On solvability of CGCARE for LQR problems with zero input-cost

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Problem statement

Consider a controllable system with state-space dynamics

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \text{ where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}.$$

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Infinite-horizon linear quadratic regulator (LQR) problem

For every initial condition $x_0 \in \mathbb{R}^{\mathtt{n}}$, find an input $u(t)$ (from admissible input space) that minimizes the functional

$$J\left(x_0, u(t)\right) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \text{ where } \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$$

and $\lim_{t\to\infty} x(t) = 0$.

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For $R > 0$ (Regular case)

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0 \qquad u(t) = -R^{-1}(B^T K_{\mathtt{max}} + S^T)x(t)$$

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For $R \geqslant 0$, $\mathtt{det}(R) = 0$ (Singular/degenerate case)

$\cancel{A^TK + KA + Q - (KB + S)R^{-1}(B^TK + S^T) = 0}$ $\cancel{u(t) = -R^{-1}(B^TK_{\mathtt{max}} + S^T)x(t)}$

Literature review

System: $\frac{d}{dt}x = Ax + Bu$ Cost matrix: $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0, \texttt{det}(R) = 0.$

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Theorem (Ferrante and Ntogramatzidis, Automatica 2014)

Singular LQR problem is solvable using a static state-feedback controller

$$\Updownarrow$$

there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *such that*

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$$\mathtt{ker}(R) \subseteq \mathtt{ker}(S + KB).$$

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Constrained Generalized Continuous Algebraic Riccati Equation - CGCARE.

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Singular LQR problems corresponding to single-input sytems can be solved using proportional-derivative (PD) controllers.

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When is CGCARE solvable for $R = 0$?

Hamiltonian pencils

- For $R = 0$, $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0 \Rightarrow Q \geqslant 0$ and $S = 0$.

Hamiltonian pencils

- Cost function: $\int_0^\infty \left(x^T Q x\right) dt$.

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$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B \\ -Q & -A^T & 0 \\ 0 & B^T & 0 \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u \end{bmatrix}.$$

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(E, H): Hamiltonian matrix pair, $(sE - H)$: Hamiltonian matrix pencil.

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$$\frac{d}{dt} \begin{bmatrix} x \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}}_{\widehat{A}} \begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\widehat{B}} u, \text{ and } 0 = \underbrace{\begin{bmatrix} 0 & B^T \end{bmatrix}}_{\widehat{C}} \begin{bmatrix} x \\ z \end{bmatrix}$$

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Lemma

CGCARE with $R = 0$ solvable

$$\Updownarrow$$

$\mathcal{P}(s) := \widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B} \equiv 0$ *(as a rational matrix).*

For the R singular case: more necessary and sufficient conditions in Bhawal, Qais, and Pal, IEEE L-CSS 2019.

Autonomy?

- $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B} \equiv 0 \Leftrightarrow \widehat{C}\widehat{A}^k\widehat{B} = 0$ for all $k \in \{0, 1, 2, \cdots\}$.

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$$\begin{aligned}
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\end{aligned}$$

- Thus for $\widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} \equiv 0$ we must have

$$Q\begin{bmatrix} B & AB & \cdots & A^{\mathtt{n}-1}B \end{bmatrix} = 0.$$

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Since (A, B) is controllable, this is not possible unless $Q = 0$.

CGCARE solvability

Theorem (Main result)

Corresponding to a singular LQR problem with $R = 0$ (zero input cost) the following statements are equivalent:

1. $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B} \not\equiv 0$.
2. *CGCARE is not solvable.*

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Corresponding to a singular LQR problem with $R = 0$ (zero input cost) the following statements are equivalent:

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3. *There exists no proportional state-feedback controller that solves the singular LQR Problem.*

Example

- $\frac{d}{dt}x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$

Cost: $\int_0^\infty \left(x^T Q x \right) dt$,
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- Let $K = \begin{bmatrix} k_1 & k_2 & k_4 \\ k_2 & k_3 & k_5 \\ k_4 & k_5 & k_6 \end{bmatrix}$.

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- We have from $A^T K + KA + Q = 0$:

$$\begin{bmatrix} 2(k_1 + k_4) & k_4 & k_1 + k_4 + k_6 \\ k_4 & 0 & k_6 \\ k_1 + k_4 + k_6 & k_6 & 2k_4 + 1 \end{bmatrix} = 0$$

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- CGCARE not solvable.

No P state-feedback controller

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- Eigenvalues of (E, H) are $\{-1, +1\}$.

$$E = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; H = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & 0 \\ 0 & B^T & 0 \end{bmatrix}$$

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- Basis for stable-eigenspace of (E, H):

$$V := \begin{bmatrix} 1 & 1 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}$$

such that $HV = EV\Gamma$, where $\Gamma = -1$. Define $V_1 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}^T$, $V_3 = 0$.

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- Design the controllers:

$$F_{\mathtt{p}} := \begin{bmatrix} V_3 & f_0 & f_1 \end{bmatrix} X_1^{-1} \text{ and } F_{\mathtt{d}} := \begin{bmatrix} 0 & 1 & -f_0 \end{bmatrix} X_1^{-1}.$$

where $f_0, f_1 \in \mathbb{R}$.

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- Design the controllers:

$$F_{\mathtt{p}} := \begin{bmatrix} V_3 & f_0 & f_1 \end{bmatrix} X_1^{-1} \text{ and } F_{\mathtt{d}} := \begin{bmatrix} 0 & 1 & -f_0 \end{bmatrix} X_1^{-1}.$$

where $f_0, f_1 \in \mathbb{R}$.

- Chosing $f_0 = 0$ and $f_1 = f$: $F_{\mathtt{p}} = \begin{bmatrix} 2f & 0 & f \end{bmatrix}$ and $F_{\mathtt{d}} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$.

PD controller

- $\frac{d}{dt}x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$

Cost: $\int_0^\infty \left(x^T Q x\right) dt$,
$Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- $V_1 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}^T$, $V_3 = 0$.
- Define $X_1 = \begin{bmatrix} V_1 & B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
- Design the controllers:

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PD controller

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- PD controller: $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \dot{x}$.
- Closed loop system: $(I_{\mathtt{n}} - BF_{\mathtt{d}}) \frac{d}{dt} x = (A + BF_{\mathtt{p}}) x$.

PD controller

- $\frac{d}{dt}x = \begin{bmatrix}1 & 0 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}x + \begin{bmatrix}0\\1\\0\end{bmatrix}u$

Cost: $\int_0^\infty \left(x^T Q x\right) dt$,
$Q := \left[\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right]$.

- $V_1 = \begin{bmatrix}1 & 1 & -2\end{bmatrix}^T$, $V_3 = 0$.
- Define $X_1 = \begin{bmatrix}V_1 & B & AB\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\-2 & 0 & 1\end{bmatrix}$.
- Design the controllers:

$$F_{\mathtt{p}} := \begin{bmatrix}V_3 & f_0 & f_1\end{bmatrix} X_1^{-1} \text{ and } F_{\mathtt{d}} := \begin{bmatrix}0 & 1 & -f_0\end{bmatrix} X_1^{-1}.$$

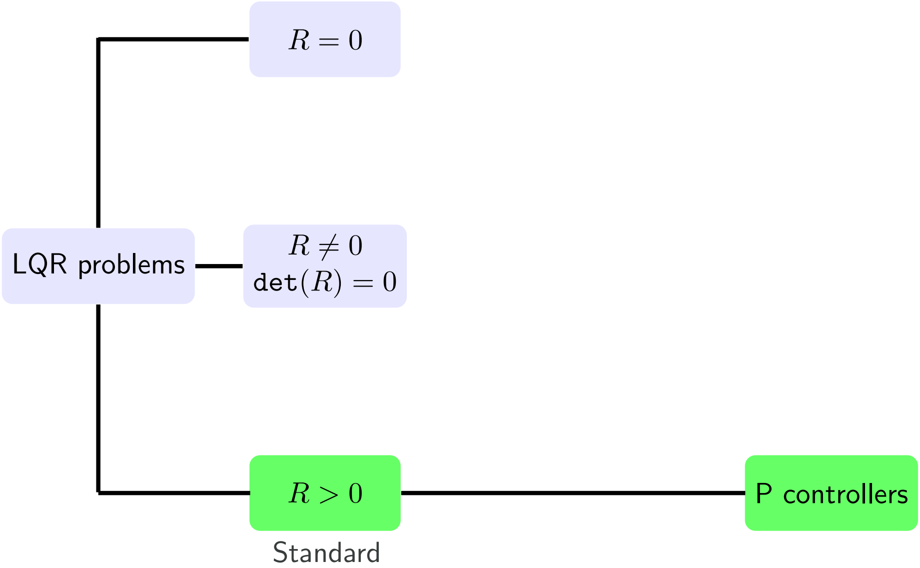
  where $f_0, f_1 \in \mathbb{R}$.
- Chosing $f_0 = 0$ and $f_1 = f$: $F_{\mathtt{p}} = \begin{bmatrix}2f & 0 & f\end{bmatrix}$ and $F_{\mathtt{d}} = \begin{bmatrix}-1 & 1 & 0\end{bmatrix}$.
- PD controller: $u = F_{\mathtt{p}}x + F_{\mathtt{d}}\dot{x}$.
- Closed loop system: $(I_{\mathtt{n}} - BF_{\mathtt{d}})\frac{d}{dt}x = (A + BF_{\mathtt{p}})x$.
  Choose f: $\mathtt{det}(s(I_{\mathtt{n}} - BF_{\mathtt{d}}) - (A + BF_{\mathtt{p}})) \neq 0$.

Conclusion

LQR problems

- $R = 0$
- $R \neq 0$, $\texttt{det}(R) = 0$
- $R > 0$

Conclusion


$R = 0$
LQR problems
$R \neq 0$
$\texttt{det}(R) = 0$
$R > 0$
P controllers
Standard

Conclusion

$R = 0$

This paper

PD controllers

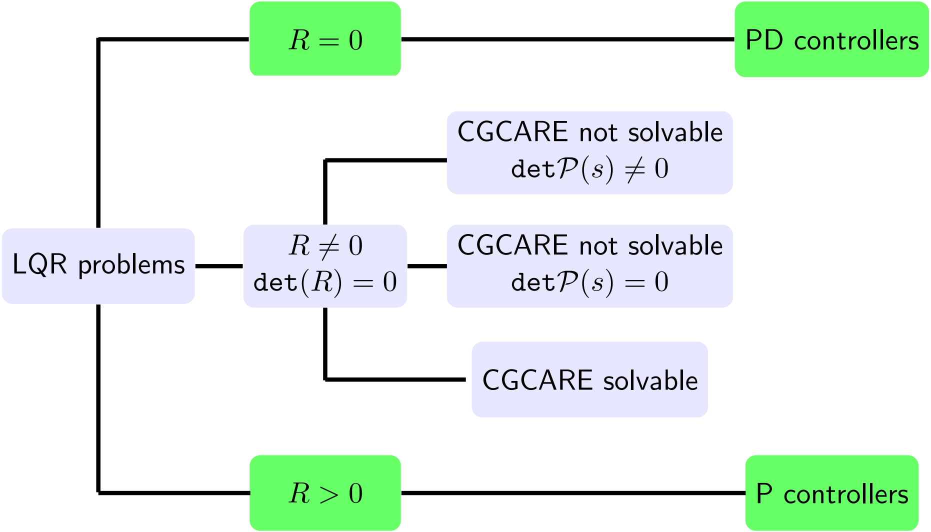
LQR problems

$R \neq 0$
$\mathtt{det}(R) = 0$

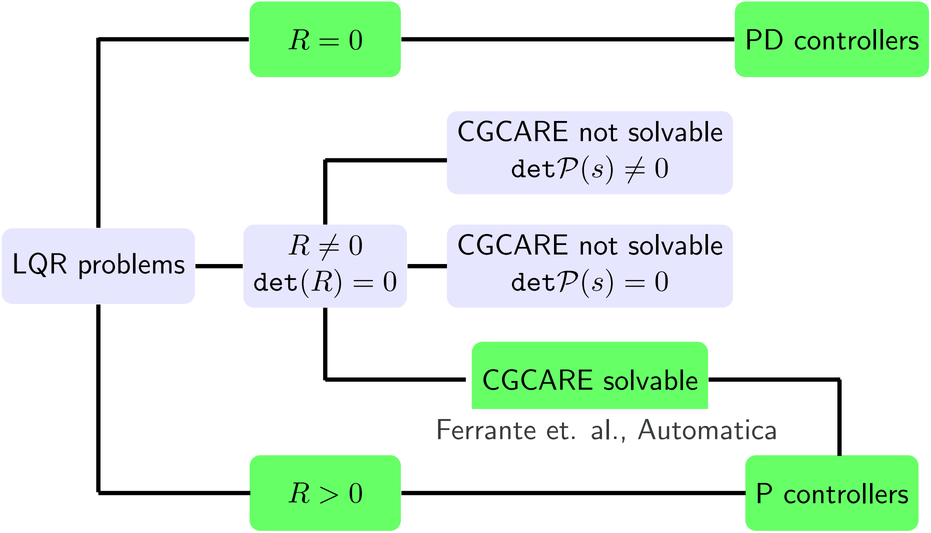
$R > 0$

P controllers

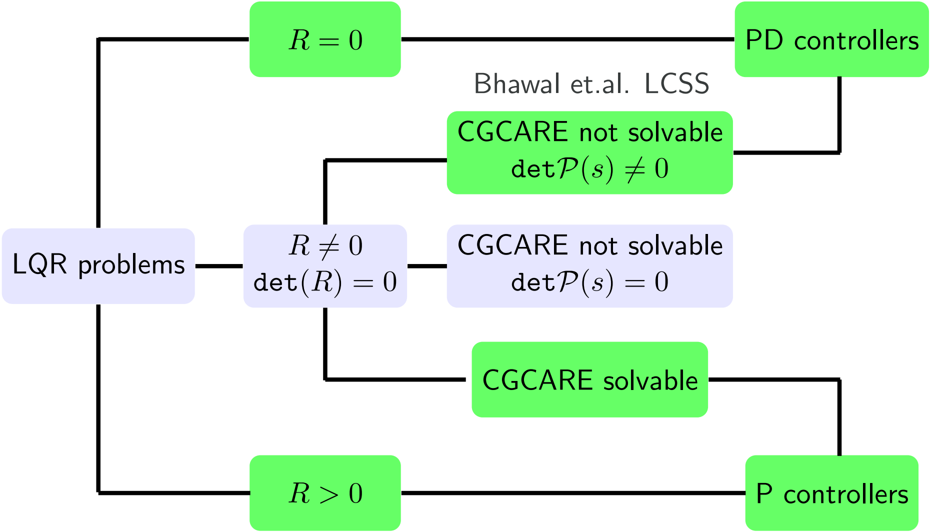
Conclusion


LQR problems
$R = 0$
PD controllers
$R \neq 0$
$\det(R) = 0$
CGCARE not solvable
$\det \mathcal{P}(s) \neq 0$
CGCARE not solvable
$\det \mathcal{P}(s) = 0$
CGCARE solvable
$R > 0$
P controllers

Conclusion

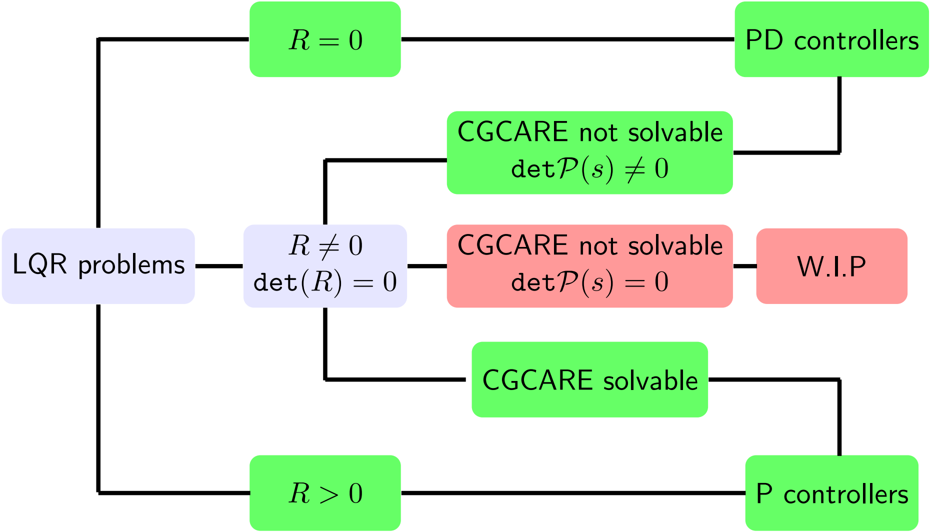

$R = 0$
PD controllers
CGCARE not solvable
$\mathtt{det}\mathcal{P}(s) \neq 0$
LQR problems
$R \neq 0$
$\mathtt{det}(R) = 0$
CGCARE not solvable
$\mathtt{det}\mathcal{P}(s) = 0$
CGCARE solvable
Ferrante et. al., Automatica
$R > 0$
P controllers

Conclusion



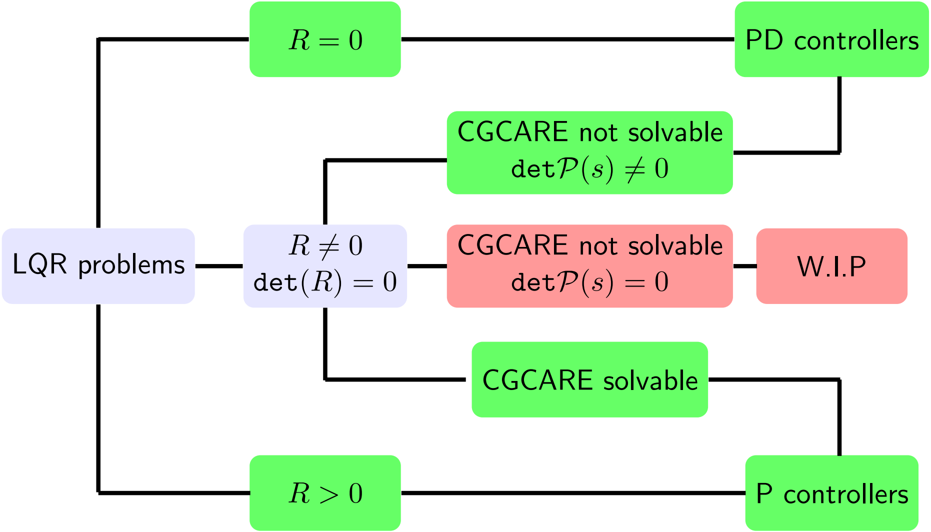
LQR problems
$R = 0$
PD controllers
Bhawal et.al. LCSS
CGCARE not solvable
$\texttt{det}\mathcal{P}(s) \neq 0$
$R \neq 0$
$\texttt{det}(R) = 0$
CGCARE not solvable
$\texttt{det}\mathcal{P}(s) = 0$
CGCARE solvable
$R > 0$
P controllers

Conclusion



$R = 0$
PD controllers
CGCARE not solvable $\det \mathcal{P}(s) \neq 0$
LQR problems
$R \neq 0$ $\det(R) = 0$
CGCARE not solvable $\det \mathcal{P}(s) = 0$
W.I.P
CGCARE solvable
$R > 0$
P controllers

Conclusion



THANK YOU