On solutions of bounded-real LMI for singularly bounded-real systems

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- System: controllable and observable

$$\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du,$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

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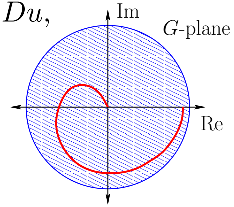

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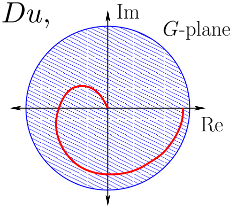

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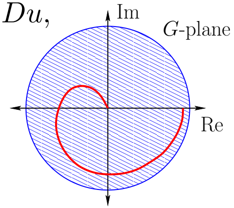

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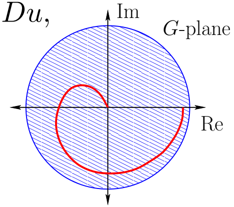

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- $\mathcal{H}_\infty$ synthesis problem, $\mathcal{H}_2$ synthesis problem, design of filters, etc.
- LMI solved using: LMI solvers (iterative), ARE solvers.

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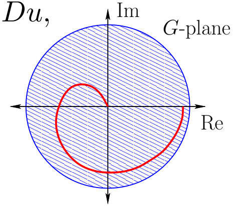

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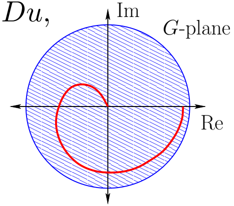

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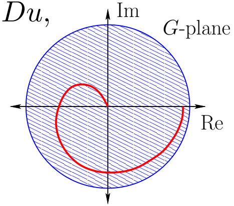

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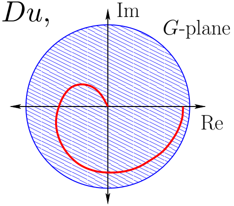

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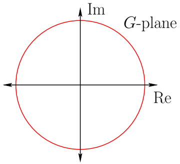

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- Will the algorithm used to find ARE solution work?
- Let's review the algorithm [P. van Dooren, SSC 1981].

- Hamiltonian matrix pair

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- Partition $\sigma(E,H)$ in two disjoint sets based on certain rules. Each of these sets are called Lambda-set of (E,H).

- Symbol for a Lambda-set: Λ.

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- $\mathtt{n}$ eigenvectors corresponding to the elements of Λ: $V_1, V_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $V_3 \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$

$$\begin{bmatrix} A & 0 & BD^T \\ -C^TC & -A^T & -C^T \\ -C & -DB^T & I_{\mathtt{p}} - DD^T \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \Gamma,$$

where $\sigma(\Gamma) = \Lambda$.

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- $\mathtt{n}$-dimensional eigenspaces corresponding to Λ

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1. V_1 is invertible.
2. $K := V_2V_1^{-1}$ is symmetric.
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Classification of bounded-real systems based on

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Bounded-real systems

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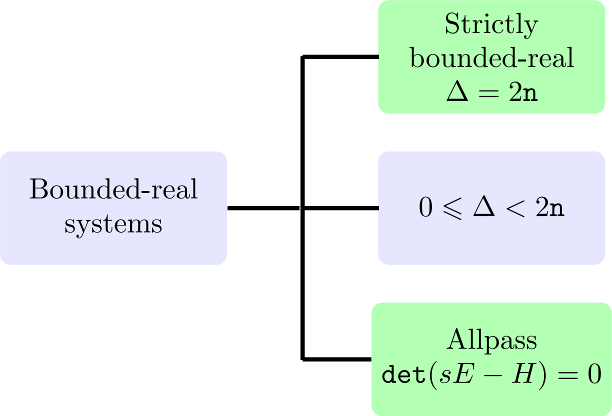

Algorithm exists: For $\Delta = 2\texttt{n}$ with $\sigma(E, H) \cap j\mathbb{R} = \emptyset$.

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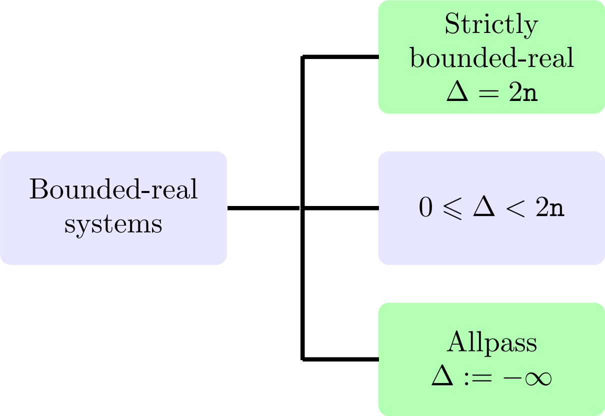

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$\Delta = -\infty$ [Bhawal et.al. TCAS 2018].

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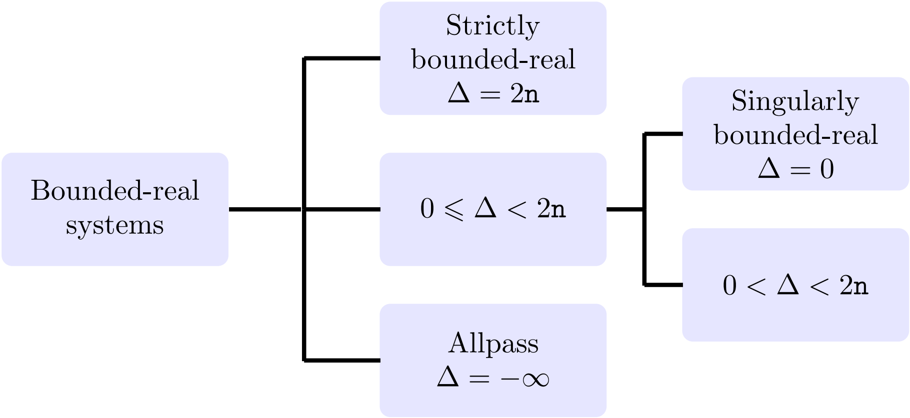

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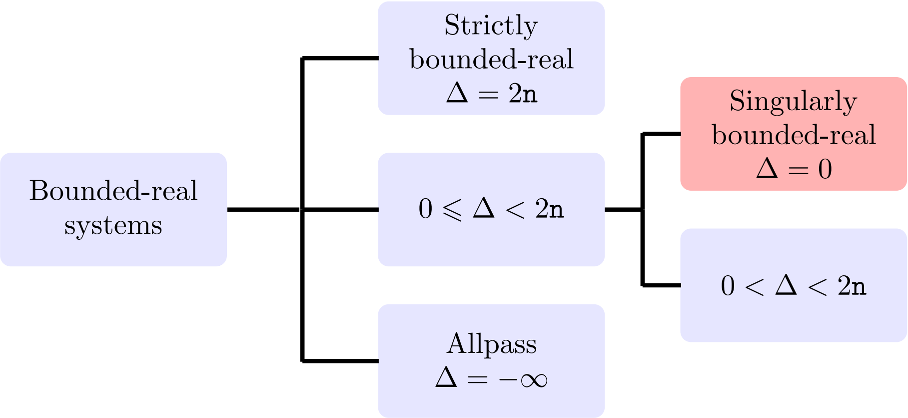

Singularly bounded-real systems: Bounded-real systems with $\Delta = 0$.

Strictly bounded-real system versus Singularly bounded-real system:

Properties	**Strictly bounded-real**	**Singularly bounded-real**
ARE	Admits	Does not admit
Lambda-set	Exists with cardinality $\mathtt{n}$	Does not exist $(\mathtt{deg}\,\mathtt{det}(sE - H) = 0)$
Solutions to bounded-real LMI	Multiple solutions	Unique solution

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Problem statement

Find an algorithm to compute the unique solution of bounded-real LMI for singularly bounded-real systems.

Known result (Doo'81)

- *Hamiltonian matrix pair (Assumption: $\sigma(E,\mathcal{H})\cap j\mathbb{R}=\emptyset$)*

$$E=\begin{bmatrix} I_{\mathtt{n}} & 0 & 0\\ 0 & I_{\mathtt{n}} & 0\\ 0 & 0 & 0\end{bmatrix},\ H=\begin{bmatrix} A & 0 & BD^T\\ -C^TC & -A^T & -C^T\\ -C & -DB^T & I_{\mathtt{p}}-DD^T\end{bmatrix}\in\mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}.$$

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Theorem

- *Hamiltonian matrix pair*

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- *Define* $\widehat{A} := \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix}$ *and* $\widehat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}$.

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- $W := \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \cdots \widehat{A}^{\mathtt{n}-1}\widehat{B} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$.

Theorem

- *Hamiltonian matrix pair*

$$E = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} A & 0 & B \\ -C^TC & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}.$$

- *Define* $\widehat{A} := \begin{bmatrix} A & 0 \\ -C^TC & -A^T \end{bmatrix}$ *and* $\widehat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}$.
- $W := \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \cdots \widehat{A}^{\mathtt{n}-1}\widehat{B} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$.
- *Define* $W =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, *where* $X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

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Then, the following statements hold.

1. X_1 *is invertible.*
2. $K := X_2 X_1^{-1}$ *is symmetric.*
3. $KB + C^T = 0$ *and* $A^T K + KA + C^T C \leqslant 0$.

- Singularly bounded real system: $(\texttt{det}(sE - H) = 30)$.

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.5 & -6 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \;\; y = -\begin{bmatrix} 5 & 4 & 2 \end{bmatrix} x + u.$$

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- $K = X_2 X_1^{-1} = \begin{bmatrix} 32 & 16 & 5 \\ 16 & 11 & 4 \\ 5 & 4 & 2 \end{bmatrix}.$

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- $A^T K + KA + C^T C = \texttt{diag}(-30, 0, 0) \leqslant 0$ and $KB + C^T = 0$.
- K satisfies $\begin{bmatrix} A^T K + KA + C^T C & KB + C^T \\ B^T K + C & 0 \end{bmatrix} \leqslant 0.$

- Hamiltonian system

$$\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix}.$$

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- Output nulling representation:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C^TC & -A^T \end{bmatrix}}_{\widehat{A}} \begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ -C^T \end{bmatrix}}_{\widehat{B}} u, \quad 0 = \underbrace{\begin{bmatrix} -C & -B^T \end{bmatrix}}_{\widehat{C}} \begin{bmatrix} x \\ z \end{bmatrix}.$$

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- $\mathtt{deg}\,\mathtt{det}(sE - H) = 0 \Rightarrow \mathtt{num}(\widehat{C}(sI - \widehat{A})^{-1}\widehat{B}) \in \mathbb{R} \setminus 0.$

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- Relative degree $= 2\mathtt{n}$. The initial few Markov parameters are zero.

- Hamiltonian system

$$\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix}.$$

Lemma

- $\frac{d}{dt}x = Ax + Bu$, $y = Cx + Du$ *(singularly bounded-real).*
- *Define* $\widehat{A} := \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix}$ *and* $\widehat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}$.

Then,

$$\widehat{C}\widehat{A}^k\widehat{B} = 0 \;\; \textit{for} \;\; k \in \{0, 1, 2, \ldots, 2\mathtt{n} - 2\}.$$

- For allpass systems: bounded-real LMI becomes
$$\begin{bmatrix} A^TK+KA+C^TC & KB+C^T \\ B^TK+C & 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} A^TK+KA+C^TC=0 \\ KB+C^T=0 \end{cases}$$

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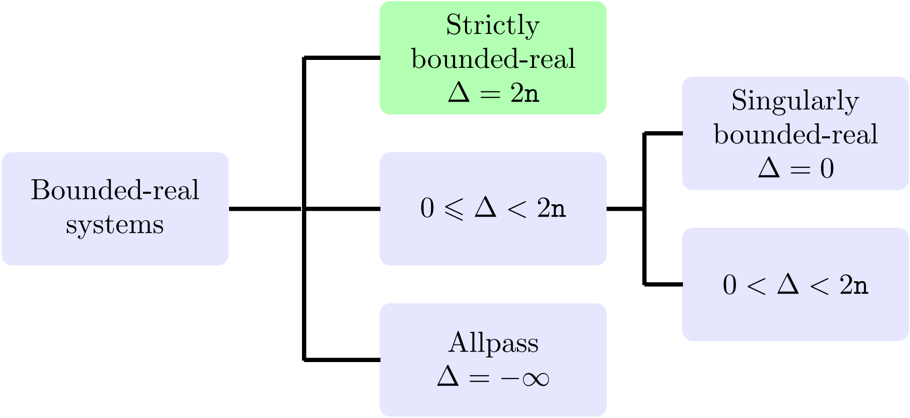
Corollary (Allpass systems)

- $\Sigma_{\texttt{all}}$: $\frac{d}{dt}x = Ax + Bu$ *and* $y = Cx + u$.
- *Define* $\widehat{A} = \begin{bmatrix} A & 0 \\ -C^TC & -A^T \end{bmatrix}$ *and* $\widehat{B} = \begin{bmatrix} B \\ -C^T \end{bmatrix}$.
- $W := \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \cdots \widehat{A}^{\mathtt{n}-1}\widehat{B} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}^{\mathtt{2n}\times\mathtt{n}}$.

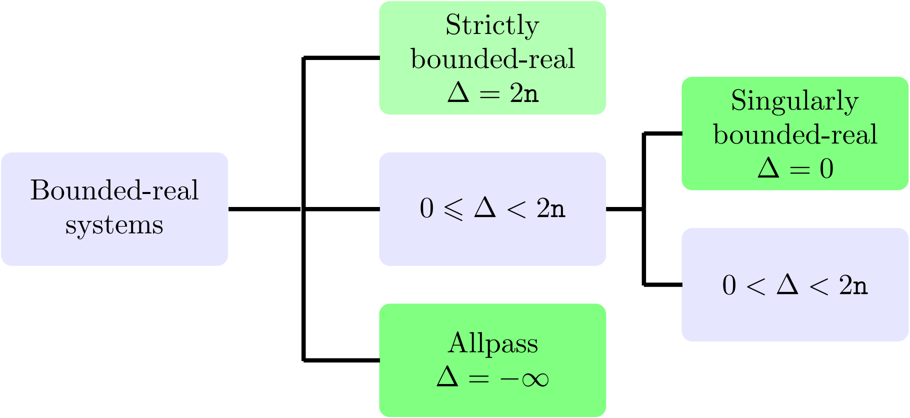
Then, the following statements hold.

1. X_1 *is invertible.*
2. $K := X_2X_1^{-1}$.
3. $KB + C^T = 0$ *and* $A^TK + KA + C^TC = 0$.

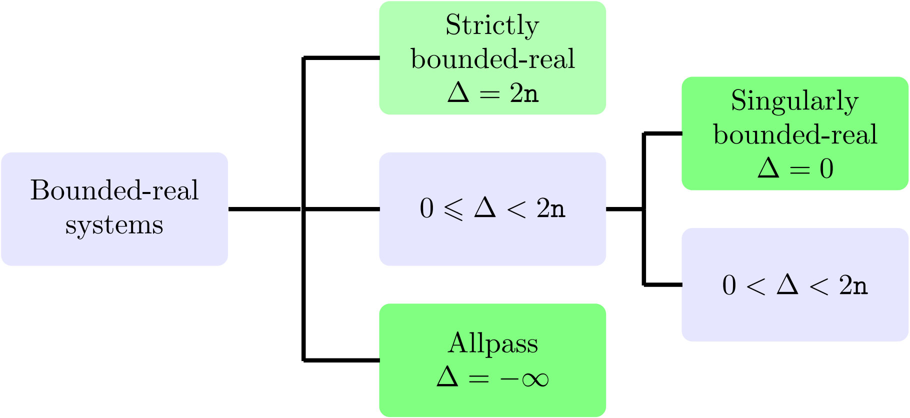
Reason: For allpass systems $\widehat{C}\widehat{A}^k\widehat{B} = 0$ for all $k \in \mathbb{N}$.



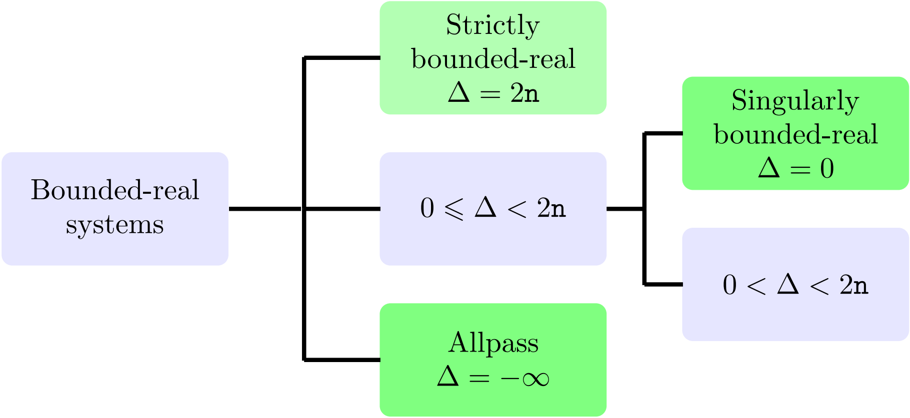
Algorithms already present for $\Delta = 2\mathtt{n}$.



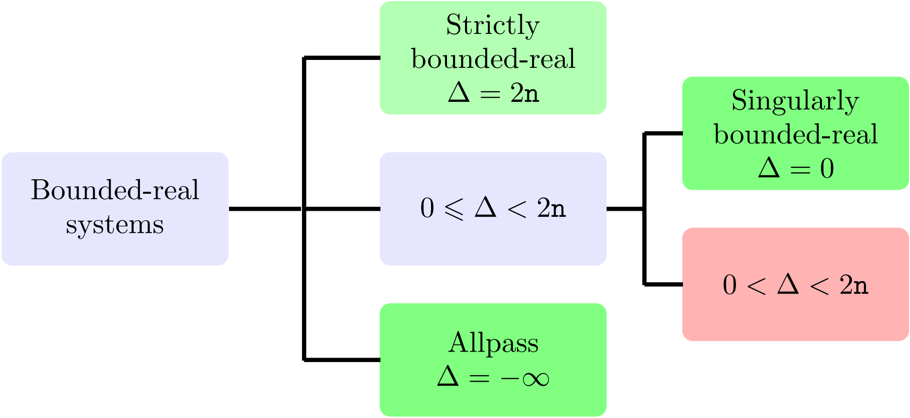
- Markov parameters of Hamiltonian system crucial.



- Markov parameters of Hamiltonian system crucial.
- Flop count $\mathcal{O}(\mathtt{n}^3)$: better than LMI solvers $\mathcal{O}(\mathtt{n}^{4.5})$.



- Markov parameters of Hamiltonian system crucial.
- Flop count $\mathcal{O}(\mathtt{n}^3)$: better than LMI solvers $\mathcal{O}(\mathtt{n}^{4.5})$.
- Algorithm works for LQR, passivity, as well.



- Algorithms required for $0 < \Delta < 2\mathtt{n}$. (Paper under review)

Thank you
Questions?