A) = 0$ for any $\lambda \in \mathbb{C}$.
  This happens if and only if $\exists\ a_{ii} = b_{ii} = 0$.

- $\texttt{det}(sB - A) \neq 0 \Rightarrow (sB - A)$ called regular matrix pencil.

Generalized eigenvalue problem: A few properties

- $\texttt{deg}\,\texttt{det}(sB - A) = \texttt{n}$ if and only if B is nonsingular.

$$Ax = \lambda Bx$$
$$B^{-1}Ax = \lambda x$$

Never ever ever ... Too costly

- $\texttt{det}(sB - A) = 0 \Rightarrow \texttt{det}(\lambda B - A) = 0$ for any $\lambda \in \mathbb{C}$.
  This happens if and only if $\exists\ a_{ii} = b_{ii} = 0$.

- $\texttt{det}(sB - A) \neq 0 \Rightarrow (sB - A)$ called regular matrix pencil.

Generalized eigenvalue problem: A few properties

- $\texttt{deg}\,\texttt{det}(sB - A) = \texttt{n}$ if and only if B is nonsingular.

$$Ax = \lambda Bx$$
$$B^{-1}Ax = \lambda x$$

Never ever ever ... Too costly

- $\texttt{det}(sB - A) = 0 \Rightarrow \texttt{det}(\lambda B - A) = 0$ for any $\lambda \in \mathbb{C}$.
  This happens if and only if $\exists\ a_{ii} = b_{ii} = 0$.

- $\texttt{det}(sB - A) \neq 0 \Rightarrow (sB - A)$ called regular matrix pencil.

Generalized eigenvalue problem: A few properties

- $\texttt{deg}\,\texttt{det}(sB - A) = \texttt{n}$ if and only if B is nonsingular.

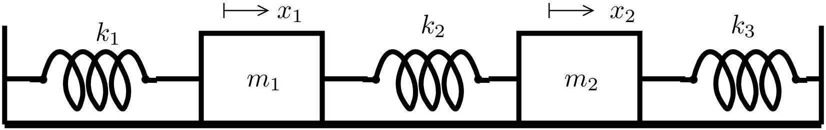
$$Ax = \lambda Bx$$
$$B^{-1}Ax = \lambda x$$

Never ever ever ... Too costly

- $\texttt{det}(sB - A) = 0 \Rightarrow \texttt{det}(\lambda B - A) = 0$ for any $\lambda \in \mathbb{C}$.
  This happens if and only if $\exists\ a_{ii} = b_{ii} = 0$.

- $\texttt{det}(sB - A) \neq 0 \Rightarrow (sB - A)$ called regular matrix pencil.

Application: Mechanical structures



$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume the solution is of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin \omega t$. This leads to

$$\left(K - \omega^2 M\right) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Application: Mechanical structures

- Dynamic response analysis of an assemblage of structural elements

$$\left(K - \omega^2 M\right)\phi = 0$$

K is the stiffness matrix and M is the mass-matrix.

- The $\mathtt{n}$ solutions are

$$K\Phi = M\Phi\Omega^2$$

$\Phi = \begin{bmatrix}\Phi_1 & \cdots & \Phi_{\mathtt{n}}\end{bmatrix}$ where each Φ_i are M-orthonormalized eigenvectors (free vibration modes).

$\Omega^2 = \mathtt{diag}\left(\omega_1^2, \ldots, \omega_{\mathtt{n}}^2\right)$. where each ω_i is free vibration frequency.

Application: Mechanical structures

- Dynamic response analysis of an assemblage of structural elements

$$\left(K - \omega^2 M\right)\phi = 0$$

K is the stiffness matrix and M is the mass-matrix.

- The $\mathtt{n}$ solutions are

$$K\Phi = M\Phi\Omega^2$$

$\Phi = \begin{bmatrix}\Phi_1 & \cdots & \Phi_{\mathtt{n}}\end{bmatrix}$ where each Φ_i are M-orthonormalized eigenvectors (free vibration modes).

$\Omega^2 = \mathtt{diag}\left(\omega_1^2, \ldots, \omega_{\mathtt{n}}^2\right)$. where each ω_i is free vibration frequency.

Invariant subspaces

- $\mathcal{V} \subseteq \mathbb{R}^n$ is called A-invariant if $Ax \in \mathcal{V}$ for all $x \in \mathcal{V}$.
- Vector space formed by the span of eigenvectors are naturally A invariant.

  For example: Let $\{v_1, v_2\}$ are eigenvectors of A corresponding to eigenvalues $\{\lambda_1, \lambda_2\}$.

  $$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 \in \langle v_1, v_2 \rangle.$$

  $\mathcal{V} = \langle v_1, v_2 \rangle$ is an A-invariant subspace.
- What is its analogy in case of generalized eigenvalue problem?

Invariant subspaces

- $\mathcal{V} \subseteq \mathbb{R}^{\mathtt{n}}$ is called A-invariant if $Ax \in \mathcal{V}$ for all $x \in \mathcal{V}$.
- Vector space formed by the span of eigenvectors are naturally A invariant.

  For example: Let $\{v_1, v_2\}$ are eigenvectors of A corresponding to eigenvalues $\{\lambda_1, \lambda_2\}$.

  $$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 \in \langle v_1, v_2 \rangle.$$

  $\mathcal{V} = \langle v_1, v_2 \rangle$ is an A-invariant subspace.
- What is its analogy in case of generalized eigenvalue problem?

Invariant subspaces

- $\mathcal{V} \subseteq \mathbb{R}^{\mathtt{n}}$ is called A-invariant if $Ax \in \mathcal{V}$ for all $x \in \mathcal{V}$.
- Vector space formed by the span of eigenvectors are naturally A invariant.

  For example: Let $\{v_1, v_2\}$ are eigenvectors of A corresponding to eigenvalues $\{\lambda_1, \lambda_2\}$.

  $$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 \in \langle v_1, v_2 \rangle.$$

  $\mathcal{V} = \langle v_1, v_2 \rangle$ is an A-invariant subspace.
- What is its analogy in case of generalized eigenvalue problem?

Invariant subspaces

- $\mathcal{V} \subseteq \mathbb{R}^{\mathtt{n}}$ is called A-invariant if $Ax \in \mathcal{V}$ for all $x \in \mathcal{V}$.

- Vector space formed by the span of eigenvectors are naturally A invariant.

  For example: Let $\{v_1, v_2\}$ are eigenvectors of A corresponding to eigenvalues $\{\lambda_1, \lambda_2\}$.

  $$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 \in \langle v_1, v_2 \rangle.$$

  $\mathcal{V} = \langle v_1, v_2 \rangle$ is an A-invariant subspace.

- What is its analogy in case of generalized eigenvalue problem?

Deflating subspaces

- $\mathcal{X}$ is called deflating subspace of a matrix pair (A, B) if

$$\texttt{dim}\,(B\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

- Is it an arbitrary definition? NO.

- We check for the standard eigenvalue problem case.
For $B = I$, when does

$$\texttt{dim}\,(I\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

Only when $\mathcal{X}$ is A-invariant.

Deflating subspaces

- $\mathcal{X}$ is called deflating subspace of a matrix pair (A, B) if

$$\texttt{dim}\,(B\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

- Is it an arbitrary definition? NO.

- We check for the standard eigenvalue problem case.
  For $B = I$, when does

$$\texttt{dim}\,(I\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

  Only when $\mathcal{X}$ is A-invariant.

Deflating subspaces

- $\mathcal{X}$ is called deflating subspace of a matrix pair (A, B) if

$$\texttt{dim}\,(B\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

- Is it an arbitrary definition? NO.
- We check for the standard eigenvalue problem case.
  For $B = I$, when does

$$\texttt{dim}\,(I\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

Only when $\mathcal{X}$ is A-invariant.

Deflating subspaces

- $\mathcal{X}$ is called deflating subspace of a matrix pair (A, B) if

$$\texttt{dim}\,(B\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

- Is it an arbitrary definition? NO.

- We check for the standard eigenvalue problem case.
  For $B = I$, when does

$$\texttt{dim}\,(I\mathcal{X} + A\mathcal{X}) = \texttt{dim}\,(\mathcal{X})\,.$$

Only when $\mathcal{X}$ is A-invariant.

How to compute deflating subspaces?

- Recall there exists $Q^T AZ = U$ and $Q^T BZ = T$.

First ℓ columns of Z spans ℓ dimensional deflating subspace.

- We want $\mathtt{dim}\,(A\mathcal{X} + B\mathcal{X}) = \mathtt{dim}\,(\mathcal{X})$.
  $\mathtt{dim}\left(QUZ^T\mathcal{X} + QTZ^T\mathcal{X}\right) = \mathtt{dim}\left(UZ^T\mathcal{X} + TZ^T\mathcal{X}\right)$
- We want 1 dimensional deflating subspace. Chose $Z = \langle z_1 \rangle$.

$$UZ^T(\alpha z_1) + TZ^T(\beta z_1) = \alpha U e_1 + \beta T e_1 = \alpha \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= (\alpha a_{11} + \beta b_{11}) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\mathtt{dim}\,(A\langle z_1\rangle + B\langle z_1\rangle) = 1 = \mathtt{dim}(\langle z_1\rangle)$.

How to compute deflating subspaces?

- Recall there exists $Q^T AZ = U$ and $Q^T BZ = T$.

First ℓ columns of Z spans ℓ dimensional deflating subspace.

- We want $\texttt{dim}\,(A\mathcal{X} + B\mathcal{X}) = \texttt{dim}\,(\mathcal{X})$.

$\texttt{dim}\left(QUZ^T\mathcal{X} + QTZ^T\mathcal{X}\right) = \texttt{dim}\left(UZ^T\mathcal{X} + TZ^T\mathcal{X}\right)$

- We want 1 dimensional deflating subspace. Chose $Z = \langle z_1 \rangle$.

$$UZ^T(\alpha z_1) + TZ^T(\beta z_1) = \alpha U e_1 + \beta T e_1 = \alpha \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= (\alpha a_{11} + \beta b_{11}) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\texttt{dim}\,(A\langle z_1\rangle + B\langle z_1\rangle) = 1 = \texttt{dim}(\langle z_1\rangle)$.

How to compute deflating subspaces?

- Recall there exists $Q^T AZ = U$ and $Q^T BZ = T$.

First ℓ columns of Z spans ℓ dimensional deflating subspace.

- We want $\mathtt{dim}\,(A\mathcal{X} + B\mathcal{X}) = \mathtt{dim}\,(\mathcal{X})$.
  $\mathtt{dim}\left(QUZ^T\mathcal{X} + QTZ^T\mathcal{X}\right) = \mathtt{dim}\left(UZ^T\mathcal{X} + TZ^T\mathcal{X}\right)$
- We want 1 dimensional deflating subspace. Chose $Z = \langle z_1 \rangle$.

$$UZ^T(\alpha z_1) + TZ^T(\beta z_1) = \alpha U e_1 + \beta T e_1 = \alpha \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= (\alpha a_{11} + \beta b_{11}) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\mathtt{dim}\,(A\langle z_1 \rangle + B\langle z_1 \rangle) = 1 = \mathtt{dim}(\langle z_1 \rangle)$.

How to compute deflating subspaces?

- Recall there exists $Q^T AZ = U$ and $Q^T BZ = T$.

First ℓ columns of Z spans ℓ dimensional deflating subspace.

- We want $\mathtt{dim}\,(A\mathcal{X} + B\mathcal{X}) = \mathtt{dim}\,(\mathcal{X})$.
  $\mathtt{dim}\left(QUZ^T\mathcal{X} + QTZ^T\mathcal{X}\right) = \mathtt{dim}\left(UZ^T\mathcal{X} + TZ^T\mathcal{X}\right)$
- We want 1 dimensional deflating subspace. Chose $Z = \langle z_1 \rangle$.

$$UZ^T(\alpha z_1) + TZ^T(\beta z_1) = \alpha U e_1 + \beta T e_1 = \alpha \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= (\alpha a_{11} + \beta b_{11}) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\mathtt{dim}\,(A\langle z_1 \rangle + B\langle z_1 \rangle) = 1 = \mathtt{dim}(\langle z_1 \rangle)$.

How to compute deflating subspaces?

- Recall there exists $Q^T AZ = U$ and $Q^T BZ = T$.

> First ℓ columns of Z spans ℓ dimensional deflating subspace.

- We want $\texttt{dim}\,(A\mathcal{X} + B\mathcal{X}) = \texttt{dim}\,(\mathcal{X})$.
  $\texttt{dim}\left(QUZ^T\mathcal{X} + QTZ^T\mathcal{X}\right) = \texttt{dim}\left(UZ^T\mathcal{X} + TZ^T\mathcal{X}\right)$
- We want 1 dimensional deflating subspace. Chose $Z = \langle z_1 \rangle$.

$$UZ^T(\alpha z_1) + TZ^T(\beta z_1) = \alpha U e_1 + \beta T e_1 = \alpha \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= (\alpha a_{11} + \beta b_{11}) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\texttt{dim}\,(A\langle z_1 \rangle + B\langle z_1 \rangle) = 1 = \texttt{dim}(\langle z_1 \rangle).$$

Application of deflating subspace

- Given a, b, c, solve

$$ax^2 + bx + c = 0.$$

Too easy

- Try this: Find $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$A^T K + KA + KBR^{-1}B^T K - Q = 0.$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$, $Q = Q^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $R = R^T \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

- Deflating subspaces of Generalized eigenvalue problem helps.

Application of deflating subspace

- Given a, b, c, solve

$$ax^2 + bx + c = 0.$$

Too easy

- Try this: Find $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$A^T K + KA + KBR^{-1}B^T K - Q = 0.$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$, $Q = Q^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $R = R^T \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

- Deflating subspaces of Generalized eigenvalue problem helps.

Application of deflating subspace

- Given a, b, c, solve

$$ax^2 + bx + c = 0.$$

Too easy

- Try this: Find $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$A^T K + KA + KBR^{-1}B^T K - Q = 0.$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$, $Q = Q^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $R = R^T \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

- Deflating subspaces of Generalized eigenvalue problem helps.

Riccati equation

- LQR Optimal control problem:
  Given the stabilizable system

$$\frac{d}{dt}x = Ax + Bu$$

  find the control $u(t) = -Kx(t)$ minimizing the functional

$$J = \int_0^\infty [x(t)^T Qx(t) + u(t)^T Ru(t)]dt,$$

  where (A, Q) is detectable, $Q \geqslant 0$ and $R > 0$.
- Solutions of Riccati equation solves the LQR problem

$$A^T K + KA + KBR^{-1}B^T K - Q = 0.$$

Riccati equation

- LQR Optimal control problem:
  Given the stabilizable system

$$\frac{d}{dt}x = Ax + Bu$$

  find the control $u(t) = -Kx(t)$ minimizing the functional

$$J = \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt,$$

  where (A, Q) is detectable, $Q \geqslant 0$ and $R > 0$.

- Solutions of Riccati equation solves the LQR problem

$$A^T K + KA + KBR^{-1}B^T K - Q = 0.$$

Riccati equation and deflating subspaces

- P. Van Dooren[2] showed that deflating subspace of a special matrix pencils solves the problem.

$$s\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} A & 0 & B \\ Q & -A^T & 0 \\ 0 & -B^T & R \end{bmatrix}}_{H}$$

- Chosen n-dimensional deflating subspace* of the pair (H, E).

$$V = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \in \mathbb{R}^{(2n+p)\times n}$$

Then,

$$K = X_2 X_1^{-1}$$

[2] P. Van Dooren, A Generalized eigenvalue approach for solving Riccati equations, *SIAM Journal on Scientific and Statistical Computing*, 2.2 (1981): 121-135

Riccati equation and deflating subspaces

- P. Van Dooren[2] showed that deflating subspace of a special matrix pencils solves the problem.

$$s \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} A & 0 & B \\ Q & -A^T & 0 \\ 0 & -B^T & R \end{bmatrix}}_{H}$$

- Chosen $\mathtt{n}$-dimensional deflating subspace* of the pair (H, E).

$$V = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times\mathtt{n}}$$

Then,

$$K = X_2 X_1^{-1}$$

[2] P. Van Dooren, A Generalized eigenvalue approach for solving Riccati equations, *SIAM Journal on Scientific and Statistical Computing*, 2.2 (1981): 121-135

Canonical form

- Standard eigenvalue problems have canonical forms like Jordan canonical form. What do we have for the generalized case?
- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

- L_μ is a $\mu \times (\mu + 1)$ matrix of the form

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

- $\tilde{L}_v$ is a $(v + 1) \times v$ matrix with λ along the diagonal and -1 along the first subdiagonal.
- N is a nilpotent Jordan matrix.
- J is in Jordan (or in rational) form.

Canonical form

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form

$$P(\lambda B - A)Q =$$

$$\begin{bmatrix} L_{\mu_1} & & & & & & & \\ & \ddots & & & & & & \\ & & L_{\mu_k} & & & & & \\ & & & \tilde{L}_{v_1} & & & & \\ & & & & \ddots & & & \\ & & & & & \tilde{L}_{v_\ell} & & \\ & & & & & & \lambda N - I & \\ & & & & & & & \lambda I - J \end{bmatrix}$$

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{blockdiag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

Canonical form

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

- L_μ is a $\mu \times (\mu + 1)$ matrix of the form

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

- $\tilde{L}_v$ is a $(v+1) \times v$ matrix with λ along the diagonal and -1 along the first subdiagonal.
- N is a nilpotent Jordan matrix.
- J is in Jordan (or in rational) form.

Canonical form

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

- L_μ is a $\mu \times (\mu + 1)$ matrix of the form

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

- $\tilde{L}_v$ is a $(v+1) \times v$ matrix with λ along the diagonal and -1 along the first subdiagonal.
- N is a nilpotent Jordan matrix.
- J is in Jordan (or in rational) form.

Canonical form

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

- L_μ is a $\mu \times (\mu + 1)$ matrix of the form

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

- $\tilde{L}_v$ is a $(v + 1) \times v$ matrix with λ along the diagonal and -1 along the first subdiagonal.
- N is a nilpotent Jordan matrix.
- J is in Jordan (or in rational) form.

Canonical form

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

- L_μ is a $\mu \times (\mu + 1)$ matrix of the form

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

- $\tilde{L}_v$ is a $(v+1) \times v$ matrix with λ along the diagonal and -1 along the first subdiagonal.
- N is a nilpotent Jordan matrix.
- J is in Jordan (or in rational) form.

Canonical form

- There exists nonsingular P and Q such that any pencil $\lambda B - A$ has a canonical quasi-diagonal form $P(\lambda B - A)Q =$

$$\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$$

- L_μ is a $\mu \times (\mu + 1)$ matrix of the form

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

- $\tilde{L}_v$ is a $(v+1) \times v$ matrix with λ along the diagonal and -1 along the first subdiagonal.
- N is a nilpotent Jordan matrix.
- J is in Jordan (or in rational) form.

Some features of Kronecker canonical form

- $P(\lambda B - A)Q =$
  $\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$
- Matrices L_μ and $\tilde{L}_v$ will not appear if the pencil is regular, i.e. $\texttt{det}(sB - A) \neq 0$.
- For regular pencils:

$$P(\lambda B - A)Q = \texttt{block diag}\left(\lambda N - I, \lambda I - J\right)$$

- For pencils with only finite eigenvalues (B is invertible):

$$P(\lambda B - A)Q = (\lambda I - J).$$

Guess what will be PB^{-1}?

Some features of Kronecker canonical form

- $P(\lambda B - A)Q =$ $\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$
- Matrices L_μ and $\tilde{L}_v$ will not appear if the pencil is regular, i.e. $\texttt{det}(sB - A) \neq 0$.
- For regular pencils:

$$P(\lambda B - A)Q = \texttt{block diag}\,(\lambda N - I, \lambda I - J)$$

- For pencils with only finite eigenvalues (B is invertible):

$$P(\lambda B - A)Q = (\lambda I - J)\,.$$

Guess what will be PB^{-1}?

Some features of Kronecker canonical form

- $P(\lambda B - A)Q =$ $\texttt{block diag}\left(L_{\mu_1}, \cdots, L_{\mu_k}, \tilde{L}_{v_1}, \cdots, \tilde{L}_{\mu_\ell}, \lambda N - I, \lambda I - J\right).$
- Matrices L_μ and $\tilde{L}_v$ will not appear if the pencil is regular, i.e. $\texttt{det}(sB - A) \neq 0$.
- For regular pencils:

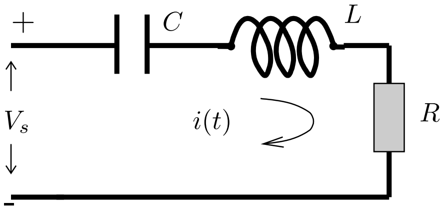
$$P(\lambda B - A)Q = \texttt{block diag}\,(\lambda N - I, \lambda I - J)$$

- For pencils with only finite eigenvalues (B is invertible):

$$P(\lambda B - A)Q = (\lambda I - J)\,.$$

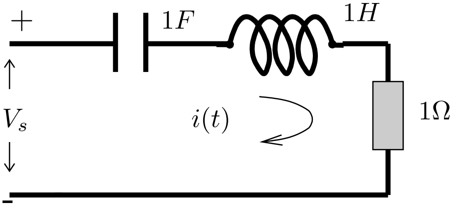
Guess what will be PB^{-1}?

Application



$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i(t) \\ v_L(t) \\ v_C(t) \\ v_R(t) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{C} & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} i(t) \\ v_L(t) \\ v_C(t) \\ v_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_s(t)$$

Application: RLC network



$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt} \begin{bmatrix} i(t) \\ v_L(t) \\ v_C(t) \\ v_R(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} i(t) \\ v_L(t) \\ v_C(t) \\ v_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_s(t)$$

Application: Descriptor system

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}\frac{d}{dt}\begin{bmatrix}i(t)\\v_L(t)\\v_C(t)\\v_R(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\-1 & 0 & 0 & 1\\0 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}i(t)\\v_L(t)\\v_C(t)\\v_R(t)\end{bmatrix} + \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}V_s(t)$$

Use Scilab: $[P, Q, i] = \texttt{pencan}(E, A)$.

$$P(\lambda E - A)Q = \lambda\begin{bmatrix}I & \\ & N\end{bmatrix} - \begin{bmatrix}J & \\ & I\end{bmatrix}$$

$$\Rightarrow \lambda E - A = \lambda P^{-1}\begin{bmatrix}I & \\ & N\end{bmatrix}Q^{-1} - P^{-1}\begin{bmatrix}J & \\ & I\end{bmatrix}Q^{-1}$$

Application: Descriptor system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt}x(t) = \begin{bmatrix} -0.692 & 0.647 & 0 & 0 \\ -1.215 & -0.308 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.984 \\ -0.148 \\ -0.669 \\ -0.669 \end{bmatrix} V_s(t)$$

You can easily compute the solutions to the differential equation.

Try Scilab: $[P, Q] = \texttt{kroneck}(E, A)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.816 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.057 \end{bmatrix} \dot{x}(t) = \frac{1}{\sqrt{6}} \begin{bmatrix} 2\sqrt{3} & \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \end{bmatrix} x(t) + \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{6} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{bmatrix} V_s(t)$$

Application: Descriptor system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} x(t) = \begin{bmatrix} -0.692 & 0.647 & 0 & 0 \\ -1.215 & -0.308 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.984 \\ -0.148 \\ -0.669 \\ -0.669 \end{bmatrix} V_s(t)$$

You can easily compute the solutions to the differential equation.

Try Scilab: $[P, Q] = \texttt{kroneck}(E, A)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.816 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.057 \end{bmatrix} \dot{x}(t) = \frac{1}{\sqrt{6}} \begin{bmatrix} 2\sqrt{3} & \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \end{bmatrix} x(t) + \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{6} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{bmatrix} V_s(t)$$

Conclusion

Properties	**Standard Eigenvalue Problem**	**Generalized Eigenvalue Problem**
Eigenvalues	λ	Pairs (α, β)
Number of eigenvalues	$= \mathtt{n}$	$\leqslant \mathtt{n}$
Geometry	Invariant subspaces	Deflating subspaces
Canonical Forms	Jordan	Kronecker canonical form

- Does the generalization end here?

  $(\lambda^2 A_2 + \lambda A_1 + A_0)x = 0$ or $(\sum_{i=0}^{k} \lambda^i A_i)x = 0$ or $R(\lambda)x = 0$.

  Canonical form: Smith canonical form.

Conclusion

Properties	Standard Eigenvalue Problem	Generalized Eigenvalue Problem
Eigenvalues	λ	Pairs (α, β)
Number of eigenvalues	$= \mathtt{n}$	$\leqslant \mathtt{n}$
Geometry	Invariant subspaces	Deflating subspaces
Canonical Forms	Jordan	Kronecker canonical form

- Does the generalization end here?
  $(\lambda^2 A_2 + \lambda A_1 + A_0)x = 0$ or $(\sum_{i=0}^{k} \lambda^i A_i)x = 0$ or $R(\lambda)x = 0$.
  Canonical form: Smith canonical form.

Conclusion

Properties	**Standard Eigenvalue Problem**	**Generalized Eigenvalue Problem**
Eigenvalues	λ	Pairs (α, β)
Number of eigenvalues	$= \mathtt{n}$	$\leqslant \mathtt{n}$
Geometry	Invariant subspaces	Deflating subspaces
Canonical Forms	Jordan	Kronecker canonical form

- Does the generalization end here?
  $(\lambda^2 A_2 + \lambda A_1 + A_0)x = 0$ or $(\sum_{i=0}^{k} \lambda^i A_i)x = 0$ or $R(\lambda)x = 0$.
  Canonical form: Smith canonical form.

Thank You