New results and algorithms for computing storage functions: the lossless/all-pass cases

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Dissipative systems and storage function

- A dissipative system
  1. has no source of energy.
  2. can only absorb energy.
  3. can store energy.

- $\underbrace{\text{Power supplied}}_{Q_\Sigma(w)} = \underbrace{\text{Rate-change-stored-energy}}_{\frac{d}{dt}Q_\Psi(w)} + \underbrace{\text{Dissipated power}}_{\Delta Q_\Sigma(w)}$

- Dissipative systems : $\frac{d}{dt}Q_\Psi(w) \leqslant Q_\Sigma(w)$.

- Power supplied with respect to system variable: $Q_\Sigma(w) = w^T \Sigma w$.

  e.g. Supply rate: $w = (u, y)$: $Q_\Sigma(w) = w^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} w = 2u^T y$.

- Stored energy: Storage function[1] $Q_\Psi(w) = x^T K x$.

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Lossless systems and algebraic Riccati equation (ARE)

- System: $\dot{x} = Ax + Bu \qquad y = Cx + Du.$
- ARE helps to calculate extremal storage functions ($x^T Kx$):

$$A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$$

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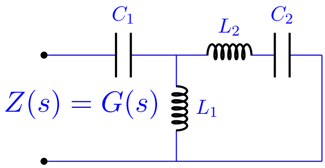

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System and its adjoint

Linear differential behavior $\mathfrak{B}$

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}.$$

System

Behavior $\mathfrak{B}$ :: states x

Minimal i/s/o representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Adjoint system

Behavior $\mathfrak{B}^{\perp_\Sigma}$ :: co-states z

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System: $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$

- The behavior $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ has first order representation

$$\underbrace{\left(\xi \underbrace{\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D + D^T \end{bmatrix}}_{H} \right)}_{\text{Hamiltonian pencil } R(\xi)} \begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0$$

- For lossless systems[2] $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \mathfrak{B}$.

  McMillan degree of $\mathfrak{B}$ = McMillan degree of $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$.

- x and z must have static relations between them.

  Use these static relations to find the storage function for $\mathfrak{B}$.

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Theorem

- Lossless behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$: (A, B, C, D) minimal realization.
- $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \ker R(\frac{d}{dt})$, $R(\xi)$: Hamiltonian pencil with $D + D^T = 0$.

Then, there exists a unique $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$\frac{d}{dt}x^T K x = 2u^T y \qquad \text{for all } \begin{bmatrix} u \\ y \end{bmatrix} \in \mathfrak{B}.$$

if and only if

$$\text{rank} \begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} = \text{rank } R(\xi).$$

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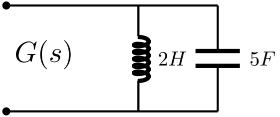
Algorithm and Example:

1. Lossless behavior $\mathfrak{B}$ with transfer function $G(s) = \frac{0.2s}{s^2+0.1}$.

$$\frac{d}{dt}\begin{bmatrix} i_L \\ v_C \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{5} & 0 \end{bmatrix}\begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} i \quad \text{and} \quad v = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} i_L \\ v_C \end{bmatrix} + 0\, i$$



2. Hamiltonian pencil:

$$R(\xi) = \left[\begin{array}{cc|cc|c} \xi & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{5} & \xi & 0 & 0 & -\frac{1}{5} \\ \hline 0 & 0 & \xi & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} & \xi & -1 \\ \hline 0 & -1 & 0 & \frac{1}{5} & 0 \end{array}\right].$$

Find MPB of $R(\xi)$: $R(\xi)M(\xi) = 0$

$$\therefore \begin{bmatrix} -K & I & 0 \end{bmatrix} \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix} = 0$$

Find MPB of left nullspace of $M_1(\xi) : N(\xi)M_1(\xi) = 0$

$$\therefore \begin{bmatrix} N_{11} & N_{21} \\ N_{12}(\xi) & N_{22}(\xi) \end{bmatrix} M_1(\xi) = 0$$

$K = -N_{21}^{-1}N_{11}$

Storage function $= x^T K x$

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$$M(\xi) = \begin{bmatrix} 1 \\ 2\xi \\ 2 \\ 10\xi \\ 1+10\xi^2 \end{bmatrix}$$

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$$N(\xi) = \left[\begin{array}{cc|cc} 2 & -5 & -1 & 1 \\ -50 & -20 & 25 & 4 \\ \hline 2\xi & 5 & -\xi & -1 \end{array}\right]$$

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$$K = -N_{21}^{-1}N_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Storage function $= 2i_L^2 + 5v_C^2$

Theorem: Bezoutian based method

- Controllable lossless behavior $\mathfrak{B}$:

$$w = M(\tfrac{d}{dt})\ell =: \begin{bmatrix} n(\frac{d}{dt}) \\ d(\frac{d}{dt}) \end{bmatrix} \ell, w \in \mathfrak{B}, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}}) \quad \text{and } G(s) = \tfrac{n(s)}{d(s)}$$

- Construct Bezoutian

$$z_b(\zeta,\eta) := \tfrac{n(\zeta)d(\eta)+n(\eta)d(\zeta)}{\zeta+\eta} = \begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{n-1} \end{bmatrix}^T Z_b \begin{bmatrix} 1 \\ \eta \\ \vdots \\ \eta^{n-1} \end{bmatrix}$$

Then, $x^T Z_b x$ is the unique storage function for the Σ-lossless system, where $x = (\ell, \dot{\ell}, \ddot{\ell}, \ldots, \frac{d}{dt}^{n-1}\ell)$.

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- Bezoutian has the form $\Psi(\zeta,\eta) = \frac{n(\zeta)d(\eta)+n(\eta)d(\zeta)}{\zeta+\eta} =: \frac{\Phi(\zeta,\eta)}{\zeta+\eta}$
- Rewrite the Bezoutian: $(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta)$

$$\Phi(\zeta,\eta) = \phi_0(\eta) + \zeta\phi_1(\eta) + \ldots + \zeta^n\phi_n(\eta)$$
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$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -0.1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{and} \quad v = \begin{bmatrix} 0 & 0.2 \end{bmatrix} x + 0\,u$$

- $\Phi(\zeta,\eta) = n(\zeta)d(\eta) + n(\eta)d(\zeta) = \underbrace{0.02\eta}_{\Phi_0(\eta)} + (\underbrace{0.02 + 0.2\eta^2}_{\Phi_1(\eta)})\zeta + (\underbrace{0.2\eta}_{\Phi_2(\eta)})\zeta^2$

- $\Psi_0(\xi) = \frac{\Phi_0(\xi)}{\xi} = 0.02 \qquad \Psi_1(\xi) = \frac{\Phi_1(\xi)-\Psi_0(\xi)}{\xi} = 0.2\xi$

$$\Psi(\zeta,\eta) = 0.02 + 0.2\zeta\eta$$

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$$K = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.2 \end{bmatrix}$$

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- Lossless system: $G(s) = \frac{r_0}{s} + \sum_{i=1}^{\mathtt{m}} \frac{r_i s}{s^2+\omega_i^2}$ where $r_0, r_i > 0, \omega_i > 0$
- Minimal state space realisation

$$A = \text{diag}\ (0, A_1, A_2, ..., A_m) \text{ where } A_i = \begin{bmatrix} 0 & -r_i \\ \frac{\omega_i^2}{r_i} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} r_0 & r_1 & 0 & r_2 & 0 & \cdots & r_m & 0 \end{bmatrix}^T \in \mathbb{R}^{2\mathtt{m}}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\mathtt{m}}.$$

Then, unique storage function is $x^T K x$ where

$$K := \text{diag}\ \left(\frac{1}{r_0}, K_1, K_2, \ldots, K_m\right) \text{ where } K_i := \begin{bmatrix} \frac{1}{r_i} & 0 \\ 0 & \frac{r_i}{\omega_i^2} \end{bmatrix}$$

Theorem: Partial fraction based method

- Lossless system: $G(s) = \frac{r_0}{s} + \sum_{i=1}^{\mathtt{m}} \frac{r_i s}{s^2 + \omega_i^2}$ where $r_0, r_i > 0, \omega_i > 0$
- Minimal state space realisation

$$A = \text{diag}\ (0, A_1, A_2, ..., A_m) \text{ where } A_i = \begin{bmatrix} 0 & -r_i \\ \frac{\omega_i^2}{r_i} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} r_0 & r_1 & 0 & r_2 & 0 & \cdots & r_m & 0 \end{bmatrix}^T \in \mathbb{R}^{2\mathtt{m}}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\mathtt{m}}.$$

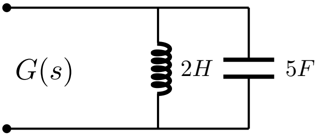
Then, unique storage function is $x^T K x$ where

$$K := \text{diag}\ \left(\frac{1}{r_0}, K_1, K_2, \ldots, K_m\right) \text{ where } K_i := \begin{bmatrix} \frac{1}{r_i} & 0 \\ 0 & \frac{r_i}{\omega_i^2} \end{bmatrix}$$

Example

- We stick to same example: $G(s) = \frac{0.2s}{s^2+0.1}$

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{5} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix} i \quad \text{and} \quad v = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + 0\, i$$



- Here $r_1 = 0.2, \omega_1^2 = 0.1$. Hence

$$K_1 := \begin{bmatrix} \frac{1}{r_1} & 0 \\ 0 & \frac{r_1}{\omega_1^2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Experimental results

- Parameters under inspection: Computation time and error.
- Averaged over three sets of randomly generated lossless transfer function.
- Error in computation:

$$\text{Err}(K) := \left\| \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} \right\|_2 .$$

Computation error

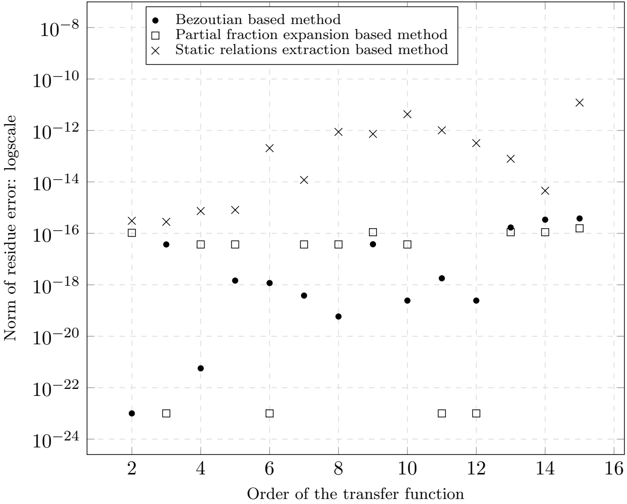


Figure: Plot of error residue versus system's order.

Computation time

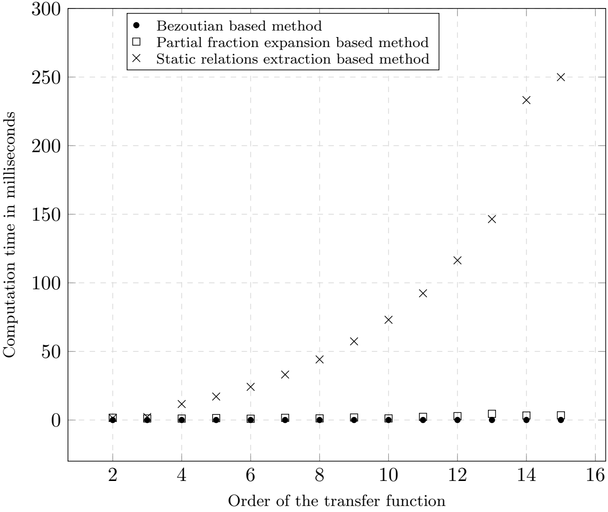


Figure: Plot of computation time versus system's order.

Conclusion

1. Reported three methods to compute storage function of lossless systems.
    - **Static relation based**: static relations between x and z.
    - **Bezoutian based**: computed using Bezoutian of two polynomials.
    - **Partial fraction based**: based on Foster realization of LC circuits.
2. Bezoutian based method is more efficient.
3. Methods have been extended to MIMO case (C. Bhawal et.al., TCAS-I under review).

Thank You

Questions?