On the link between storage functions of allpass systems and Gramians

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Dissipative systems and storage function

- A dissipative system
  1. has no source of energy.
  2. can only absorb energy.
  3. can store energy.

- $\underbrace{\text{Rate-change-stored-energy}}_{\frac{d}{dt}Q_\Psi(w)} + \underbrace{\text{Dissipated power}}_{Q_\Delta(w)} = \underbrace{\text{Power supplied}}_{Q_\Sigma(w)}$

- Dissipative systems : $\frac{d}{dt}Q_\Psi(w) \leqslant Q_\Sigma(w)$.

- Power supplied with respect to system variable: $Q_\Sigma(w) = w^T \Sigma w$.

  e.g. Supply rate: $w = (u, y)$: $Q_\Sigma(w) = w^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} w = 2u^T y$.

- Stored energy: Storage function[1] $Q_\Psi(w) = x^T K x$.

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Passivity and bounded-real systems

Dissipative systems

$$\frac{d}{dt}\left(x^T K x\right) \leqslant Q_\Sigma(w), \ \ where\ w = (u, y).$$

	Positive-real	**Bounded-real**
Supply rate Σ	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$
Dissipation inequality	$\frac{d}{dt}\left(x^T K x\right) \leqslant 2u^T y$	$\frac{d}{dt}\left(x^T K x\right) \leqslant u^T u - y^T y$

Passivity and bounded-real systems

Conservative systems

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Conservative counterpart	$\frac{d}{dt}\left(x^T K x\right) = 2u^T y$	$\frac{d}{dt}\left(x^T K x\right) = u^T u - y^T y$
Name	Lossless	Allpass
Example	$G(s) = \frac{s}{s^2+1}$ (LC circuits)	$G(s) = \frac{s-1}{s+1}$ (allows all frequencies to pass)

Allpass systems

- System $\mathfrak{B} : \frac{d}{dt}x = Ax + Bu,\ y = Cx + Du$, (Minimal)
  $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $C, B^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

  Necessary condition for allpass: $I - D^T D = 0$.

- Dissipation equation: $\frac{d}{dt}\left(x^T K x\right) = u^T u - y^T y$.

- Linear Matrix Equations (LME):
  Allpass if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

$$\begin{bmatrix} A^T K + KA + C^T C & KB + C^T \\ B^T K + C & 0 \end{bmatrix} = 0$$

- Rewriting the LME:
  Allpass if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

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  Necessary condition for allpass: $D + D^T = 0$.

- Dissipation equation: $\frac{d}{dt}\left(x^T K x\right) = 2u^T y$.

- Linear Matrix Equations (LME):
  Lossless if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} = 0$$

- Rewriting the LME:
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Allpass and Lossless systems

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Supply rate Σ	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$
Conservative counterpart	$\frac{d}{dt}\left(x^T K x\right) = 2u^T y$	$\frac{d}{dt}\left(x^T K x\right) = u^T u - y^T y$
Example	$G(s) = \frac{s}{s+1}$ (LC circuits)	$G(s) = \frac{s-1}{s+1}$ (allows all frequency)
Poles of the system	On imaginary axis	Symmetric about real and imaginary axis
K satisfies LME	$A^T K + KA = 0$ $KB - C^T = 0$	$A^T K + KA + C^T C = 0$ $KB + C^T = 0$

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Zeros and poles of the system	Interlace on $j\mathbb{R}$	Mirrored about $j\mathbb{R}$
K satisfies LME	$A^T K + KA = 0$ $KB - C^T = 0$	$A^T K + KA + C^T C = 0$ $KB + C^T = 0$

Lossless: Lyapunov equation $A^T K + KA = 0$ has non-unique solutions.

Link between storage functions of lossless systems and allpass systems

Theorem

- *$\mathfrak{B}_{all}$ is an allpass system and $\mathfrak{B}_{\ell}$ is its lossless counterpart.*

	Lossless	**Allpass**
Minimal i/s/o representation	$\frac{d}{dt}x = Ax + Bu$ $y = Cx + Du$	$\frac{d}{dt}x = \hat{A}x + \hat{B}\left(\frac{u+y}{\sqrt{2}}\right)$ $\frac{u-y}{\sqrt{2}} = \hat{C}x + \hat{D}\left(\frac{u+y}{\sqrt{2}}\right)$

$\hat{A} := \left(A - B(I+D)^{-1}C\right)$, $\hat{B} = \frac{1}{\sqrt{2}}\left(B + B(I+D)^{-1}(I-D)\right)$, $\hat{C} = -\sqrt{2}(I+D)^{-1}C$ and $\hat{D} := (I+D)^{-1}(I-D)$.

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- $\mathfrak{B}_{all}$ *is an allpass system and* $\mathfrak{B}_\ell$ *is its lossless counterpart.*

Then, the storage function of $\mathfrak{B}_{all}$ *and* $\mathfrak{B}_\ell$ *is the same.*

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Allpass systems and Observability Gramian

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- *Stable, minimal, allpass system*

$$\frac{d}{dt}x = Ax + Bu \quad \textit{and} \quad y = Cx + Du,$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ *and* $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

- *Assume* Q *to be the observability Gramian of the system.*

Then, x^TQx *is the unique storage function of the system.*

Corollary

- *Allpass system* $\mathfrak{B}_{all}$ *and its lossless counterpart* $\mathfrak{B}_{\ell}$.

Then, observability Gramian of $\mathfrak{B}_{all}$ *is the storage function of* $\mathfrak{B}_{\ell}$.

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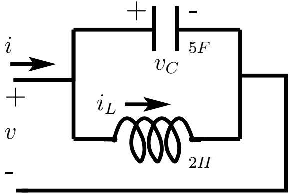
Link between storage functions of lossless systems and allpass systems

Lossless system $G(s) = \dfrac{0.2s}{s^2 + 0.1}$.

i/s/o representation:

$$\frac{d}{dt}\begin{bmatrix} i_L \\ v_C \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{5} & 0 \end{bmatrix}\begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} i,$$

$$v = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} i_L \\ v_C \end{bmatrix}.$$



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$$\left(\frac{u-y}{\sqrt{2}}\right) = \begin{bmatrix} 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} i_L \\ v_C \end{bmatrix} + 1\left(\frac{u+y}{\sqrt{2}}\right).$$

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Observability Gramian matrix: $Q = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

Stored energy $= 2i_L^2 + 5v_c^2$. (Recall: power $= 2vi$)

$$A^T Q + QA = 0 \qquad \hat{A}^T Q + Q\hat{A} + \hat{C}^T\hat{C} = 0$$

$$QB - C^T = 0. \qquad Q\hat{B} + \hat{C}^T = 0.$$

$x^T Q x$ is the storage function of both the systems.

Storage functions of allpass and lossless systems

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Observability Gramian matrix: $Q = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

Stored energy $= 2i_L^2 + 5v_c^2$. (Recall: power $= 2vi$)

$$A^TQ + QA = 0 \qquad \hat{A}^TQ + Q\hat{A} + \hat{C}^T\hat{C} = 0$$

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x^TQx is the storage function of both the systems.

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Allpass systems and Balanced realization

Definition

System representation is said to be in balanced state space basis if controllability Gramian P = observability Gramian Q.

Theorem

- *Stable, allpass system:* $\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du$ *in a balanced state space basis.*

  Then, Storage function $K = I$.

- Allpass systems[2] have $PQ = I$.
- Balanced transformation: $P = Q$. Then $Q^2 = I$.
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Positive-real system and its adjoint

System $\mathfrak{B}$ :: states x	Adjoint system $\mathfrak{B}^{\perp_\Sigma}$ :: co-states λ
Minimal i/s/o representation	Minimal i/s/o representation
$\dot{x} = Ax + Bu$	$\dot{\lambda} = -A^T\lambda + C^T f$
$y = Cx + Du$	$e = B^T\lambda - D^T f$

Interconnect: u to f and y to e $\implies$ $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ formed.
First order representation of $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$:

$$\underbrace{\left(\frac{d}{dt}\underbrace{\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D+D^T \end{bmatrix}}_{H}\right)}_{\text{Hamiltonian pencil } R(\xi)} \begin{bmatrix} x \\ \lambda \\ u \end{bmatrix} = 0$$

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Lossless system and its adjoint

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Static relations: states and costates

- Lossless system $\mathfrak{B}$: (A, B, C, D) minimal realization.
- $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$: $\left(E\frac{d}{dt} - H\right)\begin{bmatrix} x \\ \lambda \\ u \end{bmatrix} = 0$ and $y = Cx + Du$.
- Q: observability Gramian of allpass counterpart of $\mathfrak{B}$.

Then, there exists a unique $K = K^T \in \mathbb{R}^{n \times n}$ such that

$$\frac{d}{dt} x^T K x = 2u^T y \qquad \text{for all } \begin{bmatrix} u \\ y \end{bmatrix} \in \mathfrak{B}.$$

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$$\operatorname{rank} \begin{bmatrix} R(\xi) \\ \begin{matrix} -K & I & 0 \end{matrix} \end{bmatrix} = \operatorname{rank} R(\xi).$$

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Static relations: example

- Lossless behavior $\mathfrak{B}$ with transfer function $G(s) = \frac{0.2s}{s^2+0.1}$.

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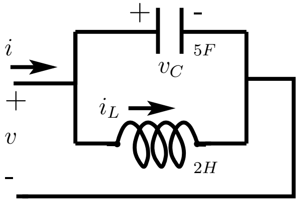
Hamiltonian pencil:

$$R(\xi) = \left[\begin{array}{cc|cc|c} \xi & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{5} & \xi & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & \xi & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} & \xi & -1 \\ 0 & -1 & 0 & \frac{1}{5} & 0 \end{array}\right].$$



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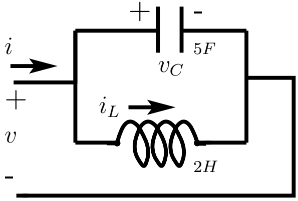

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$$\begin{bmatrix} -1 & 5(\xi-2) & 0.5 & (2-\xi) & 5\xi^2 - 10\xi + 0.5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \left[\begin{array}{cc|cc|c} \xi & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{5} & \xi & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & \xi & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} & \xi & -1 \\ 0 & -1 & 0 & \frac{1}{5} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|c} -2 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} -Q & I & 0 \end{bmatrix}$$

Conclusion

1. One-to-one correspondence between lossless and allpass system: The storage function remains same.
2. Observability Gramian is the storage function for allpass/lossless systems.
3. Easy computation of storage functions of lossless systems.
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Questions?