A geometric characterization of the slow space of the Hamiltonian system arising from the singular LQR problem

Imrul Qais [*] **Debasattam Pal** [*] **Chayan Bhawal** [**]

[*] *Indian Institute of Technology Bombay, Mumbai, India (e-mail: imrul@ee.iitb.ac.in, debasattam@ee.iitb.ac.in).*
[**] *Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany (e-mail: bhawal@mpi-magdeburg.mpg.de).*

Abstract: In this paper we first characterize the slow space (see Definition 9) of a given state-space system. We provide this characterization in terms of an eigenspace of the corresponding Rosenbrock matrix pair. We also characterize the "good" slow space (see Definition 11) in terms of a stable eigenspace of the Rosenbrock matrix pair. Moreover, we show how the dimensions of these subspaces can be calculated from the determinant of the Rosenbrock matrix pencil. Then, we apply these results to the Hamiltonian system arising from the singular linear quadratic regulator (LQR) problem and explore a few interesting properties of the good slow space of this Hamiltonian system. Finally, we provide a feedback law to achieve the smooth optimal solutions.

Keywords: Matrix pencil, generalized eigenvalues and eigenspaces, Rosenbrock matrix pair, slow space, LQR problem, Hamiltonian system.

1. INTRODUCTION

Singular LQR problem is one of the classical problems in systems and control theory (Francis (1979), Hautus and Silverman (1983), Willems et al. (1986)). This problem is still an area of active research (Kalaimani et al. (2013), Reis et al. (2015), Ferrante and Ntogramatzidis (2018),Bhawal and Pal (2019)). The following is the formal statement of the infinite-horizon LQR problem:

Problem 1. Consider the stabilizable system defined by $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition $x(0) = x_0$, find an input $u(t)$ that minimizes the functional

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \qquad (1)$$

with $\lim_{t\to\infty} x(t) = 0$, where $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, S \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, such that $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right] \geqslant 0$.

Problem 1 is called a *regular* LQR problem if $R > 0$, and a *singular* LQR problem if $R \geqslant 0$ with R being singular. It is well-known in the literature that the regular LQR problem admits an algebraic Riccati equation (ARE) given as:

$$A^TK + KA + Q - (KB + S)R^{-1}(B^TK + S^T) = 0. \qquad (2)$$

If $K_{\mathtt{max}}$ is the maximal solution of equation (2), that is, $K_{\mathtt{max}} - K \geqslant 0$ for any other solution K of the ARE, then the LQR Problem 1 can be solved by using the feedback law $u = Fx$, where $F := -R^{-1}(B^TK_{\mathtt{max}} + S^T)$. Notice from equation (2) that the existence of an ARE crucially depends on the invertibility of R. Naturally, a singular LQR problem does not admit an ARE and consequently can not be solved using the feedback law as mentioned before. Hautus and Silverman (1983) deals with the solution of the singular LQR problem, but a feedback solution for the problem has not been provided there. In Reis et al. (2015), the notion of deflating subspaces has been used to provide a linear implicit control law which, unfortunately, often turns out to be not feedback implementable. The theory presented in Reis et al. (2015) assumes that the state and the input of the system are from the space of locally square-integrable functions. This assumption prevents the presence of impulses in the input and the states. Reis et al. (2015) also imposes a restriction on the initial condition of the system. Such a restriction on the initial condition is not desirable, because an initial condition of a system should ideally be free. For single-input systems, Bhawal and Pal (2019) provides a solution for any arbitrary initial condition. They also provide a PD feedback law for the optimal solution. Since the initial condition is free, the optimal trajectories corresponding to certain initial conditions are impulsive in nature. Hence, the function space assumed in Bhawal and Pal (2019) allows impulses in the input and the states. This solution is based on the notion of the *slow subspace* and the *fast subspace* of the *Hamiltonian system* arising from the singular LQR problem (Bhawal et al. (2019b)). We wish to extend the result based on the notion of slow and fast subspaces of the Hamiltonian system to the case of multi-input systems in our future research. Therefore, in this paper we characterize the slow space of the Hamiltonian system in terms of an eigenspace of the Hamiltonian matrix pencil. We also provide a static feedback law for the smooth optimal solutions.

We characterize the slow space of the Hamiltonian system in two steps. First, we characterize the slow space and the good slow space for the system $\Sigma : \frac{d}{dt}x = Ax + Bu$, $y = Cx + Du$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$. In Hautus and Silverman (1983), it has been shown that the slow space is the largest subspace $\mathcal{V}$ for which there exists $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C + DF)\mathcal{V} = 0$. The *good* slow space is the largest such subspace with an additional condition that $(A + BF)|_{\mathcal{V}}$ is Hurwitz. In this paper, we give an explicit characterization of the slow and the good slow spaces of the system, for the case when $\mathtt{p} = \mathtt{m}$ (for a Hamiltonian system, this condition is always satisfied), in terms of an eigenspace and a stable eigenspace of the corresponding Rosenbrock matrix pair $(\left[\begin{smallmatrix} I_{\mathtt{n}} & 0\\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right])$, respectively. We also show how to obtain the dimension of these spaces from the determinant of the Rosenbrock matrix pencil. Then, we apply these results to the Hamiltonian system to obtain its good slow space. We also show how the good slow spaces of the primal (see Definition 15) and the Hamiltonian are related.

The results presented in this paper are extensions of a few results presented in Bhawal et al. (2019b). In Bhawal et al. (2019b) the results have been proved for the single-input case only, whereas in this paper we provide the proofs for the multi-input case. Furthermore, in Bhawal et al. (2019b) the matrix D in the output equation of the system Σ has been assumed to be zero. But, in this paper we have not made such an assumption. Also, for the single-input case of the singular LQR problem $R = 0$, but in the multi-input case R is not-necessarily zero. Hence, the structure of the Hamiltonian system corresponding to the multi-input case is significantly different from that of the single-input case. These issues make the extension nontrivial.

The organization of the paper is as follows: In Section 2 we provide the notation used in this paper and discuss the necessary preliminaries. Section 3 deals with the characterization of the slow and the good slow spaces. Then, we apply these results to the Hamiltonian system corresponding to the singular LQR problem in Section 4. We conclude in Section 5 with some possible direction for future work.

2. NOTATION AND PRELIMINARIES

We first present the various notation used throughout this paper, then we discuss a few preliminaries required for the development of the theory presented in this paper.

2.1 Notation

The symbols $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ are used for the sets of real numbers, complex numbers, and natural numbers, respectively. $\mathbb{R}_+$ denotes the set of non-negative real numbers. We use the symbols $\overline{\mathbb{C}}_+$ and $\mathbb{C}_-$ for the closed right-half and the open left-half of the complex plane, respectively. The symbol $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}$. We use $\bullet$ when a dimension need not be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows. We use the symbol $I_{\mathtt{n}}$ for an $\mathtt{n}\times\mathtt{n}$ identity matrix and the symbol $0_{\mathtt{n},\mathtt{m}}$ for an $\mathtt{n}\times\mathtt{m}$ matrix with all entries zero. Symbol $\mathtt{col}(B_1, B_2,\ldots, B_{\mathtt{n}})$ represents a matrix of the form $[B_1^T \; B_2^T \; \cdots \; B_{\mathtt{n}}^T]^T$. The symbol $\mathtt{det}(A)$ represents the determinant of a square matrix A. Symbol $\mathtt{rank}\, A$ denotes the rank of a matrix A. We use the symbol $\mathtt{roots}(p(s))$ to denote the set of roots (over $\mathbb{C}$) of a polynomial $p(s)$ with real or complex coefficients (counted with multiplicity). The symbol $\mathtt{deg}(p(s))$ is used to denote the degree of the polynomial $p(s)$. The symbol $\sigma(\Gamma)$ denotes the set of eigenvalues of a square matrix Γ (counted with multiplicity). The symbol $|\Gamma|$ denotes the cardinality of a set Γ (counted with multiplicity). We use the symbols $A|_{\mathcal{S}}$ to denote the restriction of a matrix A to a subspace $\mathcal{S}$ (with respect to a suitable basis) and $\sigma(A|_{\mathcal{S}})$ to represent the set of eigenvalues of A restricted to the subspace $\mathcal{S}$. We use the symbol $\mathtt{dim}\,(\mathcal{S})$ to denote the dimension of a space $\mathcal{S}$. The symbol $\mathtt{img}\, A$ and $\mathtt{ker}\, A$ denote the image and nullspace of a matrix A, respectively. The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ is represented by the symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. The symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})\,|_{\mathbb{R}_+}$ represents the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}^{\mathtt{n}}$ that are restrictions of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$.

2.2 Regular matrix pencils and their canonical form

Linear matrix pencils and their eigenvectors are crucially used throughout this paper. Hence, we define eigenvalues and eigenvectors of a linear matrix pencil next.

Definition 2. Consider a regular matrix pencil $(sU_1 - U_2) \in \mathbb{R}[s]^{\mathtt{n}\times\mathtt{n}}$, i.e., $\mathtt{det}(sU_1 - U_2) \neq 0$. Let $\lambda \in \mathtt{roots}\,(\mathtt{det}(sU_1 - U_2))$. Then λ is called an *eigenvalue* of (U_1, U_2) and every nonzero vector $v \in \mathtt{ker}\,(\lambda U_1 - U_2)$ is called an *eigenvector* of the matrix pair (U_1, U_2) corresponding to the eigenvalue λ. Further, every nonzero vector $\tilde{v} \in \mathtt{ker}\,(\lambda U_1 - U_2)^i$, where $i \in \{2, 3, \ldots\}$, is called a *generalized eigenvector* of the matrix pair (U_1, U_2) corresponding to the eigenvalue λ.

We use the symbol $\sigma(U_1, U_2)$ to denote the set of eigenvalues of (U_1, U_2) (with $\lambda \in \sigma(U_1, U_2)$ included in the set as many times as its algebraic multiplicity).

In this paper, we extensively use one of the canonical forms of a linear matrix pencil (see Dai (1989) for more on different canonical forms). We review the result that leads to such a canonical form next (Dai, 1989, Lemma 1-2.2).

Proposition 3. A matrix pair (U_1, U_2) is regular, i.e., $\mathtt{det}(sU_1 - U_2) \neq 0$ if and only if there exist nonsingular matrices Z_1 and Z_2 such that $Z_1U_1Z_2 = \mathtt{diag}(I_{\mathtt{n}_1}, N)$ and $Z_1U_2Z_2 = \mathtt{diag}(U, I_{\mathtt{n}_2})$, where $\mathtt{n}_1 + \mathtt{n}_2 = \mathtt{n}, U \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$, and $N \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent.

A matrix pair (U_1, U_2) in the form $\left(\begin{bmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{bmatrix}, \begin{bmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{bmatrix}\right)$ is said to be in a canonical form. Further, note that $\mathtt{det}(sU_1 - U_2) = k \times \mathtt{det}(sI_{\mathtt{n}_1} - U)$, where $k \in \mathbb{R}\setminus\{0\}$. The following two lemmas provide some important properties related to the generalized eigenspaces of a matrix pair.

Lemma 4. Consider the matrix pair $\left(\begin{bmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{bmatrix}, \begin{bmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{bmatrix}\right)$, where $U \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$, and $N \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent. Let $\widetilde{W_1} \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{k}}, \widetilde{W_2} \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{k}}$, and $\widetilde{\Gamma} \in \mathbb{R}^{\mathtt{k}\times\mathtt{k}}$ be such that

$$\begin{bmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{bmatrix}\begin{bmatrix} \widetilde{W_1}\\ \widetilde{W_2}\end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{bmatrix}\begin{bmatrix} \widetilde{W_1}\\ \widetilde{W_2}\end{bmatrix}\widetilde{\Gamma}. \qquad (3)$$

Then, $\widetilde{W_2} = 0$.

Proof. From equation (3), we get $\widetilde{W_2} = N\widetilde{W_2}\widetilde{\Gamma}$. Now, if we keep substituting $\widetilde{W_2} = N\widetilde{W_2}\widetilde{\Gamma}$ on the right-hand side of the equation, then clearly we have $\widetilde{W_2} = N^i\widetilde{W_2}\widetilde{\Gamma}^i$ for all $i \in N$. But, N is a nilpotent matrix. Therefore. $\widetilde{W_2} = 0$.□

Lemma 5. Let the matrix pair (U_1, U_2) with $U_1, U_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be such that $\mathtt{degdet}(sU_1 - U_2) =: \mathtt{n}_1 \neq 0$. Then,

(1) There exist a full column-rank matrix $W \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}_1}$ and $\Gamma \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ with $\mathtt{det}(sI_{\mathtt{n}_1} - \Gamma) = \mathtt{det}(sU_1 - U_2)$ such that $U_2W = U_1W\Gamma$.
(2) There exist $T_1, T_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ non-singular such that $T_1U_1T_2 = \mathtt{diag}(I_{\mathtt{n}_1}, N)$ and $T_1U_2T_2 = \mathtt{diag}(\Gamma, I_{\mathtt{n}_2})$, where $\mathtt{n}_1 + \mathtt{n}_2 = \mathtt{n}$ and $N \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent.

Proof. (1): According to Proposition 3, there exist $Z_1, Z_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ non-singular such that $Z_1U_1Z_2 = \mathtt{diag}(I_{\mathtt{n}_1}, N)$ and $Z_1U_2Z_2 = \mathtt{diag}(U, I_{\mathtt{n}_2})$, where $U \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$, $\mathtt{det}(sI_{\mathtt{n}_1} - U) = \mathtt{det}(sU_1 - U_2)$, and $N \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent. Evidently, if a matrix $\Gamma \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ is similar to the matrix U, then there exists $W_1 \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ non-singular such that $UW_1 = W_1\Gamma$. Then, clearly, $\begin{bmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{bmatrix}\begin{bmatrix} W_1\\ 0\end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{bmatrix}\begin{bmatrix} W_1\\ 0\end{bmatrix}\Gamma$ holds. Consequently, the equation $U_2W = U_1W\Gamma$ is satisfied, where $W := Z_2^{-1}\begin{bmatrix} W_1\\ 0\end{bmatrix}$. Since $W_1 \in$

$\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ is non-singular, we must have that $W \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}_1}$ is full-column-rank. Also, $\mathtt{det}(sI_{\mathtt{n}_1} - \Gamma) = \mathtt{det}(sI_{\mathtt{n}_1} - U) = \mathtt{det}(sU_1 - U_2)$. This completes the proof of Statement (1). (2): From the proof of Statement (1) we have $W_1^{-1}UW_1 = \Gamma$. Define $\widetilde{T} := \mathtt{diag}(W_1, I_{\mathtt{n}_2}), T_1 := \widetilde{T}^{-1}Z_1$, and $T_2 := Z_2\widetilde{T}$. It is evident that T_1 and T_2 are non-singular. Further, it is easy to verify that $T_1U_1T_2 = \mathtt{diag}(I_{\mathtt{n}_1}, N)$ and $T_1U_2T_2 = \mathtt{diag}(\Gamma, I_{\mathtt{n}_2})$. □

2.3 (A, B)-invariant subspace

Notion of (A, B)-invariance is pivotal for the theory developed in this paper. Thus, we present the formal definition of (A, B)-invariance next.

Definition 6. Consider $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. A subspace $\mathcal{V} \subseteq \mathbb{R}^{\mathtt{n}}$ is said to be (A, B)-invariant if there exists $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

The following Proposition (see (Wonham, 1985, Lemma 4.2)) provides us with a method to determine whether a given subspace is (A, B)-invariant.

Proposition 7. A subspace $\mathcal{V} \subseteq \mathbb{R}^{\mathtt{n}}$ is (A, B)-invariant if and only if $A\mathcal{V} \subseteq \mathcal{V} + \mathtt{img}B$.

Corresponding to an (A, B)-invariant subspace $\mathcal{V}$, we define the set

$$\mathbb{F}(\mathcal{V}) := \{F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}} \mid (A + BF)\mathcal{V} \subseteq \mathcal{V}\}.$$

Suppose $\mathcal{V}$ is an (A, B)-invariant subspace, and assume that $\mathcal{S} \subseteq \mathcal{V}$ is an (A, B)-invariant subspace as well. Then, the following Proposition from (Wonham, 1985, Lemma 5.7) shows a relation between $\mathbb{F}(\mathcal{S})$ and $\mathbb{F}(\mathcal{V})$.

Proposition 8. Assume that $\mathcal{V}$ and $\mathcal{S}$ are (A, B)-invariant subspaces such that $\mathcal{S} \subseteq \mathcal{V}$. If $F_0 \in \mathbb{F}(\mathcal{S})$, then there exists $F \in \mathbb{F}(\mathcal{V}) \cap \mathbb{F}(\mathcal{S})$ such that $F|_{\mathcal{S}} = F_0|_{\mathcal{S}}$.

2.4 The slow subspace

Definition 9. Consider the system $\frac{d}{dt}x(t) = Ax(t)+Bu(t)$, $y(t) = Cx(t) + Du(t)$. A state $x_0 \in \mathbb{R}^{\mathtt{n}}$ is called *weakly unobservable* if there exists an input $u \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}$ such that $y(t; x_0, u) \equiv 0$ for all $t \geqslant 0$, where $y(t; x_0, u)$ is the output of the system corresponding to the initial condition x_0 and the input $u(t)$. The collection of all such weakly unobservable states is called the *weakly unobservable subspace* or the *slow space* of the state-space and is denoted by $\mathcal{O}_w$.

The following property of the slow space is crucially used in this paper (see (Hautus and Silverman, 1983, Theorem 3.10)).

Proposition 10. The slow space $\mathcal{O}_w$ is the largest subspace $\mathcal{V}$ of the state-space for which there exists a feedback $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that

$$(A + BF)\mathcal{V} \subseteq \mathcal{V} \text{ and } (C + DF)\mathcal{V} = 0. \tag{4}$$

In other words, if $\mathcal{V}$ is any subspace that satisfies equation(4), then $\mathcal{V} \subseteq \mathcal{O}_w$.

An important subspace of the slow space, $\mathcal{O}_w$ is the "good" slow space $\mathcal{O}_{wg}$. We formally define this subspace next.

Definition 11. The good slow space $\mathcal{O}_{wg}$ is the largest subspace $\mathcal{V}$ of the state-space for which there exists a feedback $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that

$$\begin{aligned} &(A + BF)\mathcal{V} \subseteq \mathcal{V},\ (C + DF)\mathcal{V} = 0, \\ \text{and}\qquad &\sigma((A + BF)|_{\mathcal{V}}) \subseteq \mathbb{C}_-. \end{aligned} \tag{5}$$

In other words, if $\mathcal{V}$ is any subspace that satisfies equation (5), then $\mathcal{V} \subseteq \mathcal{O}_{wg}$.

3. CHARACTERIZATION OF THE SLOW SPACE

Consider the system Σ given by the input-state-output representation

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t), \tag{6}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, and $D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$.Corresponding to this system, we define the matrices

$$U_1 := \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})} \text{ and } U_2 := \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{7}$$

The pair (U_1, U_2) is called the Rosenbrock matrix pair and the matrix $(sU_1 - U_2)$ is the Rosenbrock matrix pencil corresponding to the system Σ. Throughout this paper we assume that the matrix pencil $(sU_1 - U_2)$ is regular, i.e., $\mathtt{det}(sU_1 - U_2) \neq 0$.
We present this section in two parts. In the first part, we characterize the slow space of Σ, and in the second part we characterize the good slow space of Σ.

3.1 Characterization of the slow space in terms of an eigenspace of the Rosenbrock matrix pair

In the following theorem we characterize the slow space, $\mathcal{O}_w$, of the system Σ in terms of the generalized eigenspace of the Rosenbrock matrix pair (U_1, U_2). This theorem also provides us with the dimension of the subspace $\mathcal{O}_w$.

Theorem 12. Consider the system Σ defined in equation (6) and the corresponding Rosenbrock matrix pair (U_1, U_2) as defined in equation (7). Assume that $\mathtt{det}(sU_1 - U_2) \neq 0$ and $\mathtt{degdet}(sU_1 - U_2) =: \mathtt{n_s}$. Let $V_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_2 \in \mathbb{R}^{\mathtt{m}\times\mathtt{n_s}}$ be such that $\mathtt{col}(V_1, V_2)$ is full column-rank and

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{U_2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix}}_{U_1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} J, \tag{8}$$

where $J \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ and $\mathtt{det}(sI_{\mathtt{n_s}} - J) = \mathtt{det}(sU_1 - U_2)$. (Such V_1, V_2, and J exist due to Lemma 5.) Let $\mathcal{O}_w$ be the slow space of Σ. Then, the following statements hold:

(1) V_1 is full column-rank.
(2) $\mathcal{O}_w = \mathtt{img}V_1$.
(3) $\mathtt{dim}(\mathcal{O}_w) = \mathtt{n_s}$.

Proof. (1): To the contrary assume that V_1 is not full column-rank. Then, there exists a non-singular matrix $T \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ such that $\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] T = \left[\begin{smallmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{smallmatrix}\right]$, where $V_{11} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}_1}$, $\mathtt{n}_1 := \mathtt{rank}V_1$, and $V_{22} \in \mathbb{R}^{\mathtt{m}\times(\mathtt{n_s}-\mathtt{n}_1)}$. Define $\widehat{J} := T^{-1}JT$. So, from equation (8) it follows that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \underbrace{\begin{bmatrix} \widehat{J}_{11} & \widehat{J}_{12} \\ \widehat{J}_{21} & \widehat{J}_{22} \end{bmatrix}}_{\widehat{J}}, \tag{9}$$

where the partition in $\widehat{J}$ conforms to the partition in $\left[\begin{smallmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{smallmatrix}\right]$. Equation (9) can equivalently be written as

$$\begin{aligned} AV_{11} + BV_{21} = V_{11}\widehat{J}_{11},\ BV_{22} = V_{11}\widehat{J}_{12}, \\ CV_{11} + DV_{21} = 0, \text{ and } DV_{22} = 0. \end{aligned} \tag{10}$$

From equation (10) it is clear that the $\widehat{J}_{21}$ and $\widehat{J}_{22}$ blocks in $\widehat{J}$ are free; in particular, $\widehat{J}_{21} = 0 \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{n}_1)\times\mathtt{n}_1}$ and $\widehat{J}_{22} = 0 \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{n}_1)\times(\mathtt{n_s}-\mathtt{n}_1)}$ also satisfy equation (9). In that case $0 \in \sigma(\widehat{J})$. This is a contradiction, because $\widehat{J} = T^{-1}JT$

and thus $\mathtt{det}(sI_{\mathtt{n_s}} - \widehat{J}) = \mathtt{det}(sI_{\mathtt{n_s}} - J) = \mathtt{det}(sU_1 - U_2)$. But, we have assumed that $\mathtt{det}(sU_1 - U_2) \neq 0$. So, $\mathtt{det}(sI_{\mathtt{n_s}} - \widehat{J}) \neq 0$. Hence, $0 \notin \sigma(\widehat{J})$. This implies that our assumption that V_1 is not full column-rank cannot be true. Hence, V_1 is full column-rank.

(2): From Statement (1) of this theorem we get that V_1 is full column-rank. Thus, there exists $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $V_2 = FV_1$. So, equation (8) can also be written as

$$(A + BF)V_1 = V_1 J \text{ and } (C + DF)V_1 = 0. \quad (11)$$

From equation (11) and Proposition 10 it follows that $\mathtt{img}V_1 \subseteq \mathcal{O}_w$. Next, we prove that $\mathtt{img}V_1 = \mathcal{O}_w$. To the contrary assume that $\mathtt{img}V_1 \neq \mathcal{O}_w$. Thus, there exists a non-trivial subspace $\mathcal{V}_{\mathtt{e}}$ such that $\mathtt{img}V_1 \oplus \mathcal{V}_{\mathtt{e}} = \mathcal{O}_w$. Define $\ell := \mathtt{dim}\mathcal{V}_{\mathtt{e}}$ and let $V_{\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times\ell}$ be full column-rank such that $\mathtt{img}V_{\mathtt{e}} = \mathcal{V}_{\mathtt{e}}$. Now, from Proposition 8 and Proposition 10 it is evident that there exists $F_{\mathtt{e}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $F_{\mathtt{e}}|_{\mathtt{img}V_1} = F|_{\mathtt{img}V_1}$ and

$$(A + BF_{\mathtt{e}})\mathcal{O}_w \subseteq \mathcal{O}_w \text{ and } (C + DF_{\mathtt{e}})\mathcal{O}_w = 0. \quad (12)$$

Thus, from equation (11) it follows that

$$(A + BF_{\mathtt{e}})V_1 = V_1 J \text{ and } (C + DF_{\mathtt{e}})V_1 = 0. \quad (13)$$

Also, since $\mathtt{img}V_{\mathtt{e}} \subseteq \mathcal{O}_w$, from equation (12) it is clear that there exist $T_1 \in \mathbb{R}^{\mathtt{n_s}\times\ell}$ and $T_{\mathtt{e}} \in \mathbb{R}^{\ell\times\ell}$ such that

$$(A + BF_{\mathtt{e}})V_{\mathtt{e}} = [V_1 \ \ V_{\mathtt{e}}] \begin{bmatrix} T_1 \\ T_{\mathtt{e}} \end{bmatrix} \text{ and } (C + DF_{\mathtt{e}})V_{\mathtt{e}} = 0. \quad (14)$$

Recall that $F_{\mathtt{e}}|_{\mathtt{img}V_1} = F|_{\mathtt{img}V_1}$. Thus, $F_{\mathtt{e}}V_1 = FV_1 = V_2$. So, combining equation (12) and equation (14) together, we get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{bmatrix} \begin{bmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{bmatrix}, \quad (15)$$

where $V_{2\mathtt{e}} := F_{\mathtt{e}}V_{\mathtt{e}}$. Considering the fact that $\mathtt{degdet}(sU_1 - U_2) = \mathtt{n_s}$, from Statement (2) of Lemma 5, it is clear that there exist nonsingular matrices $Y, Z \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}$ such that

$$U_1 = Y \left[\begin{smallmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{smallmatrix}\right] Z \text{ and } U_2 = Y \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] Z, \quad (16)$$

where $N \in \mathbb{R}^{(\mathtt{n}+\mathtt{m}-\mathtt{n_s})\times(\mathtt{n}+\mathtt{m}-\mathtt{n_s})}$ is a nilpotent matrix. Thus, using equation (16) in equation (15) we further get that

$$\begin{aligned} Y \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] Z \left[\begin{smallmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{smallmatrix}\right] &= Y \left[\begin{smallmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{smallmatrix}\right] Z \left[\begin{smallmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{smallmatrix}\right] \left[\begin{smallmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{smallmatrix}\right] \\ \text{or, } \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] Z \left[\begin{smallmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{smallmatrix}\right] &= \left[\begin{smallmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{smallmatrix}\right] Z \left[\begin{smallmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{smallmatrix}\right] \left[\begin{smallmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{smallmatrix}\right]. \quad (17) \end{aligned}$$

Define $Z \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] =: \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix}$ and $Z \left[\begin{smallmatrix} V_{\mathtt{e}} \\ V_{2\mathtt{e}} \end{smallmatrix}\right] =: \begin{bmatrix} \widehat{V}_{\mathtt{e}} \\ \widehat{V}_{2\mathtt{e}} \end{bmatrix}$, where $\widehat{V}_1 \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ and $\widehat{V}_{\mathtt{e}} \in \mathbb{R}^{\mathtt{n_s}\times\ell}$. We rewrite equation (17) as

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{V}_1 & \widehat{V}_{\mathtt{e}} \\ \widehat{V}_2 & \widehat{V}_{2\mathtt{e}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \widehat{V}_1 & \widehat{V}_{\mathtt{e}} \\ \widehat{V}_2 & \widehat{V}_{2\mathtt{e}} \end{bmatrix} \begin{bmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{bmatrix}. \quad (18)$$

Thus, from Lemma 4, we have $\begin{bmatrix} \widehat{V}_2 & \widehat{V}_{2\mathtt{e}} \end{bmatrix} = 0$. Since Z is non-singular, $\mathtt{rank} \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} = \mathtt{rank} \begin{bmatrix} \widehat{V}_1 \\ 0 \end{bmatrix} = \mathtt{rank} \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] = \mathtt{n_s}$. Thus, $\widehat{V}_1$ is non-singular, which further implies that $\mathtt{img}\widehat{V}_{\mathtt{e}} \subseteq \mathtt{img}\widehat{V}_1$. As a consequence

$$\mathtt{img} \begin{bmatrix} \widehat{V}_{\mathtt{e}} \\ \widehat{V}_{2\mathtt{e}} \end{bmatrix} = \mathtt{img} \begin{bmatrix} \widehat{V}_{\mathtt{e}} \\ 0 \end{bmatrix} \subseteq \mathtt{img} \begin{bmatrix} \widehat{V}_1 \\ 0 \end{bmatrix} \Rightarrow \mathtt{img} \left[\begin{smallmatrix} V_{\mathtt{e}} \\ V_{2\mathtt{e}} \end{smallmatrix}\right] \subseteq \mathtt{img} \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right].$$

Therefore $\mathtt{img}V_{\mathtt{e}} \subseteq \mathtt{img}V_1$. This is a contradiction. Therefore there does not exist any nontrivial subspace $\mathcal{V}_{\mathtt{e}}$ such that $\mathtt{img}V_1 \oplus \mathcal{V}_{\mathtt{e}} = \mathcal{O}_w$. This, again, is a contradiction to the assumption that $\mathtt{img}V_1 \neq \mathcal{O}_w$. Hence, $\mathcal{O}_w = \mathtt{img}V_1$.

(3): Since $\mathtt{rank}V_1 = \mathtt{n_s}$, from Statement (2) of this theorem it directly follows that $\mathtt{dim}(\mathcal{O}_w) = \mathtt{n_s}$. □

3.2 Characterization of the good slow space in terms of a stable eigenspace of the Rosenbrock matrix pair

In this section we characterize the good slow space of the state-space system Σ given by equation (6). Notice that, we can partition $\sigma(J)$ as $\sigma(J) = \sigma_{\mathtt{g}}(J) \cup \sigma_{\mathtt{b}}(J)$, where $\sigma_{\mathtt{g}}(J) \subseteq \mathbb{C}_-$ and $\sigma_{\mathtt{b}}(J) \subseteq \overline{\mathbb{C}}_+$. Define $\mathtt{n_g} := |\sigma_{\mathtt{g}}(J)|$. Clearly, there exists a non-singular matrix $T \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ such that $T^{-1}JT = \begin{bmatrix} J_{\mathtt{g}} & 0 \\ 0 & J_{\mathtt{b}} \end{bmatrix}$, where $J_{\mathtt{g}} \in \mathbb{R}^{\mathtt{n_g}\times\mathtt{n_g}}$, $J_{\mathtt{b}} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{n_g})\times(\mathtt{n_s}-\mathtt{n_g})}$, $\sigma(J_{\mathtt{g}}) = \sigma_{\mathtt{g}}(J)$, and $\sigma(J_{\mathtt{b}}) = \sigma_{\mathtt{b}}(J)$. Define

$$\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] T =: \begin{bmatrix} V_{1\mathtt{g}} & V_{1\mathtt{b}} \\ V_{2\mathtt{g}} & V_{2\mathtt{b}} \end{bmatrix}, \quad (19)$$

where $V_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_g}}$ and $V_{2\mathtt{b}} \in \mathbb{R}^{\mathtt{m}\times(\mathtt{n_s}-\mathtt{n_g})}$. So, from equation (8) it follows that

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} T &= \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} TT^{-1}JT \\ \text{or, } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{1\mathtt{g}} & V_{1\mathtt{b}} \\ V_{2\mathtt{g}} & V_{2\mathtt{b}} \end{bmatrix} &= \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1\mathtt{g}} & V_{1\mathtt{b}} \\ V_{2\mathtt{g}} & V_{2\mathtt{b}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & 0 \\ 0 & J_{\mathtt{b}} \end{bmatrix}. \quad (20) \end{aligned}$$

In the following lemma we show that the good slow space, $\mathcal{O}_{wg}$ of Σ is given by the subspace $\mathtt{img}V_{1\mathtt{g}}$.

Lemma 13. Consider the system Σ and the corresponding Rosenbrock matrix pair (U_1, U_2) as defined in equation (6) and equation (7), respectively. Assume that $\mathtt{det}(sU_1 - U_2) \neq 0$. Consider the matrix $V_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_g}}$ as defined in equation (19) and $\mathtt{n_g} = |\sigma_{\mathtt{g}}(J)|$. Then, the following statements hold:

(1) $V_{1\mathtt{g}}$ is full column-rank.
(2) $\mathcal{O}_{wg} = \mathtt{img}V_{1\mathtt{g}}$.
(3) $\mathtt{dim}(\mathcal{O}_{wg}) = \mathtt{n_g}$.

Proof. (1): From Statement (1) of Theorem 12, we know that V_1 is full column-rank. Now, since T is non-singular, it is evident that $V_1T = [V_{1\mathtt{g}} \ V_{1\mathtt{b}}]$ is full column-rank. Consequently, $V_{1\mathtt{g}}$ is full column-rank.

(2): Since $V_{1\mathtt{g}}$ is full column-rank, there exists $F_{\mathtt{g}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $V_{2\mathtt{g}} = F_{\mathtt{g}}V_{1\mathtt{g}}$. Thus, from equation (20) it follows that $(A + BF_{\mathtt{g}})V_{1\mathtt{g}} = V_{1\mathtt{g}}J_{\mathtt{g}}$ and $(C + DF_{\mathtt{g}})V_{1\mathtt{g}} = 0$. Recall that $\sigma(J_{\mathtt{g}}) \subseteq \mathbb{C}_-$. Thus, $\mathtt{img}V_{1\mathtt{g}} \subseteq \mathcal{O}_{wg}$. Now, to the contrary, we assume that $\mathtt{img}V_{1\mathtt{g}} \neq \mathcal{O}_{wg}$. So, there exists a non-trivial subspace $\mathcal{V}_{\mathtt{eg}}$ such that $\mathtt{img}V_{1\mathtt{g}} \oplus \mathcal{V}_{\mathtt{eg}} = \mathcal{O}_{wg}$. Define $\mathtt{n_{eg}} := \mathtt{dim}\mathcal{V}_{\mathtt{eg}}$ and let $V_{\mathtt{eg}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_{eg}}}$ be such that $\mathtt{img}V_{\mathtt{eg}} = \mathcal{V}_{\mathtt{eg}}$. Next, by Definition 11, there exists $F_{\mathtt{eg}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF_{\mathtt{eg}})\mathcal{O}_{wg} \subseteq \mathcal{O}_{wg}$, $(C + DF_{\mathtt{eg}}) = 0$, $\sigma((A+BF_{\mathtt{eg}})|_{\mathcal{O}_{wg}}) \subseteq \mathbb{C}_-$, and $F_{\mathtt{eg}}|_{\mathtt{img}V_{1\mathtt{g}}} = F_{\mathtt{g}}|_{\mathtt{img}V_{1\mathtt{g}}}$. Thus, there exist $T_1 \in \mathbb{R}^{\mathtt{n_g}\times\mathtt{n_{eg}}}$ and $T_2 \in \mathbb{R}^{\mathtt{n_{eg}}\times\mathtt{n_{eg}}}$ such that

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{U_2} \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} = \underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix}}_{U_1} \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}, \quad (21)$$

where $\sigma(T_2) \subseteq \mathbb{C}_-$, $V_{2\mathtt{eg}} := F_{\mathtt{eg}}V_{\mathtt{eg}}$, and $V_{2\mathtt{g}} = F_{\mathtt{g}}V_{1\mathtt{g}} = F_{\mathtt{eg}}V_{1\mathtt{g}}$ ($\because F_{\mathtt{eg}}|_{\mathtt{img}V_{1\mathtt{g}}} = F_{\mathtt{g}}|_{\mathtt{img}V_{1\mathtt{g}}}$). Now, similar to the proof of Statement (2) of Theorem 12, we use equation (16) in equation (21) to obtain

$$\left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] Z \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} = \left[\begin{smallmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{smallmatrix}\right] Z \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}. \quad (22)$$

Define $Z \begin{bmatrix} V_{1\mathtt{g}} \\ V_{2\mathtt{g}} \end{bmatrix} =: \begin{bmatrix} \widehat{V}_{1\mathtt{g}} \\ \widehat{V}_{2\mathtt{g}} \end{bmatrix}$ and $Z \begin{bmatrix} V_{\mathtt{eg}} \\ V_{2\mathtt{eg}} \end{bmatrix} =: \begin{bmatrix} \widehat{V}_{\mathtt{eg}} \\ \widehat{V}_{2\mathtt{eg}} \end{bmatrix}$, where $\widehat{V}_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_g}}$ and $\widehat{V}_{\mathtt{eg}} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_{eg}}}$. Thus, from equation (22), we get

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{\mathtt{eg}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{eg}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{\mathtt{eg}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}. \quad (23)$$

From Lemma 4, it follows that $\begin{bmatrix} \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{eg}} \end{bmatrix} = 0$. Thus, equation (23) reduces to

$$J \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{\mathtt{eg}} \end{bmatrix} = \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}. \quad (24)$$

From equation (24), it is evident that $\sigma(J_{\mathtt{g}}) \cup \sigma(T_2) \subseteq \sigma(J)$. But, we have assumed that $\sigma(J) \cap \mathbb{C}_- = \sigma(J_{\mathtt{g}})$. Hence, $\sigma(T_2) \subseteq \overline{\mathbb{C}}_+$. This is a contradiction. Accordingly, there does not exist any non-trivial subspace $\mathcal{V}_{\mathtt{eg}}$ such that $\mathtt{img} V_{1\mathtt{g}} \oplus \mathcal{V}_{\mathtt{eg}} = \mathcal{O}_{wg}$. Hence, $\mathtt{img} V_{1\mathtt{g}} = \mathcal{O}_{wg}$.

(3): Since $V_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n} \times \mathtt{n_g}}$, from Statement (1) and Statement (2) of this theorem, it follows that $\mathtt{dim}(\mathcal{O}_{wg}) = \mathtt{n_g}$. $\square$

4. APPLICATION TO THE HAMILTONIAN SYSTEM ARISING FROM THE SINGULAR LQR PROBLEM

In this section, we apply the results developed in Section 3 to a special system, namely the Hamiltonian System arising from the linear quadratic regulator (LQR) problem (Problem 1). Recall that, the LQR problem is called a regular LQR problem when R is non-singular, and it is called a singular LQR problem when R is singular. Since, $R \geqslant 0$, there exists an orthogonal $U \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}$ such that $U^T R U = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{R} \end{bmatrix}$, where $\widehat{R} \in \mathbb{R}^{\mathtt{r} \times \mathtt{r}}$ and $\mathtt{r} := \mathtt{rank} R$. If we define $BU =: [\, B_1 \; B_2 \,]$ and $SU =: [\, S_1 \; S_2 \,]$, where $B_2, S_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{r}}$, then we have that $S_1 = 0$ (see (Bhawal et al., 2019a, Lemma 1)). Hence, without loss of generality, any singular LQR problem can be written as:

Problem 14. Consider the stabilizable system given by

$$\frac{d}{dt} x(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \quad (25)$$

where $A \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$, $B_1 \in \mathbb{R}^{\mathtt{n} \times (\mathtt{m}-\mathtt{r})}$, and $B_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{r}}$. Then, for every initial condition $x(0) = x_0$, find an input $u(t) := \left[\begin{smallmatrix} u_1(t) \\ u_2(t) \end{smallmatrix}\right]$ that minimizes the functional

$$J(x_o, u) := \int_0^\infty \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt, \quad (26)$$

with $\lim_{t \to \infty} x(t) = 0$, $Q \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}, S_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{r}}, \widehat{R} \in \mathbb{R}^{\mathtt{r} \times \mathtt{r}}$, $\left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{smallmatrix}\right] \geqslant 0$, and $\widehat{R} > 0$.

A Cholesky factorization of the cost matrix $\left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{smallmatrix}\right]$ gives us an auxiliary output equation for the system defined in equation (25). We call this system, the *primal* for the LQR Problem 14. The following is the formal definition of the primal system for Problem 14.

Definition 15. Consider the LQR Problem 14. Let

$$\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} = \begin{bmatrix} C^T \\ 0 \\ D_2^T \end{bmatrix} [\, C \; 0 \; D_2 \,], \quad (27)$$

where $C \in \mathbb{R}^{\mathtt{p} \times \mathtt{n}}$, $D_2 \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$, and $\mathtt{p} := \mathtt{rank} \left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{smallmatrix}\right]$. Define the system $\Sigma_{pr} : \frac{d}{dt} x(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t)$ and $y(t) = Cx(t) + D_2 u_2(t)$. We call the system Σ_{pr}, the *primal* for the LQR Problem 14.

As mentioned before, another important system arising from an LQR problem is the Hamiltonian system. We obtain this system using Pontryagin's maximum principle (PMP) to Problem 14 [1]:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B_1 & B_2 \\ -Q & -A^T & 0 & -S_2 \\ 0 & B_1^T & 0 & 0 \\ S_2^T & B_2^T & 0 & \widehat{R} \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \end{bmatrix}, \quad (28)$$

where $E \in \mathbb{R}^{(\mathtt{n}+\mathtt{m}) \times (\mathtt{n}+\mathtt{m})}$ is partitioned conforming to the partition in H. $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$ is called the state-costate pair. It follows from Pontryagin's maximum principle that if (x^*, u^*) is an optimal trajectory of the primal Σ_{pr}, then there exists z^* such that (x^*, z^*, u^*) belongs to the Hamiltonian system. Hence, the trajectories of the Hamiltonian system are of special interest. Recall that, $\widehat{R}$ is non-singular, and hence u_2 can be eliminated from equation (28) to obtain

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_{\mathtt{r}}} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0 \end{bmatrix}}_{H_{\mathtt{r}}} \begin{bmatrix} x \\ z \\ u_1 \end{bmatrix}, \quad (29)$$

where $\widetilde{A} := A - B_2 \widehat{R}^{-1} S_2^T$, $\widetilde{Q} := Q - S_2 \widehat{R}^{-1} S_2^T$, $A_z := B_2 \widehat{R}^{-1} B_2^T$, and $\widetilde{B} := B_1$. This system is called the *reduced Hamiltonian system.* Since $\left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{smallmatrix}\right] \geqslant 0$, by the notion of Schur complement it is evident that $\widetilde{Q} = Q - S_2 \widehat{R}^{-1} S_2^T \geqslant 0$. Throughout this paper, we assume that $\mathtt{det}(sE_{\mathtt{r}} - H_{\mathtt{r}}) \neq 0$. Notice that the reduced Hamiltonian system admits an output-nulling representation, $\Sigma_{\mathtt{Ham}}$ given by

$$\frac{d}{dt} \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] = \underbrace{\begin{bmatrix} \widetilde{A} & -A_z \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}}_{A_{\mathtt{r}}} \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] + \underbrace{\begin{bmatrix} \widetilde{B} \\ 0 \end{bmatrix}}_{B_{\mathtt{r}}} u_1 \text{ and } 0 = \underbrace{\begin{bmatrix} 0 & \widetilde{B}^T \end{bmatrix}}_{C_{\mathtt{r}}} \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]. \quad (30)$$

It turns out that the good slow space of $\Sigma_{\mathtt{Ham}}$ is pivotal in solving the LQR Problem 14 (see Bhawal et al. (2019b), Bhawal et al. (2019b)). Thus, in this section we compute the good slow space of $\Sigma_{\mathtt{Ham}}$. Clearly, $(E_{\mathtt{r}}, H_{\mathtt{r}})$ as defined in equation (29) is the Rosenbrock matrix pair of $\Sigma_{\mathtt{Ham}}$. Say, $\Lambda := \sigma(E_{\mathtt{r}}, H_{\mathtt{r}}) \cap \mathbb{C}_-$, $\mathtt{n_s} := |\Lambda|$, $V_1, V_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{n_s}}$, and $V_3 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r}) \times \mathtt{n_s}}$ be such that the columns of the matrix $V_{\mathtt{e}} := \left[\begin{smallmatrix} V_1 \\ V_2 \\ V_3 \end{smallmatrix}\right]$ form a basis for the $\mathtt{n_s}$-dimensional stable eigenspace of $(E_{\mathtt{r}}, H_{\mathtt{r}})$, i.e.,

$$\underbrace{\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0 \end{bmatrix}}_{H_{\mathtt{r}}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{V_{\mathtt{e}}} = \underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_{\mathtt{r}}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} J, \quad (31)$$

where $J \in \mathbb{R}^{\mathtt{n_s} \times \mathtt{n_s}}, \sigma(J) = \Lambda$. Thus, we can directly apply Lemma 13 to infer that the good slow space $\mathcal{O}_{wg}$ of $\Sigma_{\mathtt{Ham}}$ is given by $\mathcal{O}_{wg} = \mathtt{img} \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$. How the subspace $\mathcal{O}_{wg}$ can be used to solve the regular LQR problem is well-known in the literature (see (Ionescu et al., 1999, Chapter 5)).

[1] It should ne noted that if the system starts from an arbitrary initial condition, then the optimal control for the singular LQR problem is impulsive in nature and hence PMP becomes inapplicable there. But, in this paper we deal with the initial conditions for which the system admits only smooth optimal trajectories. Hence, we may apply PMP here. However, in order to solve the problem for an arbitrary initial condition the system given by equation (28) is crucial even though PMP is not applicable (Bhawal et al. (2019b), Bhawal and Pal (2019)).

In Bhawal and Pal (2019) $\mathcal{O}_{wg}$ has been used to solve the singular LQR problem for the single-input case. How to use this subspace to solve the singular LQR problem for the multi-input case is a matter of our future research.

Next, we divide this section in two parts to explore some interesting properties of $\mathcal{O}_{wg}$. We first show a relation between the good slow space ($\mathcal{V}_g$) of the primal Σ_{pr} and the subspace $\mathcal{O}_{wg}$. In the second part, we show that the subspace $\mathtt{img}V_{\mathtt{e}}$ is *disconjugate* (see Definition 18).

4.1 Relation between the spaces $\mathcal{V}_g$ and $\mathcal{O}_{wg}$

The following lemma is crucially used to establish a relation between $\mathcal{V}_g$ and $\mathcal{O}_{wg}$.

Lemma 16. Consider the LQR Problem 14 and the corresponding primal Σ_{pr} as defined in Definition 15. Further define the system Σ_{aux} : $\frac{d}{dt}x(t) = \widetilde{A}x(t) + \widetilde{B}u(t)$, $y(t) = \widetilde{C}x(t)$, where $\widetilde{A}, \widetilde{B}$ are as defined in equation (29) and $\widetilde{C} := C - D_2\widehat{R}^{-1}S_2^T$. Let $\mathcal{V}_g$ and $\mathcal{W}_g$ be the good slow spaces of Σ_{pr} and Σ_{aux}, respectively. Then, $\mathcal{V}_g = \mathcal{W}_g$.

Proof. We prove this lemma in two steps. First we show that $\mathcal{W}_g \subseteq \mathcal{V}_g$, and then we show that $\mathcal{V}_g \subseteq \mathcal{W}_g$.
($\mathcal{W}_g \subseteq \mathcal{V}_g$): Say $\mathtt{dim}(\mathcal{W}_g) =: \mathtt{g}_1$ and $W_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}_1}$ be such that $\mathcal{W}_g = \mathtt{img}W_g$. Clearly, there exist $F \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ and $J_1 \in \mathbb{R}^{\mathtt{g}_1\times\mathtt{g}_1}$ such that $(\widetilde{A}+\widetilde{B}F)W_g = W_gJ_1$ and $\widetilde{C}W_g = 0$, where $\sigma(J_1) \subseteq \mathbb{C}_-$. Hence, from definition of $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$, it immediately follows that $(A + B_1F - B_2\widehat{R}^{-1}S_2^T)W_g = W_gJ_1 \Rightarrow (A + \begin{bmatrix} B_1 & B_2\end{bmatrix}\begin{bmatrix} F \\ -\widehat{R}^{-1}S_2^T\end{bmatrix})W_g = W_gJ_1$. Also, $(C - D_2\widehat{R}^{-1}S_2^T)W_g = 0 \Rightarrow (C + \begin{bmatrix} 0 & D_2\end{bmatrix}\begin{bmatrix} F \\ -\widehat{R}^{-1}S_2^T\end{bmatrix})W_g = 0$. Consequently, $\mathcal{W}_g \subseteq \mathcal{V}_g$.

($\mathcal{V}_g \subseteq \mathcal{W}_g$): Say, $\mathtt{dim}(\mathcal{V}_g) =: \mathtt{g}_2$ and $V_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}_2}$ be such that $\mathcal{V}_g = \mathtt{img}V_g$. Thus, there exist $F_1 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}, F_2 \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$, and $J_2 \in \mathbb{R}^{\mathtt{g}_2\times\mathtt{g}_2}$ such that

$$(A+\begin{bmatrix} B_1 & B_2\end{bmatrix}\begin{bmatrix} F_1 \\ F_2\end{bmatrix})V_g=V_gJ_2 \text{ and } (C+\begin{bmatrix} 0 & D_2\end{bmatrix}\begin{bmatrix} F_1 \\ F_2\end{bmatrix})V_g=0, \quad (32)$$

where $\sigma(J_2) \subseteq \mathbb{C}_-$. Now, $(C + \begin{bmatrix} 0 & D_2\end{bmatrix}\begin{bmatrix} F_1 \\ F_2\end{bmatrix})V_g = 0 \Rightarrow D_2F_2V_g = -CV_g \Rightarrow D_2^TD_2F_2V_g = -D_2^TCV_g$. Notice, from equation (27), that $D_2^TD_2 = \widehat{R}$ and $C^TD_2 = S_2$. Thus, we have

$$F_2V_g = -\widehat{R}^{-1}S_2^TV_g. \quad (33)$$

Next, using equation (33) in equation (32), we get $(\widetilde{A} + \widetilde{B}F_1)V_g = V_gJ_2$ and $\widetilde{C}V_g = 0$. Thus, $\mathcal{V}_g \subseteq \mathcal{W}_g$. Hence, we finally conclude that $\mathcal{V}_g = \mathcal{W}_g$. $\square$

Next, we state and prove the lemma which establishes a relation between $\mathcal{V}_g$ and $\mathcal{O}_{wg}$.

Lemma 17. Let $\mathcal{V}_g$ and $\mathcal{O}_{wg}$ be the good slow spaces of the primal Σ_{pr} (defined in Definition 15) and the Hamiltonian system $\Sigma_{\mathtt{Ham}}$ (defined by equation (30)), respectively. Define the subspace

$$\mathcal{V}_{\mathtt{gHam}} := \{\begin{bmatrix} v \\ 0\end{bmatrix} \in \mathbb{R}^{2\mathtt{n}} \mid v \in \mathcal{V}_g\}.$$

Then, $\mathcal{V}_{\mathtt{gHam}} \subseteq \mathcal{O}_{wg}$.

Proof. Let $\mathtt{g} := \mathtt{dim}\mathcal{V}_{\mathtt{g}}$ and $V_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$ be such that $\mathcal{V}_g = \mathtt{img}V_g$. Thus, using Lemma 16, we infer that there exists $F \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ such that

$$(\widetilde{A} + \widetilde{B}F)V_g = V_gJ_g \text{ and } \widetilde{C}V_g = 0, \quad (34)$$

where $\sigma(J_g) \subseteq \mathbb{C}_-$. Also, since $\widetilde{C} = C - D_2\widehat{R}^{-1}S_2^T$, it is easy to verify that $\widetilde{C}^T\widetilde{C} = \widetilde{Q}$. Hence, defining $V_{3g} := FV_g$, we have the following:

$$\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0\end{bmatrix}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3g}\end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3g}\end{bmatrix}J_g$$

$$\text{or, } \begin{bmatrix} A_{\mathtt{r}} & B_{\mathtt{r}} \\ C_{\mathtt{r}} & 0\end{bmatrix}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3g}\end{bmatrix} = \begin{bmatrix} I_{2\mathtt{n}} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3g}\end{bmatrix}J_g, \quad (35)$$

where $A_{\mathtt{r}}, B_{\mathtt{r}}$, and $C_{\mathtt{r}}$ are as defined in equation (30). From equation (35), it is clear that $A_{\mathtt{r}}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} + B_{\mathtt{r}}V_{3g} = (A_{\mathtt{r}} + B_{\mathtt{r}}\begin{bmatrix} F & 0_{(\mathtt{m}-\mathtt{r}),\mathtt{n}}\end{bmatrix})\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} = \begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix}J_g$; and $C_{\mathtt{r}}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} = 0$. Thus, from Definition 11, it is evident that $\mathtt{img}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} \subseteq \mathcal{O}_{wg}$. But, notice that $\mathtt{img}\begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} = \mathcal{V}_{\mathtt{gHam}}$. Hence, $\mathcal{V}_{\mathtt{gHam}} \subseteq \mathcal{O}_{wg}$. $\square$

4.2 Disconjugacy of $\mathtt{img}V_{\mathtt{e}}$

In this section, we show that the subspace $\mathtt{img}V_{\mathtt{e}}$ (defined in equation (31)) is disconjugate. We present the definition of disconjugacy next.

Definition 18. Let $\mathcal{W}$ be an eigenspace of the matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$ as defined in equation (29). Assume that the columns of the matrix W form a basis for the eigenspace $\mathcal{W}$. Further, conforming to the partition in $H_{\mathtt{r}}$, say W be partitioned as $\begin{bmatrix} W_1 \\ W_2 \\ W_3\end{bmatrix}$. Then, $\mathcal{W}$ is said to be *disconjugate* if W_1 is full column-rank.

From the definition of disconjugacy, it is clear that disconjugacy of $\mathtt{img}V_{\mathtt{e}}$ is equivalent to V_1 (defined in equation (31)) being full column-rank. We show at the end of this section that if the system starts from an initial condition from $\mathtt{img}V_1$, then the singular LQR problem can be solved using a smooth input. Thus, disconjugacy of $\mathtt{img}V_{\mathtt{e}}$ provides us with the basis and dimension of the subspace of the state-space for which the problem can be solved using smooth input if the system starts from that subspace. Disconjugacy of $\mathtt{img}V_{\mathtt{e}}$ also enables us to provide a feedback law if the initial condition is from $\mathtt{img}V_1$ (Theorem 22). To prove that V_1 is full column-rank, we need two auxiliary results. We present these results one by one. The first auxiliary result is a well-known result about the left- and right-eigenspaces of the Hamiltonian matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$ (Ionescu et al. (1999)). For the sake of easy referencing, we present this as a proposition next.

Proposition 19. Let the columns of the matrix $V_{\mathtt{e}} = \mathtt{col}(V_1, V_2, V_3)$ form a basis for the eigenspace of the matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$ corresponding to the eigenvalues in Λ, where $(E_{\mathtt{r}}, H_{\mathtt{r}})$ is as defined in equation (29), and $\Lambda := \sigma(E_{\mathtt{r}}, H_{\mathtt{r}}) \cap \mathbb{C}_-$. Then the following statements hold:

(1) Rows of the matrix $\begin{bmatrix} V_2^T & -V_1^T & V_3^T\end{bmatrix}$ form a basis for the left-eigenspace of $(E_{\mathtt{r}}, H_{\mathtt{r}})$ corresponding to eigenvalues in $-\Lambda$. (Note: $\lambda \in -\Lambda \Leftrightarrow -\lambda \in \Lambda$)
(2) $V_1^TV_2 = V_2^TV_1$.

Next, recall from Lemma 17 that $\mathtt{img}\begin{bmatrix} V_g \\ 0\end{bmatrix} \subseteq \mathcal{O}_{wg}$. Thus, there exist $V_{12}, V_{22} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$, where $\mathtt{n_s} := \mathtt{dim}(\mathcal{O}_{wg})$, such that $\mathcal{O}_{wg} = \mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix}$. We use this fact in the following lemma which plays a pivotal role in proving the disconjugacy of the subspace $\mathtt{img}V_{\mathtt{e}}$.

Lemma 20. Let $V_{12}, V_{22} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$ be such that $\mathcal{O}_{wg} = \mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix}$, where $\mathtt{n_s} := \mathtt{dim}(\mathcal{O}_{wg}), \mathtt{g} := \mathtt{dim}(\mathcal{V}_\mathtt{g})$, $V_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$, and $\mathtt{img}V_g = \mathcal{V}_g$. Then, the following are true:

(1) V_{22} is full column-rank.
(2) $V_{22}^T V_{12} > 0$.
(3) $[V_g \ \ V_{12}]$ is full column-rank.

Proof. (1): Since $\mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix} = \mathcal{O}_{wg}$, by Definition 11, there exist $F_\mathtt{e} \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times 2\mathtt{n}}$, $\Gamma_{12} \in \mathbb{R}^{\mathtt{g}\times(\mathtt{n_s}-\mathtt{g})}$, and $\Gamma_{22} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times(\mathtt{n_s}-\mathtt{g})}$ such that

$$(A_\mathtt{r}+B_\mathtt{r}F_\mathtt{e})\begin{bmatrix} V_{12} \\ V_{22}\end{bmatrix}=\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix}\begin{bmatrix} \Gamma_{12} \\ \Gamma_{22}\end{bmatrix}, \text{ and } C_\mathtt{r}\begin{bmatrix} V_{12} \\ V_{22}\end{bmatrix}=0, \quad (36)$$

where $A_\mathtt{r}, B_\mathtt{r}$, and $C_\mathtt{r}$ are as defined in equation (30). Define $V_{32} := F_\mathtt{e}\begin{bmatrix} V_{12} \\ V_{22}\end{bmatrix}$. Then, combining equation (35) and equation (36), we get

$$\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0 \end{bmatrix}\begin{bmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \\ V_{3g} & V_{32}\end{bmatrix}=\begin{bmatrix} I_\mathtt{n} & 0 & 0 \\ 0 & I_\mathtt{n} & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \\ V_{3g} & V_{32}\end{bmatrix}\underbrace{\begin{bmatrix} J_g & \Gamma_{12} \\ 0 & \Gamma_{22}\end{bmatrix}}_{J_s}. \quad (37)$$

Now, since $\mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22}\end{bmatrix} = \mathcal{O}_{wg}$, it is evident that $\sigma(J_s) \subseteq \mathbb{C}_-$, which, in turn, implies that $\sigma(\Gamma_{22}) \subseteq \mathbb{C}_-$. Next, from equation (37), we get

$$\widetilde{A}V_{12} - A_z V_{22} + \widetilde{B}V_{32} = V_g\Gamma_{12} + V_{12}\Gamma_{22}, \quad (38)$$
$$-\widetilde{Q}V_{12} - \widetilde{A}^T V_{22} = V_{22}\Gamma_{22}, \quad (39)$$
$$\widetilde{B}^T V_{22} = 0. \quad (40)$$

Clearly, $\mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \\ V_{3g} & V_{32}\end{bmatrix}$ is an eigenspace of the Hamiltonian matrix pair $(E_\mathtt{r}, H_\mathtt{r})$. Thus, from Statement 2 of Proposition 19, it follows that $\begin{bmatrix} V_g & V_{12}\end{bmatrix}^T\begin{bmatrix} 0_{\mathtt{n},\mathtt{g}} & V_{22}\end{bmatrix} = \begin{bmatrix} 0_{\mathtt{n},\mathtt{g}} & V_{22}\end{bmatrix}^T\begin{bmatrix} V_g & V_{12}\end{bmatrix}$, which further implies that

$$V_{22}^T V_g = 0 \text{ and } V_{22}^T V_{12} = V_{12}^T V_{22}. \quad (41)$$

Next, we pre-multiply equation (38) by V_{22}^T and equation (39) by $-V_{12}^T$, and then add them together to get

$$\begin{aligned} V_{22}^T\widetilde{A}V_{12} - V_{22}^T A_z V_{22} + V_{22}^T\widetilde{B}V_{32} + V_{12}^T\widetilde{Q}V_{12} + V_{12}^T\widetilde{A}^T V_{22} \\ = V_{22}^T V_\mathtt{g}\Gamma_{12} + V_{22}^T V_{12}\Gamma_{22} - V_{12}^T V_{22}\Gamma_{22}. \quad (42)\end{aligned}$$

By using equation (40) and equation (41), equation (42) can be further reduced to obtain

$$V_{22}^T\widetilde{A}V_{12} - V_{22}^T A_z V_{22} + V_{12}^T\widetilde{Q}V_{12} + V_{12}^T\widetilde{A}^T V_{22} = 0. \quad (43)$$

Now, we prove Statement (1) of this lemma by contradiction. Thus, we assume that V_{22} is not full column-rank. So, there exists $w \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times 1}, w \neq 0$ such that $V_{22}w = 0$. Therefore, on pre- and post-multiplication of equation (43) by w^T and w, respectively, we get $w^T V_{12}^T\widetilde{Q}V_{12}w = 0$. But, since $\widetilde{Q} \geqslant 0$, we have $\widetilde{Q}V_{12}w = 0$. Hence,

$$\mathtt{ker}V_{22} \subseteq \mathtt{ker}\widetilde{Q}V_{12}. \quad (44)$$

Post-multiplying equation (39) by w, we get $-\widetilde{Q}V_{12}w - \widetilde{A}^T V_{22}w = V_{22}\Gamma_{22}w$. But, recall that $V_{22}w = 0$ and $\widetilde{Q}V_{12}w = 0$. Consequently, $V_{22}\Gamma_{22}w = 0$. Hence, $\mathtt{ker}V_{22}$ is Γ_{22}-invariant.
So, there exists a full column-rank matrix $T \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times\bullet}$ such that $V_{22}T = 0$ and $\Gamma_{22}T = T\Gamma, \sigma(\Gamma) \subseteq \sigma(\Gamma_{22}) \subseteq \mathbb{C}_-$. Moreover, from equation (44), we have $\widetilde{Q}V_{12}T = 0$. Now, post-multiplying equation (38) by T and using the fact that $V_{22}T = 0$ and $\Gamma_{22}T = T\Gamma$, we have

$$\widetilde{A}V_{12}T + \widetilde{B}V_{32}T = V_\mathtt{g}\Gamma_{12}T + V_{12}T\Gamma. \quad (45)$$

Recall that, $\widetilde{C}^T\widetilde{C} = \widetilde{Q}$. So, from the fact that $\widetilde{Q}V_{12}T = 0$, it is clear that

$$\widetilde{C}V_{12}T = 0. \quad (46)$$

Thus, combining equation (34), equation (45), and equation (46) together we derive that

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0\end{bmatrix}\begin{bmatrix} V_\mathtt{g} & V_{12}T \\ V_{3g} & V_{32}T\end{bmatrix} = \begin{bmatrix} I_\mathtt{n} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} V_\mathtt{g} & V_{12}T \\ V_{3g} & V_{32}T\end{bmatrix}\begin{bmatrix} J_g & \Gamma_{12} \\ 0 & \Gamma\end{bmatrix}.$$

Since $\sigma(J_g), \sigma(\Gamma) \subseteq \mathbb{C}_-$, from Lemma 13, it follows that $\mathtt{img}\begin{bmatrix} V_\mathtt{g} & V_{12}T\end{bmatrix}$ is contained in the largest good slow space, $\mathcal{W}_g$, of the system $\Sigma_{aux} : \frac{d}{dt}x = \widetilde{A}x + \widetilde{B}u, y = \widetilde{C}x$. But, from Lemma 16 we know that $\mathcal{W}_g = \mathcal{V}_g = \mathtt{img}V_g$. So, $\mathtt{img}\begin{bmatrix} V_\mathtt{g} & V_{12}T\end{bmatrix} = \mathtt{img}V_g$. Thus, there exist $\alpha_1 \in \mathbb{R}^{\mathtt{g}\times 1}$ and a non-zero $\alpha_2 \in \mathbb{R}^{\bullet\times 1}$ such that

$$V_g\alpha_1 + V_{12}T\alpha_2 = 0. \quad (47)$$

Recall that, $V_{22}T = 0$. Thus, $V_{22}T\alpha_2 = 0$. Combining this with equation (47), we have $\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix}\begin{bmatrix} \alpha_1 \\ T\alpha_2\end{bmatrix} = 0$. But, T being full column-rank and $\alpha_2 \neq 0$ implies that $T\alpha_2 \neq 0$. Consequently, we have a non-zero vector $\begin{bmatrix} \alpha_1 \\ T\alpha_2\end{bmatrix}$ inside $\mathtt{ker}\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix}$. This is contradiction, because $\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22}\end{bmatrix}$ is full column-rank. Therefore, V_{22} must be full column-rank.

(2): From equation (41), we know that $V_{22}^T V_{12}$ is symmetric. Now, we prove that $V_{22}^T V_{12} > 0$ in two steps. First, we show that $V_{22}^T V_{12} \geqslant 0$, and then we show that $V_{22}^T V_{12}$ is non-singular. Pre-multiplying equation (38) by V_{22}^T and using equation (40) and equation (41), we have

$$V_{22}^T\widetilde{A}V_{12} - V_{22}^T A_z V_{22} = V_{22}^T V_{12}\Gamma_{22}. \quad (48)$$

Also, by taking transpose of equation (39), and then post-multiplying by V_{12}, we obtain

$$-V_{12}^T\widetilde{Q}V_{12} - V_{22}^T\widetilde{A}V_{12} = \Gamma_{22}^T V_{22}^T V_{12}. \quad (49)$$

Adding equation (48) and equation (49) together, we get

$$-V_{12}^T\widetilde{Q}V_{12} - V_{22}^T A_z V_{22} = V_{22}^T V_{12}\Gamma_{22} + \Gamma_{22}^T V_{22}^T V_{12}. \quad (50)$$

Since, $\widetilde{Q} \geqslant 0$ and $A_z = B_2\widehat{R}^{-1}B_2^T \geqslant 0$, we have $V_{12}^T\widetilde{Q}V_{12} + V_{22}^T A_z V_{22} \geqslant 0$. Now, notice that equation (50) is a Lyapunov equation. Recall that Γ_{22} is Hurwitz. Thus, by Lyapunov's theorem, we conclude that $V_{22}^T V_{12} \geqslant 0$.
Next, to the contrary, assume that $V_{22}^T V_{12}$ is singular. Then, we must have that $(\begin{bmatrix} \widetilde{Q} & 0 \\ 0 & A_z\end{bmatrix}\begin{bmatrix} V_{12} \\ V_{22}\end{bmatrix}, \Gamma_{22})$ is unobservable (see (Wonham, 1985, Lemma 12.2)). Thus, there exists a non-zero $v \in \mathbb{C}^{(\mathtt{n_s}-\mathtt{g})\times 1}$ such that

$$\begin{aligned} &\Gamma_{22}v = \lambda v, \text{ for some } \lambda \in \sigma(\Gamma_{22}) \subseteq \mathbb{C}_-, \text{ and} \\ &\begin{bmatrix} \widetilde{Q} & 0 \\ 0 & A_z\end{bmatrix}\begin{bmatrix} V_{12} \\ V_{22}\end{bmatrix}v = 0 \Leftrightarrow \widetilde{Q}V_{12}v = 0 \ \& \ A_z V_{22}v = 0. \quad (51)\end{aligned}$$

Now, right-multiplying equation (39) with v and using $\widetilde{Q}V_{12}v = 0$ from equation (51), we have

$$\widetilde{A}^T V_{22}v = (A - B_2\widehat{R}^{-1}S_2^T)^T V_{22}v = -\lambda V_{22}v. \quad (52)$$

Also, $A_z V_{22}v = B_2\widehat{R}^{-1}B_2^T V_{22}v = 0 \Leftrightarrow \widehat{R}^{-1}B_2^T V_{22}v = 0 \Leftrightarrow B_2^T V_{22}v = 0$. Combining this with equation (40), we get $v^T V_{22}^T\begin{bmatrix} B_1 & B_2\end{bmatrix} = 0$, because $\widetilde{B} = B_1$. Further, using $B_2^T V_{22}v = 0$ in equation (52), we get that $A^T V_{22}v =$

$-\lambda V_{22}v$. From Statement (1) of this lemma, we know that V_{22} is full column-rank. So, v being a non-zero vector implies that $V_{22}v \neq 0$. This means that $V_{22}v$ is an eigenvector of A^T corresponding to the eigenvalue $-\lambda$. But, $v^T V_{22}^T \left[\, B_1 \; B_2 \,\right] = 0$ and $\lambda \in \mathbb{C}_- \Rightarrow -\lambda \in \overline{\mathbb{C}}_+$. This contradicts the assumption that $(A, \left[\, B_1 \; B_2 \,\right])$ is stabilizable (see Problem 14). Hence, $V_{22}^T V_{12}$ is non-singular. We also showed that $V_{22}^T V_{12} \geqslant 0$. Therefore, $V_{22}^T V_{12} > 0$.

(3): Say, $\beta_1 \in \mathbb{R}^{\mathtt{g}\times 1}$ and $\beta_2 \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times 1}$ be such that $\left[\, V_g \; V_{12} \,\right] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$. Pre-multiplying this equation with V_{22}^T and using equation (41), we have $V_{22}^T V_{12}\beta_2 = 0$. But, from Statement (2) of this lemma, we know that $V_{22}^T V_{12}$ is non-singular. So, $\beta_2 = 0$. This further implies that $V_g\beta_1 = 0$, which, in turn, implies that $\beta_1 = 0$, because V_g is full column-rank. Thus, $\left[\, V_g \; V_{12} \,\right] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$. Hence, $\left[\, V_g \; V_{12} \,\right]$ is full column-rank. □

Now, we are in a position to show that the subspace $\mathtt{img}V_{\mathtt{e}}$ is disconjugate. We present this result as a theorem next.

Theorem 21. Let $(E_{\mathtt{r}}, H_{\mathtt{r}})$ be the Hamiltonian matrix pair as defined in equation (29). Also, consider the eigenspace, $\mathtt{img}V_{\mathtt{e}}$, of $(E_{\mathtt{r}}, H_{\mathtt{r}})$ as define in equation (31) . Then, $\mathtt{img}V_{\mathtt{e}}$ is disconjugate.

Proof. Recall, from equation (31), that $V_{\mathtt{e}} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$. We also know that $\mathtt{img}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathcal{O}_{wg}$. But, from the statement of Lemma 20, we have that $\mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22} \end{bmatrix} = \mathcal{O}_{wg}$. Thus, $\mathtt{img}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathtt{img}\begin{bmatrix} V_g & V_{12} \\ 0 & V_{22} \end{bmatrix} \Rightarrow \mathtt{img}V_1 = \mathtt{img}\left[\, V_g \; V_{12} \,\right]$. Now, V_1 and $\left[\, V_g \; V_{12} \,\right]$ both have $\mathtt{n_s}$ number of columns. Furthermore, from Statement (3) of Lemma 20, we get that $\left[\, V_g \; V_{12} \,\right]$ is full column-rank. Hence, we must have that V_1 is full column-rank. Therefore, $\mathtt{img}V_{\mathtt{e}}$ is disconjugate.□

The following theorem renders the optimal trajectories and a feedback law to solve Problem 14, if the initial condition is from $\mathtt{img}V_1$.

Theorem 22. Consider the singular LQR Problem 14, V_1, and J as defined in equation (31). Suppose $x_0 = V_1\alpha$, $\alpha \in \mathbb{R}^{\mathtt{n_s}\times 1}$, is an arbitrary initial condition from $\mathtt{img}V_1$. Then,

(1) (x_s, u_{s_1}, u_{s_2}) is the optimal trajectory, where $x_s := V_1 e^{Jt}\alpha$, $u_{s_1} := V_3 e^{Jt}\alpha$, and $u_{s_2} := -\widehat{R}^{-1}(S_2^T V_1 + B_2^T V_2)e^{Jt}\alpha$.
(2) There exist feedbacks $F_1 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times \mathtt{n}}$ and $F_2 \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$ such that $u_{s_1} = F_1 x_s$ and $u_{s_2} = F_2 x_s$.

Proof. (1): Define $z_0 := V_2\alpha$ and $z_s := V_2 e^{Jt}\alpha$. Then, using equation (31), it is easy to verify that $(x_s, z_s, u_{s_1}, u_{s_2})$ is a trajectory for the Hamiltonian system defined by equation (28) corresponding to the initial condition (x_0, z_0). It can also be verified that (x_s, u_{s_1}, u_{s_2}) is a trajectory for the system $\frac{d}{dt}x = Ax + B_1u_1 + B_2u_2$ corresponding to the initial condition x_0. Hence, from Pontryagin's maximum principle it follows that (x_s, u_{s_1}, u_{s_2}) is the optimal trajectory corresponding to the initial condition x_0.

(2): From Theorem 21, it follows that V_1 is full column-rank. Thus, there exist $K_1 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ and $K_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $V_3 = K_1V_1$ and $V_2 = K_2V_1$. Define $F_1 := K_1$ and $F_2 := -\widehat{R}^{-1}(S_2^T + B_2^T K_2)$. Then, it is evident that $u_{s_1} = F_1x_s$ and $u_{s_2} = F_2x_s$. This completes the proof. □

5. CONCLUSION

In this paper we have provided a characterization of the slow and the good slow spaces. This characterization automatically gives a method to compute these subspaces from an eigenspace of the corresponding Rosenbrock system matrix. Furthermore, we have shown how to obtain the dimensions of these subspaces from the degree of the determinant of the Rosenbrock matrix pencil. Then, we have applied these results to the Hamiltonian system obtained from the singular LQR problem to explore some interesting properties. In this paper we have used the good slow space of the Hamiltonian to provide a feedback which solves the singular LQR problem when the initial condition of the system belong to a certain subspace. This space has been used in Bhawal and Pal (2019) to solve the singular LQR problem for any arbitrary initial condition for the single-input case. We wish to use the results developed in this paper to solve the problem for the multi-input case.

REFERENCES

C. Bhawal and D. Pal. Almost every single-input LQR optimal control problem admits a PD feedback solution. *IEEE Control Systems Letters*, 3(2):452 – 457, 2019.

C. Bhawal, I. Qais, and D. Pal. Constrained generalized continuous algebraic Riccati equations (CGCAREs) are generically unsolvable. *IEEE Control Systems Letters*, 3(1):192–197, 2019a.

C. Bhawal, I. Qais, D. Pal, and J. Heiland. The optimal cost of a singular LQR problem, and fast/slow subspaces of the Hamiltonian system. Available at: www.ee.iitb.ac.in/%7Edebasattam/MCSSManuscript.html, 2019b.

L. Dai. *Singular Control Systems*. Springer-Verlag Berlin, Heidelberg, 1989.

A. Ferrante and L. Ntogramatzidis. On the reduction of the continuous-time generalized algebraic Riccati equation: An effective procedure for solving the singular LQ problem with smooth solutions. *Automatica*, 93:554–558, 2018.

B. Francis. The optimal linear-quadratic time-invariant regulator with cheap control. *IEEE Transactions on Automatic Control*, 24(4):616–621, 1979.

M.L.J. Hautus and L.M. Silverman. System structure and singular control. *Linear Algebra and its Application*, 50: 369–402, 1983.

V. Ionescu, C. Oară, and M. Weiss. *Generalized Riccati theory and robust control: a Popov function approach*. John Wiley, 1999.

R.K. Kalaimani, M.N. Belur, and D. Chakraborty. Singular LQ control, optimal PD controller and inadmissible initial conditions. *IEEE Transactions on Automatic Control*, 58(10):2603–2608, 2013.

T. Reis, O. Rendel, and M. Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra and its Applications*, 485:153–193, 2015.

J.C. Willems, A. Kitapçi, and L.M. Silverman. Singular optimal control: a geometric approach. *SIAM Journal on Control and Optimization*, 24(2):323–337, 1986.

W.M. Wonham. *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag, New York, 1985.