On solvability of CGCARE for LQR problems with zero input-cost

Chayan Bhawal and Debasattam Pal

Abstract— **In this paper, we show that singular LQR problems with zero input-cost cannot be solved using static state-feedback controllers. To this end we first show that for such problems the corresponding constrained generalized continuous algebraic Riccati equation (CGCARE) is not solvable. This is achieved by establishing that the Hamiltonian system in such a case does not admit a transfer function which is identically zero. Further, we also show that, unlike the multi-input case which admits both autonomous and non-autonomous Hamiltonian systems, a single-input singular LQR problem always admits an autonomous Hamiltonian system.**

Keywords: ***LQR Problem, CGCARE, Hamiltonian systems***

1. Introduction

Cheap control problem has been an important problem in optimal control for a long time [1], [2], [3], [4]. This problem continues to be an active area of research [5], [6], [7], [8]. These are those infinite-horizon optimal control problems where the weighting on the control energy tends to zero. In order to motivate such problems further, we revisit the infinite-horizon linear quadratic regulator (LQR) problem with cheap control first.

Problem 1.1. (LQR problem with cheap control) *Consider a controllable system with minimal state-space dynamics* $\frac{d}{dt}x = Ax + Bu$*, where* $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$*,* $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$*. Then, for every initial condition* x_0*, find an input* u *that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & \varepsilon^2\widehat{R} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt, \qquad (1)$$

where $\begin{bmatrix} Q & S \\ S^T & \varepsilon^2\widehat{R} \end{bmatrix} \geqslant 0$*,* $\widehat{R} \geqslant 0$ *and* $\varepsilon > 0$*.*

The results in this paper deal with the limiting case of the LQR Problem 1.1, i.e., the case when $\varepsilon = 0$. For the sake of brevity, we define $R := \varepsilon^2\widehat{R}$. Thus, for $\varepsilon = 0$, we have $R = 0$. A well-known method to compute solutions to the LQR Problem 1.1 with $R > 0$ (called the regular LQR problem) is to construct a state feedback matrix using a suitable solution K (under proper assumptions) of the following algebraic Riccati equation (ARE) [9]:

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0 \qquad (2)$$

The feedback matrix $F := R^{-1}(B^T K + S^T)$ gives the desired optimal control law $u := -Fx$ that minimizes the functional (1). However, for the case with $\varepsilon = 0$ (we call this the singular LQR problem), such feedback matrices cannot be constructed, since the ARE does not exist due to the singularity of R. This has been a long-standing problem in optimal control literature, until recently, the authors in [6] have established that singular LQR problems are solvable using static state-feedback laws, like the regular case, if and only if such a problem admits solution to a special form of ARE called the constrained generalized continuous ARE (CGCARE). For an LQR problem with $R \geqslant 0$, the CGCARE takes the following form:

$$\begin{cases} A^T K + KA + Q - (KB + S)R^{\dagger}(B^T K + S^T) = 0 \\ \mathtt{ker}(R) \subseteq \mathtt{ker}(S + KB), \end{cases} \qquad (3)$$

where $R^{\dagger}$ is a generalized-inverse of R. Hence, a relevant question is: when does the CGCARE admit a solution? In [10] several necessary and sufficient conditions for the solvability of CGCARE have already been proposed. In this paper, we use these conditions to show that the CGCARE is *not* solvable for the case when $R = 0$ (Theorem 3.1). A special case of the singular case is the scenario where the singular LQR problem arises out of a single-input system. It has been recently established in the literature that the single-input singular LQR problem can indeed be solved using a proportional-derivative (PD) state-feedback controller [11]. Such a controller can be constructed using the *maximal rank-minimizing*[1] solutions of the LMI:

$$\begin{bmatrix} A^T K + KA + Q & KB + S \\ B^T K + S^T & R \end{bmatrix} \geqslant 0 \qquad (4)$$

For the ease of reference, we call LMI (4) the *LQR LMI*. A natural question is: why would one use a PD-controller

Corresponding author email: `bhawal@mpi-magdeburg.mpg.in`. D. Pal is in the Department of Electrical Engineering, Indian Institute of Technology Bombay (IIT Bombay), India. The corresponding author C. Bhawal is currently in the Max Planck Institute for Dynamics of Complex Technical Systems Magdeburg. Majority of the corresponding author's work in this paper was performed when he was with IIT Bombay.

[1] A matrix $K_{\mathtt{max}} = K_{\mathtt{max}}^T$ is called a maximal rank-minimizing solution of an LMI $\mathscr{L}(K) \geqslant 0$ if $\mathscr{L}(K_{\mathtt{max}})$ has the least rank among all the solutions of $\mathscr{L}(K) \geqslant 0$ and for any other solution K of the LMI, $K_{\mathtt{max}}$ satisfies $K_{\mathtt{max}} - K \geqslant 0$.

for the single-input case (proposed in [11]) if a static state-feedback controller can be obtained from CGCARE solutions (proposed in [8])? In this paper we answer this question and bridge the gap between the results of [8] and [11] (Theorem 3.5).

We use the symbols $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and natural numbers, respectively. The symbol $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}$. Symbol $I_{\mathtt{n}}$ is used for an $\mathtt{n}\times\mathtt{n}$ identity matrix and $0_{\mathtt{nm}}$ is used to denote an $\mathtt{n}\times\mathtt{m}$ matrix with all entries zero. The symbol $\mathtt{det}(A)$ represents the determinant of a square matrix A. The degree of a polynomial $p(s)$ is denoted by $\mathtt{deg}(p(s))$. The symbol $\mathtt{ker}(\Gamma)$ denotes the kernel of a function Γ. A matrix of the form $\begin{bmatrix} B_1^T & B_2^T & \cdots & B_{\mathtt{n}}^T \end{bmatrix}^T$ is denoted by the symbol $\mathtt{col}(B_1, B_2, \ldots, B_{\mathtt{n}})$.

2. Preliminaries

In a singular LQR problem with $R=0$, we must have $S=0$. This condition on S needs to hold to ensure that the cost-matrix $\begin{bmatrix} Q & S \\ S^T & 0 \end{bmatrix}$ is positive-semidefinite. Hence, the LQR Problem 1.1 for $\varepsilon=0$ takes the following form:

Problem 2.1. (Singular LQR problem with zero input-cost) *Consider a controllable system Σ with minimal state-space dynamics $\frac{d}{dt}x = Ax+Bu$, where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B\in\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition x_0, find an input u that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \left(x^T Qx\right) dt, \text{ where } Q \geqslant 0 \text{ and } Q \neq 0. \tag{5}$$

Since $R^\dagger$ for the LQR Problem 2.1 is 0 and $\mathtt{ker}(R)=\mathbb{R}^{\mathtt{m}}$, the corresponding CGCARE is given by

$$\begin{cases} A^T K + KA + Q = 0 \\ \mathbb{R}^{\mathtt{m}} \subseteq \mathtt{ker}(KB) \Rightarrow KB = 0 \end{cases} \tag{6}$$

The LQR LMI corresponding to the LQR Problem 2.1 is given by

$$\begin{bmatrix} A^T K + KA + Q & KB \\ B^T K & 0 \end{bmatrix} \geqslant 0 \Rightarrow \begin{cases} A^T K + KA + Q \geqslant 0 \\ KB = 0 \end{cases} \tag{7}$$

The notion of Hamiltonian matrix pencils and Hamiltonian systems is essential for this paper, hence we define them next (see [12] for more on such pencils). Define the matrices:

$$E := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{mm}} \end{bmatrix}, \text{ and } H := \begin{bmatrix} A & 0 & B \\ -Q & -A^T & 0 \\ 0 & B^T & 0_{\mathtt{mm}} \end{bmatrix} \tag{8}$$

The matrix pair (E,H) is called the *Hamiltonian matrix pair* and the pencil $(sE-H)$ the *Hamiltonian matrix pencil*. It is well-known that the matrix pair (E,H) induces a singular-descriptor system of the form:

$$\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{mm}} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \\ u \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & 0 \\ 0 & B^T & 0_{\mathtt{mm}} \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix} \tag{9}$$

The system in equation (9) is called the *Hamiltonian system*. An output-nulling representation of this system is given by

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \widehat{A}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{B}u \quad \text{and} \quad 0 = \widehat{C}\begin{bmatrix} x \\ z \end{bmatrix}, \tag{10}$$

where $\widehat{A} := \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}$, $\widehat{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}$ and $\widehat{C} := \begin{bmatrix} 0 & B^T \end{bmatrix}$. We use the symbol Σ_{Ham} to denote the system in equation (10).

The notion of autonomy of a system is also crucial in this paper, hence we present a result next that establishes a condition to check the autonomy of a system.

Proposition 2.2. [13, Lemma 3.3] *Consider the system $\frac{d}{dt}x = Ax+Bu$ and $0 = Cx$, where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B\in\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, and $C\in\mathbb{R}^{\mathtt{p}\times\mathtt{n}}$. Then, the system is autonomous if and only if $G(s) := C(sI_{\mathtt{n}}-A)^{-1}B$ is invertible as a rational matrix.*

From Proposition 2.2 it is evident that the Hamiltonian system in equation (10) is autonomous if and only if $\widehat{C}(sI_{2\mathtt{n}}-\widehat{A})^{-1}\widehat{B}$ is invertible as a rational matrix.

3. Main results

The first main result of this paper shows that a singular LQR problem with zero input-cost cannot be solved with a proportional (P) state-feedback controller.

Theorem 3.1. *Consider the singular LQR Problem 2.1 with the corresponding CGCARE given by equation* (6)*. Let the corresponding Hamiltonian system be as given in equation* (10)*. Then, the following statements are true:*

1) *CGCARE is not solvable.*
2) *There exists no proportional state-feedback controller that solves the singular LQR Problem 2.1.*

In order to prove this theorem we need to first review a result in [11, Statement 4, Theorem 1].

Proposition 3.2. *Consider the singular LQR Problem 2.1 with the corresponding CGCARE given by equation* (6)*. Let the corresponding Hamiltonian system be as given in equation* (10)*. Then, the CGCARE is solvable if and only if*[2] *$\widehat{C}(sI_{2\mathtt{n}}-\widehat{A})^{-1}\widehat{B} = 0$ (as a rational matrix).*

Proof of Theorem 3.1: 1) To the contrary, assume that CGCARE is solvable. Then, from Proposition 3.2 it is

[2]For singular LQR problems with $R=0$, the reduced Hamiltonian system (as defined in [10, Equation 10]) and the Hamiltonian system (equation (10)) are the same.

evident that we must have $H(s) := \widehat{C}(sI_{2n} - \widehat{A})^{-1}\widehat{B} = 0$, where $\widehat{A}, \widehat{B}$, and $\widehat{C}$ are as defined in equation (10). For $H(s)$ to be identically zero, all the Markov parameters of $H(s)$ must be zero, i.e., $\widehat{C}\widehat{A}^{\ell}\widehat{B} = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$.

We first claim that if $\widehat{C}\widehat{A}^{\ell}\widehat{B} = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$, then $QA^kB = 0$ for all $k \in \mathbb{N} \cup \{0\}$. We prove this using induction. *Base case:* ($k = 0$) For $\ell = 1$, we know that

$$\begin{aligned} \widehat{C}\widehat{A}\widehat{B} = 0 &\Rightarrow \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} = 0 \\ &\Rightarrow B^T Q B = 0 \end{aligned} \tag{11}$$

From equation (11) and the fact that $Q \geqslant 0$, it is evident that $QB = 0$.

Induction step: Assume $QA^iB = 0$ for $0 \leqslant i \leqslant k-1$. We prove that $QA^kB = 0$.

$$\begin{aligned} &\widehat{C}\widehat{A}^{(2k+1)}\widehat{B} = \\ &\begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}^{2k-1} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -(QB)^T & -(AB)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}^{2k-1} \begin{bmatrix} AB \\ -QB \end{bmatrix} \\ &= \begin{bmatrix} 0 & -(AB)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}^{2k-1} \begin{bmatrix} AB \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -(QAB)^T & (A^2B)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}^{2k-3} \begin{bmatrix} A^2B \\ -QAB \end{bmatrix} \end{aligned} \tag{12}$$

Using the induction hypothesis, $QAB = 0$ in equation (12), we have

$$\widehat{C}\widehat{A}^{(2k+1)}\widehat{B} = \begin{bmatrix} 0 & (A^2B)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}^{2k-3} \begin{bmatrix} A^2B \\ 0 \end{bmatrix}$$

Proceeding in a similar way and using the assumption that $QA^iB = 0$ for all $0 \leqslant i \leqslant k-1$, we infer from equation (12) that

$$\begin{aligned} \widehat{C}\widehat{A}^{(2k+1)}\widehat{B} &= \begin{bmatrix} 0 & (-1)^k(A^kB)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} (-1)^k(A^kB) \\ 0 \end{bmatrix} \\ &= -(A^kB)^T Q (A^kB) = 0 \Rightarrow QA^kB = 0 \end{aligned}$$

This completes the mathematical induction. Thus, we can write $Q\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$. However, for a controllable system $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is a full row-rank matrix. Therefore, we must have $Q = 0$. This is a contradiction to the assumption that $Q \neq 0$ and hence $\widehat{C}(sI_{2n} - \widehat{A})^{-1}\widehat{B} \neq 0$. Thus, from Proposition 3.2 we infer that the CGCARE is not solvable.

2) From [7, Theorem 1] it is known that a singular LQR problem admits a static state-feedback controller if and only if the corresponding CGCARE is solvable. From Statement (1) of this theorem, we know that CGCARE is not solvable for the LQR Problem 2.1. Hence, there exists no proportional state-feedback controller that solves the singular LQR Problem 2.1. □

We present an example next to illustrate the results in Theorem 3.1.

Example 3.3. *Consider a system with state-space dynamics*

$$\frac{d}{dt}x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

For every initial condition x_0, find an input u that minimizes the functional

$$\int_0^{\infty} \left(x^T Q x\right) dt, \text{ where } Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $K = \begin{bmatrix} k_1 & k_2 & k_4 \\ k_2 & k_3 & k_5 \\ k_4 & k_5 & k_6 \end{bmatrix}$ be a solution of the CGCARE corresponding to this problem. Therefore we have $A^TK + KA + Q = 0$, i.e.,

$$\begin{bmatrix} 2(k_1+k_2+k_4) & k_2+k_3+k_4+k_5 & k_1+k_2+k_4+k_5+k_6 \\ k_2+k_3+k_4+k_5 & 2k_5 & k_2+k_3+k_6 \\ k_1+k_2+k_4+k_5+k_6 & k_2+k_3+k_6 & 2(k_4+k_5)+1 \end{bmatrix} = 0 \tag{13}$$

The constrained equation in this case becomes

$$KB = 0 \Rightarrow \begin{bmatrix} k_2 & k_3 & k_5 \\ k_4 & k_5 & k_6 \end{bmatrix}^T = 0 \tag{14}$$

Using solution of equation (14) *in ARE* (13), *we have*

$$\begin{bmatrix} 2k_1 & 0 & k_1 \\ 0 & 0 & 0 \\ k_1 & 0 & 1 \end{bmatrix} = 0 \tag{15}$$

Evidently, equation (15) *is not solvable. Thus, the CGCARE does not admit a solution in this case.*

In Example 3.3 one can verify that the Hamiltonian matrix pencil $(sE - H)$ is singular. This means that the Hamiltonian system Σ_{Ham} for Example 3.3 is non-autonomous (see Example 3.6). Unlike Example 3.3, the next example shows the existence of singular LQR problems that admit autonomous Hamiltonian system Σ_{Ham}.

Example 3.4. *Consider a system with state-space dynamics*

$$\frac{d}{dt}x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

For every initial condition x_0, find an input u that minimizes the functional

$$\int_0^\infty \left(x^T Qx\right) dt, \text{ where } Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

On constructing $\widehat{A}, \widehat{B}$, and $\widehat{C}$ and computing $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B}$, we get

$$H(s) := \widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B} = \begin{bmatrix} \frac{-s^4+4s^2-2}{s^2(s^4-5s^2+4)} & \frac{-1}{s^4-5s^2+4} \\ \frac{-1}{s^4-5s^2+4} & \frac{-s^2+2}{s^4-5s^2+4} \end{bmatrix}$$

It can be verified that $\mathtt{det}(H(s)) \neq 0$ and hence, $H(s)$ is invertible as a rational matrix. Thus, the Hamiltonian system is autonomous. Note that from Proposition 3.2 it is evident that the CGCARE is not solvable for this problem.

From Example 3.3 and Example 3.4 it is clear that singular LQR problems with zero input-cost can admit both autonomous and non-autonomous Hamiltonian systems. However, in the next theorem, we establish that such a scenario never arises for a *single-input* singular LQR problem. Single-input singular LQR problems always admit autonomous Hamiltonian system. Note that the single-input singular LQR problem is a special case of the singular LQR Problem 2.1. Theorem 3.1 for the single-input case takes the following form:

Theorem 3.5. *Consider the singular LQR Problem 2.1 with $B \in \mathbb{R}^{\mathrm{n}}$. Let the corresponding CGCARE and the Hamiltonian system be as given by equation (6) and equation (10), respectively. Then, the following statements are true:*

1) *CGCARE is not solvable.*
2) *There exists no proportional state-feedback controller that solves the singular LQR Problem 2.1.*
3) *Σ_{Ham} is an autonomous system.*

Proof. 1) and 2): Proof of Statement 1) and Statement 2) directly follows from Theorem 3.1.
3): To the contrary assume that Σ_{Ham} is non-autonomous. Note that for the single-input case $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B}$ is a rational function. Hence, for the system Σ_{Ham} to be non-autonomous, the Markov parameters $\widehat{C}\widehat{A}^\ell\widehat{B} \in \mathbb{R}$ must be equal to zero. From the proof of Statement 1) of Theorem 3.1 it is clear that if $\widehat{C}\widehat{A}^\ell\widehat{B} = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$, then $QA^kB = 0$ for all $k \in \mathbb{N} \cup \{0\}$. However, since the system is controllable, we must have $Q = 0$ if $QA^kB = 0$ for all $k \in \mathbb{N} \cup \{0\}$. This is a contradiction. Therefore, $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B}$ is a non-zero rational function and hence, is invertible. Using Proposition 2.2, we infer that Σ_{Ham} is autonomous. □

For the multi-input case Statement 3 is not valid. This is because for the multi-input case there can be systems with $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B} \neq 0$ but with $\mathtt{det}(\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B}) = 0$. We illustrate this with the help of Example 3.3.

Example 3.6. *Recall that the problem in Example 3.3 have*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On constructing $\widehat{A}, \widehat{B}$, and $\widehat{C}$ and computing $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B}$, we get

$$H(s) := \widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B} = \begin{bmatrix} \frac{1}{s^4-4s^2} & \frac{1}{s^3-4s} \\ \frac{-1}{s^3-4s} & \frac{-1}{s^2-4} \end{bmatrix} \tag{16}$$

Thus, $H(s)$ is not a zero matrix. However, $\mathtt{det}(H(s)) = 0$ and hence, $\widehat{C}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{B}$ is not invertible as a rational matrix. Therefore, Σ_{Ham} is not autonomous.

Interestingly, it has been established in [11] that in order to solve a single-input singular LQR problem one needs to use a proportional-derivative (PD) controller. The procedure to design the same is given in [11, Theorem 3].

Example 3.7. *Consider a system with state-space dynamics*

$$\frac{d}{dt}x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

For every initial condition x_0, find an input u that minimizes the functional

$$\int_0^\infty \left(x^T Qx\right) dt, \text{ where } Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The ARE corresponding to the CGCARE of this problem is the same as that in equation (13). The constrained equation in this case becomes

$$KB = 0 \Rightarrow \begin{bmatrix} k_2 & k_3 & k_5 \end{bmatrix}^T = 0 \tag{17}$$

Using equation (17) in equation (13), we have

$$\begin{bmatrix} 2(k_1+k_4) & k_4 & k_1+k_4+k_6 \\ k_4 & 0 & k_6 \\ k_1+k_4+k_6 & k_6 & 2k_4+1 \end{bmatrix} = 0 \tag{18}$$

Clearly, equation (18) is not solvable. Thus, the CGCARE does not admit a solution in this case. Hence, there exists no proportional state-feedback controller that solves the problem. However a PD state-feedback controller of the form $u = F_{\mathrm{p}}x + F_{\mathrm{d}}\frac{d}{dt}x$, where

$$F_{\mathrm{p}} = \begin{bmatrix} 2g & 0 & g \end{bmatrix}, F_{\mathrm{d}} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}, \text{ with } g \in \mathbb{R}$$

solves the problem (see [11] for a method to construct such controllers). The closed-loop system obtained on application of such a feedback is $(I_3 - BF_{\mathtt{d}})\frac{d}{dt}x = (A + BF_{\mathtt{p}})x$. *Here*

$$I_3 - BF_{\mathtt{d}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A + BF_{\mathtt{p}} = \begin{bmatrix} 1 & 0 & 1 \\ 1+2g & 0 & 1+g \\ 1 & 1 & 0 \end{bmatrix}$$

Note that $\mathtt{det}\left(s(I_3 - BF_{\mathtt{d}}) - (A + BF_{\mathtt{p}})\right) = -g(s+1)$. *Thus, if we chose any* $g \in \mathbb{R} \setminus 0$ *then* $\mathtt{det}\left(s(I_3 - BF_{\mathtt{d}}) - (A + BF_{\mathtt{p}})\right) \neq 0$*, i.e., the closed loop system is autonomous. Hence, for any value of* $g \in \mathbb{R} \setminus 0$*, we have a PD-controller that solves the singular LQR problem. Note that there are multiple PD-controllers that solve this problem.*

Observe that for the results in this paper to be true the system under consideration needs to be controllable but *not* necessarily observable. This becomes evident if we consider the system to be in the Kalman decomposition form. Without loss of generality, we assume the structure of (A, B, Q) to be of the following form:

$$A =: \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B =: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, Q =: \begin{bmatrix} 0 & 0 \\ 0 & Q_3 \end{bmatrix}, \tag{19}$$

where $A_{11} \in \mathbb{R}^{\mathtt{c}\times\mathtt{c}}$, $A_{22} \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}$, $B_1 \in \mathbb{R}^{\mathtt{c}\times\mathtt{m}}$, and $Q_3 = Q_3^T \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}$. From Proposition 3.2 it is clear that CGCARE is solvable if and only if $\widehat{C}\widehat{A}^{\ell}\widehat{B} = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$. Further, from the proof of Theorem 3.1 it is evident that $\widehat{C}\widehat{A}^{\ell}\widehat{B} = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$ implies that $QA^kB = 0 \Rightarrow Q_3A_{22}^kB_2 = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Since for a controllable system, (A, B) controllable $\Rightarrow$ (A_{22}, B_2) controllable, we cannot have $Q_3A_{22}^kB_2 = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Thus, even for a controllable and unobservable system, CGCARE corresponding to a zero input-cost won't have a solution. The next example illustrates this.

Example 3.8. *Find an input* u *for the system* $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$ *such that it minimizes the cost-functional* $\int_0^\infty (x^TQx)dt$*, where* $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $K := \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$*. Then observe that*

$$A^TK + KA + Q = 0 \Rightarrow \begin{bmatrix} -2k_1 & -k_2 \\ -k_2 & 1 \end{bmatrix} \neq 0$$

Thus, the CGCARE is not solvable in this example. This is indeed expected since the Hamiltonian system in this case is autonomous (observe that $\mathtt{det}(sE - H) = s^2 - 1$ *here).*

However, using the method described in [11], one can compute many PD-controllers that can solve the problem at hand. One such controller, for example, is $u = x_2 + \frac{d}{dt}x_2$.

Note that if we consider $B_2 = 0$ in equation (19), i.e., apart from (A, Q) unobservability, the system is uncontrollable as well, then $\widehat{A}^{\ell}\widehat{B} = \mathtt{col}(A_{11}^{\ell}B_1, 0_{\mathtt{qm}}, 0_{\mathtt{cm}}, 0_{\mathtt{qm}})$ for all $\ell \in \mathbb{N} \cup \{0\}$. Thus, $\widehat{C}\widehat{A}^{\ell}\widehat{B} = 0$ for all $\ell \in \mathbb{N} \cup \{0\}$. This however does not guarantee solvability of CGCARE. We illustrate this in the next example.

Example 3.9. *Find an input* u *for the system* $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$ *such that it minimizes the cost-functional* $\int_0^\infty (x^TQx)dt$*, where* $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $K := \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$*. Then observe that*

$$KB = 0 \Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

Thus, we have

$$A^TK + KA + Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$$

Thus, CGCARE is not solvable for this system. This shows that Proposition 3.2 is valid only for controllable systems.

The discussion above illustrates that the results in this paper hold for controllable systems only.

4. Conclusion

In this paper, we established that the limiting case of the cheap control problem ($\varepsilon = 0$) cannot be solved using a proportional state-feedback controller. To this end we showed that such a problem never admits solution to its corresponding CGCARE (Theorem 3.1). Interestingly, such problems may admit both autonomous and non-autonomous Hamiltonian systems (Example 3.4 and Example 3.6). However, if such a problem arises out of a single-input system, then the corresponding Hamiltonian system is always autonomous (Theorem 3.5). For single-input systems autonomy of the Hamiltonian system helps in explicit characterization of the trajectories of the Hamiltonian system. Such a characterization in turn aids us to characterize the optimal trajectories of a singular LQR problem [11, Theorem 1]. Similar to the single-input case, we think that for the multi-input case too autonomy of the Hamiltonian system can be similarly exploited. This is a matter of our future research. To summarize, the results in this paper establish that for LQR problems with $R = 0$, the results in [8] are not applicable. For the case when the singular LQR problem arises from a single-input system, the results in [11] can be used to design PD state-feedback controllers to solve the problem at hand. This bridges the gap between the results in [8] and [11].

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