A 2D-DFT based method to compute the Bezoutian and a link to Lyapunov equations

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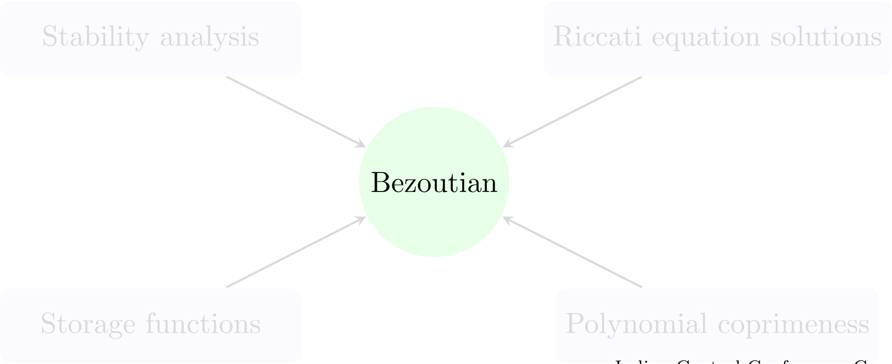
Introduction

p, q are polynomials:

$$\frac{p(x)q(y) + p(y)q(x)}{x + y} \qquad\qquad \frac{p(x)q(y) - p(y)q(x)}{x - y}$$



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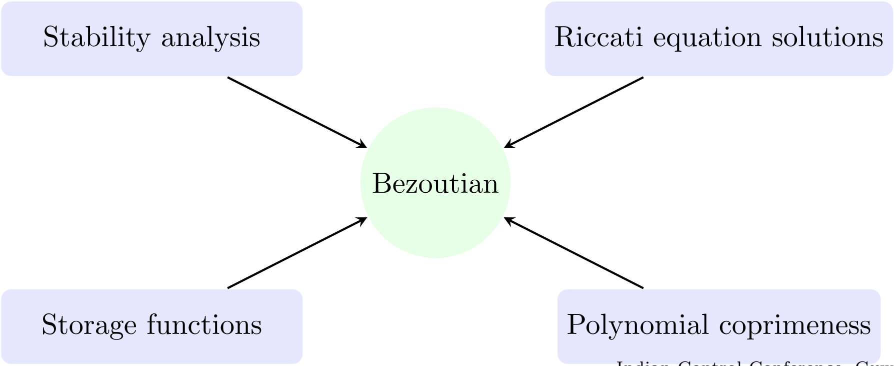

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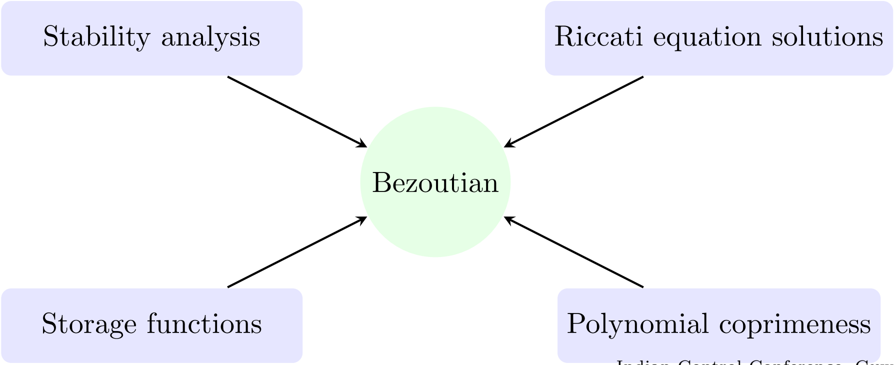

Bezoutian $b(x, y)$

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Objective and Outline

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OBJECTIVE: To compute the Bezoutian $b(x, y)$.

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1. 2D-DFT based method to compute Bezoutian.
2. Bezoutian and link to Lyapunov equation.
3. Lyapunov equation and its link to two variable polynomials.

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The Algorithm

$$\text{Let } \mathbf{X} := \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{\mathtt{N}-1} \end{bmatrix}, \mathbf{Y} := \begin{bmatrix} 1 \\ y \\ y^2 \\ \vdots \\ y^{\mathtt{N}-1} \end{bmatrix}$$

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$$b(x,y) = \mathbf{X}^T \begin{bmatrix} \tilde{B} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Y} =: \mathbf{X}^T B \mathbf{Y},\ \tilde{B} \in \mathbb{R}^{(\mathtt{N}-1)\times(\mathtt{N}-1)}$$

The Algorithm

$$(x+y)b(x,y) = \phi(x,y) \implies \left(\mathbf{X}^T R Y\right)\left(\mathbf{X}^T B \mathbf{Y}\right) = \left(\mathbf{X}^T \Phi Y\right)$$

- # Objective: Find B.
- # Different problems have different coefficient matrix Φ.
- # R remains fixed.

- Two variable polynomial multiplication $\Leftrightarrow$ 2D-Convolution.

$$R \star B = \Phi$$

where $\star$ means 2D-convolution.

- 2D-convolution $\Leftrightarrow$ Elementwise multiplication

$$\mathscr{F}(R) \otimes \mathscr{F}(B) = \mathscr{F}(\Phi)$$

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$$B = \mathscr{F}^{-1}\Big[\mathscr{F}(\Phi)./\mathscr{F}(R)\Big].$$

- The algorithm fails when any element of $\mathscr{F}(R)$ is zero.
- $(k,\ell)^{th}$ element of $\mathscr{F}(R)$ is $e^{-j\frac{2\pi}{\mathtt{N}}k} + e^{-j\frac{2\pi}{\mathtt{N}}\ell}$ i.e. pairwise sum of roots of unity.

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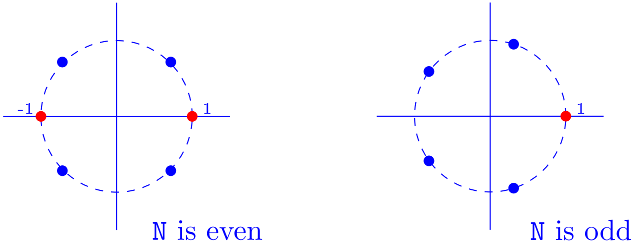

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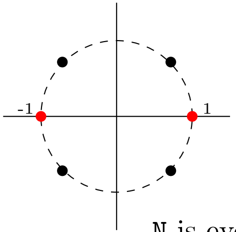

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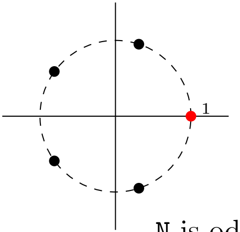

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Bezoutian and Lyapunov operators

- $b(x,y) = \mathbf{X}^T B \mathbf{Y}$ and $\phi(x,y) = \mathbf{X}^T \Phi \mathbf{Y}$

$$(x+y)b(x,y) = \phi(x,y)$$

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- Then,

$$\boxed{EB + BE^T = \Phi}$$

$$xb(x,y) + yb(x,y) = \mathbf{X}^T EB\mathbf{Y} + \mathbf{X}^T BE^T \mathbf{Y}.$$

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Relation between 2D-DFT and Lyapunov equation?

Lyapunov operator

- $\mathscr{L}_A(P) = AP + PA^T$ where $A \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}, P \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$.
- Eigenvalues of A be $\{\lambda_0, \lambda_1, \ldots, \lambda_{\mathtt{N}-1}\}$ corresponding to eigenvectors $\{v_0, v_1, \ldots, v_{\mathtt{N}-1}\}$.
- $\mathscr{L}_A(v_i v_j^*) = (\lambda_i + \lambda_j^*) v_i v_j^*$ (compare with $Ax = \lambda x$).

  Eigenmatrix of $\mathscr{L}_A(\bullet)$ is $v_i v_j^*$ with respect to eigenvalue $(\lambda_i + \lambda_j^*)$.
- For us, $A = E$ i.e. $\mathscr{L}_E(P) = EP + PE^T$.
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Bezoutian and Lyapunov operators

- $(x+y) = \mathbf{X}^T R \mathbf{Y}$ where $R = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ and $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  Elements of $\mathscr{F}(R)$ are pairwise sum of roots of unity.

- $\mathscr{L}_E(P) := EP + PE^T$ where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix.

  Eigenvalues of $\mathscr{L}_E(\bullet)$ are also pairwise sum of roots of unity.

- Then,

  Elements of $\mathscr{F}(R) \equiv$ corresponding eigenvalues of $\mathscr{L}_E(\bullet)$.

Is there a link between Lyapunov operators and two variable polynomial multiplication in general?

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Two variable polynomials and Lyapunov operator

- $\mathscr{L}_E(P) := EP + PE^T$ where E is unit cyclic matrix.
- Let $V \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ and $v(x,y) := \mathbf{X}^T V \mathbf{Y}$.
- Canonical surjection map

$$\Pi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y]/\mathbb{A} \text{ where } \mathbb{A} := \langle x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1 \rangle.$$

- Then,

$$\mathscr{L}_E(V) = \mu V \iff \Pi\Big((x+y)v(x,y)\Big) = \mu v(x,y).$$

$\mathbb{A}$ is the ideal generated by $(x^{\mathtt{N}} - 1), (y^{\mathtt{N}} - 1)$
i.e. the set of all polynomials of the form $e(x,y)(x^{\mathtt{N}} - 1) + f(x,y)(y^{\mathtt{N}} - 1)$.

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Example

- Consider $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$.
- Consider $V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\mathscr{L}_E(V) = 2V$.
- $v(x,y) = \mathbf{X}^T V \mathbf{Y} = 1 + x + y + xy$.
- $(x+y)v(x,y)/\langle x^2-1, y^2-1\rangle$

$$= (x^2 + x^2y + xy^2 + 2xy + x + y^2 + y)/\langle x^2-1, y^2-1\rangle$$
$$= 2 + 2x + 2y + 2xy$$
$$= 2v(x,y)$$

Example

- Consider $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$.

- Consider $V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\mathscr{L}_E(V) = 2V$.

- $v(x,y) = \mathbf{X}^T V \mathbf{Y} = 1 + x + y + xy.$

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Two variable polynomials and Lyapunov operator

$$\mathscr{L}_E(V) = \mu V \quad \Longleftrightarrow \quad \Pi\Big((x+y)v(x,y)\Big) = \mu v(x,y).$$

Analogous result:

- $\ell_E(p) := Ep$ where $E \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is the unit cyclic matrix
- Let $p \in \mathbb{C}^{\mathtt{N}\times 1}$ and $v(x) = \mathbf{X}^T p$
- $\pi : \mathbb{C}[x] \longrightarrow \mathbb{R}[x]/\mathfrak{a}$ where $\mathfrak{a} := \langle x^{\mathtt{N}} - 1\rangle$.

$$\ell_E(q) = \lambda q \quad \Longleftrightarrow \quad \pi\Big(xv(x)\Big) = \lambda^{(\mathtt{N}-1)} v(x).$$

Conclusion

1. Reported a method to compute Bezoutian using 2D-DFT.
2. Link between Bezoutian and Lyapunov operator.
3. Two variable interpretation of Lyapunov operator.

THANK YOU

Queries?

2D-Convolution: example

$$(1+x+y)(2+xy) = \left(\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} \right) \left(\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} \right)$$
$$= 2 + 2x + 2y + xy + x^2y + xy^2$$

We compute: 2D-convolution of coefficient matrices,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \star \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

2D-convolution formula

$$y[m,n] = \sum_{j=0}^{\mathtt{N}} \sum_{i=0}^{\mathtt{N}} x[i,j]h[m-i,n-j]$$

V.Y. Pan, *Structured Matrices and Polynomials: Unified Superfast Algorithms*, Birkhäuser, 2001.

2D-Convolution: example

$$(1+x+y)(2+xy) = \left(\begin{bmatrix}1\\x\end{bmatrix}^T \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\y\end{bmatrix}\right)\left(\begin{bmatrix}1\\x\end{bmatrix}^T \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\y\end{bmatrix}\right)$$
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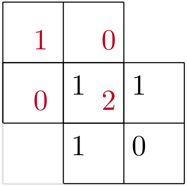
2D-Convolution

Example: $(1+x+y)(2+xy) = 2 + 2x + 2y + xy + x^2y + xy^2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \star \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 2 & & \\ & & \\ & & \end{bmatrix}$$

2D-Convolution

Example: $(1 + x + y)(2 + xy) = 2 + 2x + 2y + xy + x^2y + xy^2$

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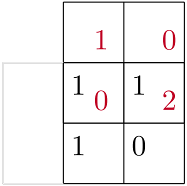

$$\begin{bmatrix} 2 & 2 & \\ & & \\ & & \end{bmatrix}$$

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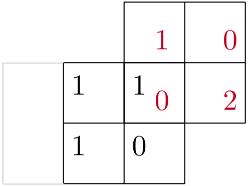

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$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \star \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

1	1 0	1
0	1 2	0

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & & \\ & & \end{bmatrix}$$

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	1 1	1 0
	1 0	0 2

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & \\ & & \end{bmatrix}$$

2D-Convolution

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	1	1 1	0
	1	0 0	2

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ & & \end{bmatrix}$$

2D-Convolution

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	1	1
1	1 0	0
0	2	

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & & \end{bmatrix}$$

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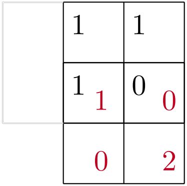

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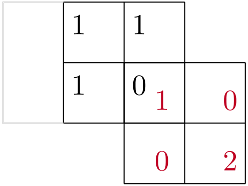

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$