On circulant Lyapunov operators, two-variable polynomials, and DFT

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Lyapunov equations and its applications

- System: $\frac{d}{dt}x = Ax$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.
- Lyapunov equation: for $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

$$A^T K + KA + Q = 0$$

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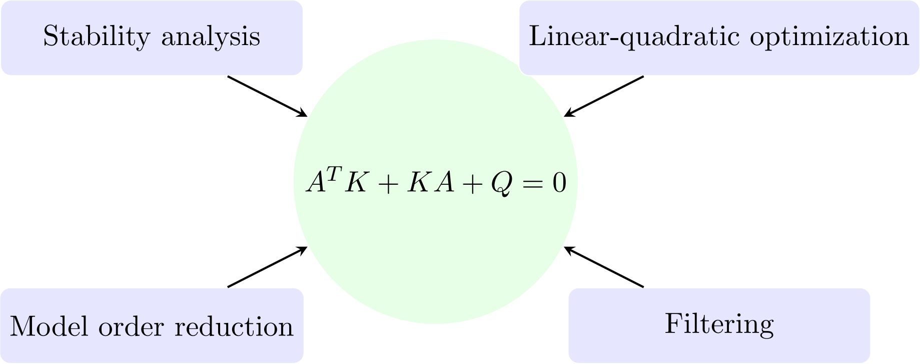

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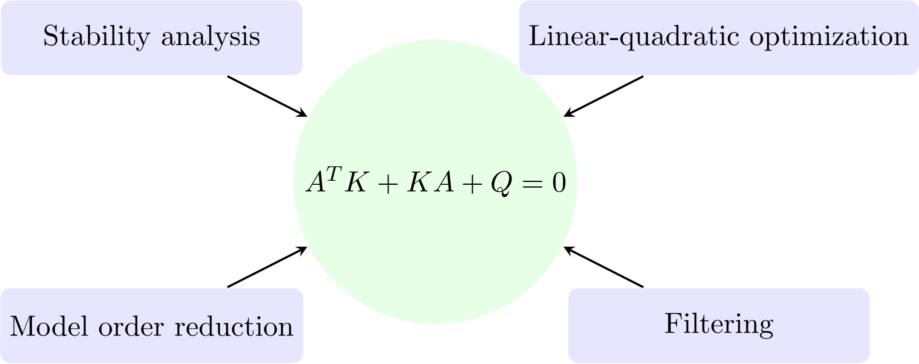

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Circulant Lyapunov operators

- $\mathcal{L}_A(P) := AP + PA^T$ is called *circulant* if A is a circulant matrix.

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{\mathtt{n}-2} & a_{\mathtt{n}-1} \\ a_{\mathtt{n}-1} & a_0 & \cdots & a_{\mathtt{n}-3} & a_{\mathtt{n}-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{\mathtt{n}-1} & a_0 \end{bmatrix} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$$

- We exploit the structure of A to design the proposed algorithm.
- Basis for the space of circulant matrices: $\{I, E, E^2, \cdots, E^{\mathtt{n}-1}\}$ with

$$E := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A = a_0 I + a_1 E + \cdots + a_{\mathtt{n}-1} E^{\mathtt{n}-1} = \sum_{k=0}^{\mathtt{n}-1} a_k E^k.$$

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Two-variable polynomials & circulant Lyap. operator

- $\mathscr{L}_A(K) = A^T K + KA$, where $A := \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k E^k$.
- Define $v(x,y) := \mathbf{X}^T V \mathbf{Y}$, where $V \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$.
- Consider the map

$$\Pi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y]/\mathbb{A},$$

where $\mathbb{A} := \langle x^{\mathtt{n}} - 1, y^{\mathtt{n}} - 1 \rangle \subset \mathbb{C}[x,y]$.

- Then the following are equivalent
  (1) $\mathscr{L}_A(V) = \gamma V$.
  (2) $\Pi\Big(\sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k) v(x,y)\Big) = \gamma v(x,y)$.

$\mathbf{X} := \mathtt{col}\left(1, x, x^2, \ldots, x^{\mathtt{n}-1}\right)$, $\mathbf{Y} := \mathtt{col}\left(1, y, y^2, \ldots, y^{\mathtt{n}-1}\right)$.

Polynomial interpretation of eigenmatrix V

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Two-variable polynomials & circulant Lyap. operator

- Polynomial ring $\mathbb{C}[x,y]$ and the ideal $\mathbb{A} := \langle x^{\mathtt{n}} - 1, y^{\mathtt{n}} - 1 \rangle$.
- Define the map $\Pi : \mathbb{C}[x,y] \to \mathbb{C}[x,y]/\mathbb{A}$.
- Given $G, R \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ define $g(x,y) := \mathbf{X}^T G \mathbf{Y}$, $r(x,y) := \mathbf{X}^T R \mathbf{Y}$.
- Suppose $g(x,y), r(x,y) \in \mathbb{C}[x,y]$ satisfy

$$\Pi\Big(\sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k) g(x,y)\Big) = r(x,y),$$

  where $k \in \mathbb{N}$ and $k < \mathtt{n}$.
- Suppose $\sum_{k=0}^{\mathtt{n}-1} 2\mathtt{a}_k \cos\left(\frac{\pi(m-n)k}{\mathtt{n}}\right) \omega^{\frac{(m+n)k}{2}} \neq 0$ for every $m, n = 0, 1, 2, \ldots, \mathtt{n}-1$.
- Then, $g(x,y)$ is unique.

Two-variable polynomials & circulant Lyap. operator

- $\mathscr{L}_A(K) = A^T K + KA$, where $A := \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k E^k$.
- Polynomial ring $\mathbb{C}[x,y]$ and the ideal $\mathbb{A} := \langle x^{\mathtt{n}} - 1, y^{\mathtt{n}} - 1 \rangle$.
- Define the map $\Pi : \mathbb{C}[x,y] \to \mathbb{C}[x,y]/\mathbb{A}$.
- Let $g(x,y), r(x,y) \in \mathbb{C}[x,y]$ be such that

$$\Pi\Big(\sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k) g(x,y) \Big) = r(x,y).$$

- Then $\mathscr{L}_A(\bullet)$ is nonsingular if and only if $g(x,y)$ is unique.

2D-DFT & circulant Lyap. operator

- Polynomial ring $\mathbb{C}[x,y]$ and the ideal $\mathbb{A} := \langle x^{\mathtt{n}} - 1, y^{\mathtt{n}} - 1 \rangle$.
- Define the map $\Pi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y]/\mathbb{A}$.
- Let $P, Q, R \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ be such that

$$p(x,y) := \mathbf{X}^T P \mathbf{Y}, \;\; q(x,y) := \mathbf{X}^T Q \mathbf{Y}, \;\; r(x,y) := \mathbf{X}^T R \mathbf{Y}.$$

- Let $\mathscr{F}(P), \mathscr{F}(Q)$ and $\mathscr{F}(R)$ be the 2D-DFT matrices of P, Q and R, respectively.
- Then the following is true:

$$\Pi\Big(p(x,y)q(x,y)\Big) = r(x,y) \text{ if and only if } \mathscr{F}(P) \odot \mathscr{F}(Q) = \mathscr{F}(R).$$

$\odot$ means Elementwise multiplication.

2D-DFT & circulant Lyap. operator

- Lyapunov equation $AP + PA^T = Q$ with $A := \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k E^k$.
- Define $J \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $\mathbf{X}^T J \mathbf{Y} = \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k)$.
- Then, the following is true:

$$\Pi\Big(\sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k) p(x,y)\Big) = q(x,y) \Leftrightarrow \mathscr{F}(J) \odot \mathscr{F}(P) = \mathscr{F}(Q).$$

- P in the Lyapunov equation can be computed using:

$$P = \mathscr{F}^{-1}\left(\mathscr{F}(Q) \oslash \mathscr{F}(J)\right),$$

where $\mathscr{F}^{-1}$ is inverse DFT operation and $\oslash$ means element-wise division.

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What if elements have zero?

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When does $\mathscr{F}(J)$ has every element nonzero?

- Circulant Lyapunov operator $\mathscr{L}_A(\bullet)$, where $A := \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k E^k$
- Define $J \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $\mathbf{X}^T J \mathbf{Y} := \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k)$.
- Λ be the set of eigenvalues of $\mathscr{L}_A(\bullet)$.
- Γ be the set of elements of $\mathscr{F}(J)$.
- Then, $\Lambda = \Gamma$.
- Further, the following are equivalent:
  1. $\mathscr{F}(J)$ has every element nonzero.
  2. $\mathscr{L}_A(\bullet)$ is nonsingular.
  3. For each $m, n = 0, 1, 2, \ldots, \mathtt{n}-1$

$$\sum_{k=0}^{\mathtt{n}-1} 2\mathtt{a}_k \cos\left(\frac{\pi(m-n)k}{\mathtt{n}}\right) \omega^{\frac{(m+n)k}{2}} \neq 0.$$

For $A = a_1 E$: $\mathtt{n}$ is odd

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2D-DFT, Two-variable polynomials and Lyap. operator

- Lyapunov operator $\mathscr{L}_A(\bullet)$ with $A := \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k E^k$
- $J \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ be such that $\sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k) =: \mathbf{X}^T J \mathbf{Y}$.
- $\mathbb{A} := \langle x^{\mathtt{n}} - 1, y^{\mathtt{n}} - 1 \rangle \subset \mathbb{C}[x, y]$.
- Assume $g(x, y), r(x, y) \in \mathbb{C}[x, y]$ such that

$$\Pi\Big(\sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k (x^k + y^k) g(x, y)\Big) = r(x, y).$$

Then the following are equivalent:

(a) $\mathscr{L}_A(\bullet)$ is nonsingular.
(b) $g(x, y)$ is unique.
(c) $\mathscr{F}(J)$ has every element nonzero.

Relation among circulant Lyapunov operator, two-variable polynomials and 2D-DFT established.

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Algorithm A 2D-DFT based algorithm to solve circulant Lyapunov equation i.e. $\mathscr{L}_A(P) = Q$.

Input: $\mathtt{a}_0, \mathtt{a}_1, \ldots, \mathtt{a}_{\mathtt{n}-1} \in \mathbb{R}$, $Q \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ and $A = \sum_{k=0}^{\mathtt{n}-1} \mathtt{a}_k E^k$.
Output: Solution $P \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ of $AP + PA^T = Q$.

Construct $v := \begin{bmatrix} \mathtt{a}_1 & \mathtt{a}_2 & \cdots & \mathtt{a}_{\mathtt{n}-1} \end{bmatrix}$ and $J := \begin{bmatrix} 2\mathtt{a}_0 & v \\ v^T & \mathbf{0} \end{bmatrix}$.
Compute $\mathscr{F}(Q) =: [\alpha_{mn}]$ and $\mathscr{F}(J) =: [\beta_{mn}]$, respectively.
for $m = 1 : \mathtt{n}$ **do**
    for $n = 1 : \mathtt{n}$ **do**
        if $\beta_{mn} \neq 0$ **then**
            $\kappa_{mn} = \alpha_{mn} / \beta_{mn}$.
        else
            $\kappa_{mn} = \tau$, where $\tau \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$.
        end if
    end for
end for
$\mathscr{F}(P) = [\kappa_{mn}]$ and $P = \mathscr{F}^{-1}\Big(\mathscr{F}(P)\Big)$.

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Lyap. equation has unique solution

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Output: Solution $P \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ of $AP + PA^T = Q$.

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Construct v := [a_1  a_2  ···  a_{n-1}] and J := [2a_0  v ; v^T  0].
Compute F(Q) =: [α_mn] and F(J) =: [β_mn], respectively.
for m = 1 : n do
    for n = 1 : n do
        if β_mn ≠ 0 then
            κ_mn = α_mn/β_mn.
        else
            κ_mn = τ, where τ ∈ C^{n×n}.
        end if
    end for
end for
F(P) = [κ_mn] and P = F^{-1}(F(P)).
```

In math notation: Construct $v := \begin{bmatrix} \mathtt{a}_1 & \mathtt{a}_2 & \cdots & \mathtt{a}_{\mathtt{n}-1} \end{bmatrix}$ and $J := \begin{bmatrix} 2\mathtt{a}_0 & v \\ v^T & \mathbf{0} \end{bmatrix}$. Compute $\mathscr{F}(Q) =: [\alpha_{mn}]$ and $\mathscr{F}(J) =: [\beta_{mn}]$; $\kappa_{mn} = \alpha_{mn}/\beta_{mn}$; $\kappa_{mn} = \tau$, where $\tau \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$; $\mathscr{F}(P) = [\kappa_{mn}]$ and $P = \mathscr{F}^{-1}\Big(\mathscr{F}(P)\Big)$.

Lyap. equation has infinitely many solutions

Examples

- Lyapunov equation $A^T P + PA + Q = 0$ with

$$A := \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = 2 \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Q := \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}, J = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- 2D-DFT matrices:

$$\mathscr{F}(J) = \begin{bmatrix} 4 & 1-j\sqrt{3} & 1+j\sqrt{3} \\ 1-j\sqrt{3} & -2-j2\sqrt{3} & -2 \\ 1+j\sqrt{3} & -2 & -2+j2\sqrt{3} \end{bmatrix}, \ \mathscr{F}(Q) = \frac{1}{2} \begin{bmatrix} 34 & -5+j3\sqrt{3} & -5-j3\sqrt{3} \\ 7+j3\sqrt{3} & -2 & 10-j6\sqrt{3} \\ 7-j3\sqrt{3} & 10+j6\sqrt{3} & -2 \end{bmatrix}.$$

- One can verify that

$$P = \mathscr{F}^{-1}\left(\mathscr{F}(Q) \oslash \mathscr{F}(J)\right) = \frac{1}{4} \begin{bmatrix} -2 & 6 & 1 \\ -2 & -1 & 4 \\ 5 & 4 & 2 \end{bmatrix}.$$

Lyap. equation has unique solution

Examples

- Lyapunov equation $A^T P + PA + Q = 0$ with

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, Q := \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, J = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

- 2D-DFT matrices:

$$\mathscr{F}(J) = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}, \mathscr{F}(Q) = \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}.$$

- Define $\mathscr{F}(P) := \begin{bmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{bmatrix}$.
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κ_2, κ_3 free

- Using the inverse 2D-DFT:

Infinitely many solutions

$$P = \mathscr{F}^{-1}\Big(\mathscr{F}(P)\Big) = \frac{1}{4}\begin{bmatrix} 1 + (\kappa_2 + \kappa_3) & 3 + (\kappa_2 - \kappa_3) \\ 3 - (\kappa_2 - \kappa_3) & 1 - (\kappa_2 + \kappa_3) \end{bmatrix}.$$

Examples

- Lyapunov equation $A^T P + PA + Q = 0$ with

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Conclusion

- Under a suitable projection map Π, two-variable polynomials are related to circulant Lyapunov operators.
- Nonsingularity of Lyapunov operators is a necessary and sufficient condition for every element of the 2D-DFT matrix corresponding to a suitably constructed matrix J to be nonzero.
- Devised an algorithm to solve circulant Lyapunov equations.

Thank you.

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