Defective Hamiltonian matrix imaginary eigenvalues and losslessness

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***Abstract*— The solutions to algebraic Riccati equation (ARE) have widespread applications in the area of control and network theory. For certain solutions of the Riccati equation, namely so-called 'semi-stabilizing' solutions, the corresponding Hamiltonian matrix has two or more purely imaginary eigenvalues. In this paper we explore the relationship between existence of such imaginary eigenvalues and lossless trajectories present in the system. It is known that under suitable conditions, such imaginary eigenvalues of the corresponding Hamiltonian matrix are 'defective', i.e., there are insufficient corresponding independent eigenvectors for the given eigenvalue. This poses theoretical and numerical difficulties in computing the solutions of the corresponding ARE. In this paper, we formulate conditions under which such imaginary eigenvalues of the Hamiltonian matrix are non-defective. As an extreme case of non-defectiveness, we first formulate conditions under which a Hamiltonian matrix is normal, i.e. the matrix commutes with its transpose. We also provide conditions under which imaginary eigenvalues of the Hamiltonian matrix are defective.**
Keywords: controllability, observability, defective eigenvalues, normal matrices, all-pass, diagonalizability

1. Introduction

The algebraic Riccati equation (ARE) finds its application in various areas of control theory and network analysis problems. Depending on the problem at hand, different forms of the ARE are found in the literature. In this paper, we deal with the *continuous symmetric algebraic Riccati equation* (see [10, Chapter 7] for more details). This form of the ARE arises in classical problems of system theory such as the linear quadratic regulator problem, optimal H_2 filter design, H_∞ control, differential games, passive network synthesis procedures, and spectral factorization. A standard method to solve the continuous symmetric ARE is to find suitable invariant subspaces of a certain matrix called the *Hamiltonian matrix*. A typical Hamiltonian matrix has the following form:

$$H := \begin{bmatrix} M & -S \\ -Q & -M^T \end{bmatrix}. \tag{1}$$

In all applications arising in dynamical systems, the matrices M, S, Q of the Hamiltonian matrix H in equation (1) depend on the system dynamics and certain performance index that needs to be optimized. Table I shows the different forms of H that typically find application in dynamical systems with an input-state-output (i/s/o) representation of the form $\dot{x} = Ax + Bu$ and $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. Interestingly, most of the classical results to find solutions of an ARE using the Hamiltonian matrix H assume that none of the eigenvalues of H are on the imaginary axis. Hamiltonian matrices that admit imaginary axis eigenvalues pose difficulties in Riccati equation solution computation. Work in this area shows that under certain conditions the imaginary axis eigenvalues of the Hamiltonian matrices are defective[1]: see [5],[6]. This paper pursues further the question: when do Hamiltonian matrices admit non-defective eigenvalues? This paper formulates conditions that result in non-defective eigenvalues of a Hamiltonian matrix.

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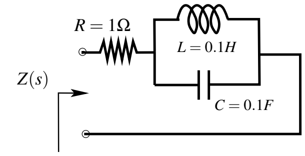

Fig. 1. A controllable RLC circuit

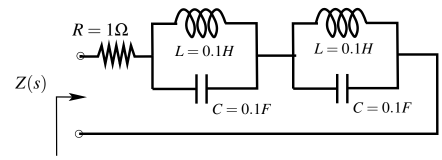

Fig. 2. An uncontrollable RLC circuit

Another motivating factor to study the imaginary axis eigenvalues of a Hamiltonian matrix is the fact that such modes of the Hamiltonian matrix, loosely speaking, reveal certain 'stationary trajectories' of the system. These are those trajectories along which the system no longer remains dissipative and hence such trajectories are aptly known in the literature as *lossless trajectories*. In the context of RLC circuits, imaginary eigenvalues of H correspond to no loss of energy at that frequency. Thus in order to characterize the lossless trajectories of a dissipative system it is essential to study the various features of imaginary eigenvalues of H. The characterization of lossless trajectories critically depend on the algebraic and geometric multiplicity of the imaginary eigenvalues of H. Hence, in this paper we formulate conditions under which the imaginary eigenvalues of H become non-defective. Interestingly, defectiveness of the imaginary eigenvalues of H is critically linked to controllability of a system. For example, consider the circuit described by Figure 1. Note that the circuit is controllable. The corresponding Hamiltonian matrix contains imaginary eigenvalues $\pm 10j$. Each of the imaginary axis eigenvalues has an algebraic multiplicity of 2 and geometric multiplicity equal to 1. Hence, there is only one Jordan block of size 2×2 corresponding to the eigenvalue $\pm 10j$. However, if one considers the system described by Figure 2 which is uncontrollable, each of the eigenvalues $\pm 10j$ in the Hamiltonian matrix now has algebraic multiplicity of 4 and geometric

[1]An eigenvalue $\lambda \in \mathbb{C}$ of a matrix A is called defective if the algebraic multiplicity of $\lambda >$ the geometric multiplicity of λ.

TABLE I

DIFFERENT FORMS OF HAMILTONIAN MATRIX: SEE EQUATION (1)

Problem	Performance Index	M	S	Q
Passivity	$u^T y$	$A-B(D+D^T)^{-1}C$	$B(D+D^T)^{-1}B^T$	$C^T(D+D^T)^{-1}C$
Bounded-real	$u^T u - y^T y$	$A+B(I-D^TD)^{-1}D^TC$	$B(I-D^TD)^{-1}B^T$	$C^T(I-DD^T)^{-1}C$
LQR	x^TQx+u^TRu	A	BB^T	Q

multiplicity equal to 3. Thus, there exists three distinct Jordan blocks for eigenvalues $\pm 10j$, one of size 2×2 and the other two blocks of size 1×1. These examples show the effect of controllability on the partial multiplicities of the imaginary eigenvalues of H. Hence, this paper focuses on formulating the link between uncontrollability, defectiveness of imaginary eigenvalues of H, and open loop poles of the system on the imaginary axis.

The organization of the paper is as follows: Section 2 contains notation and preliminaries needed in this paper, while Section 3 contains main results regarding non-defectiveness of imaginary eigenvalues of the given Hamiltonian matrix. Section 4 has a few concluding remarks about the results in the paper.

2. Notation and Preliminaries

A. Notation

We use $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ to denote the sets of real numbers, complex numbers and natural numbers, respectively. The symbols $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$ are used to denote the set of complex numbers with non-negative and non-positive real parts, respectively. The symbol $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ matrices with elements from $\mathbb{R}$. Symbol I_n is used for $n\times n$ identity matrix. We use the symbol $\sigma(A)$ to denote the multiset[1] of eigenvalues of $A\in\mathbb{R}^{n\times n}$. The symbols $\mathtt{ker}(A)$ and $\mathtt{img}(A)$ denote the kernel and image of the matrix A, respectively. The symbol $\mathscr{R}_1\oplus\mathscr{R}_2$ represents a vector-space that is a direct sum of the subspaces $\mathscr{R}_1$ and $\mathscr{R}_2$.

B. Dissipativity and Hamiltonian matrix

The notion of dissipativity plays a crucial role in this paper and we review it next.

Definition 2.1. *Consider a system $\mathfrak{B}$ with an i/s/o representation $\dot{x}=Ax+Bu$ and $y=Cx+Du$, where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$ and $D\in\mathbb{R}^{p\times m}$. Let $\Sigma=\Sigma^T\in\mathbb{R}^{(m+p)\times(m+p)}$. The system $\mathfrak{B}$ is called* dissipative *with respect to Σ if there exists $K=K^T\in\mathbb{R}^{n\times n}$ such that*

$$\frac{d}{dt}\left(x^TKx\right)\leqslant\begin{bmatrix}u\\y\end{bmatrix}^T\Sigma\begin{bmatrix}u\\y\end{bmatrix}. \qquad (2)$$

In inequality (2), Σ is called the *supply rate* of the system. In particular, the supply rate $\begin{bmatrix}I_m & 0\\ 0 & -I_p\end{bmatrix}$ is called the *bounded-real supply rate* and, relevant only when the number of inputs and outputs equal each other (i.e. $m=p$), the supply rate $\begin{bmatrix}0 & I_m\\ I_m & 0\end{bmatrix}$ is called the *passivity supply rate*. Most of the results in this paper are for systems that are dissipative with respect to the passivity supply rate. From Table I it can be seen that the Hamiltonian matrix corresponding to the passivity supply rate is

$$H=\begin{bmatrix}A-B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T\\ -C^T(D+D^T)^{-1}C & -(A-B(D+D^T)^{-1}C)^T\end{bmatrix}. \qquad (3)$$

In inequality (2), the quadratic term x^TKx is called a *storage function* of the system. With a slight abuse of nomenclature, we call K to be a storage function of $\mathfrak{B}$, as well. Interestingly, it is known in the literature that for a controllable and dissipative system $\mathfrak{B}$, the set of storage functions admits a maximal and a minimal element. More precisely, for a dissipative and controllable system $\mathfrak{B}$ with a *minimal* i/s/o representation as in the above definition, there exists a storage function $K_{\max}$ and $K_{\min}$ such that for all storage functions K of $\mathfrak{B}$, we have $K_{\min}\leqslant K\leqslant K_{\max}$. The extremal storage functions $K_{\max}$ and $K_{\min}$ have a system theoretic significance. The minimum amount of energy required for taking a system from zero state to a state $x=a$ is given by $a^TK_{\max}a$. Similarly, the maximum amount of energy that can be extracted out from a system with initial state $x=a$ is given by $a^TK_{\min}a$. Interestingly, the states x of a system for which $x^T(K_{\max}-K_{\min})x=0$ reveals certain trajectories of the system for which the energy supplied or extracted from the system is equal to the energy stored in the system. These are called *lossless trajectories* and are of utmost importance for this paper.

In order to prove the main results of this paper we crucially use a result in [5]. We review the result in the form of a proposition next. Before we present the proposition we need the notion of a 'complementary'-set (a 'c-set', for short) [5] and we define it next.

Definition 2.2. *Consider a matrix $P\in\mathbb{R}^{n\times n}$ such that $\sigma(P)=\sigma(-P^T)$. Let $\Lambda\subset\sigma(P)$ and define $-\bar{\Lambda}:=\{\lambda\in\mathbb{C}\,|\,-\bar{\lambda}\in\Lambda\}$. A set $\Lambda\subset\sigma(P)$ is called* <u>c-set</u> *if $\Lambda\cap-\bar{\Lambda}=\emptyset$ and $\Lambda\cup-\bar{\Lambda}=\sigma(P)\backslash i\mathbb{R}$.*

Note that from Definition 2.2 it is evident that if Λ is a c-set then $-\bar{\Lambda}$ is also a c-set. We call $-\bar{\Lambda}$ to be the *complementary c-set* of Λ.

Proposition 2.3. *[5, Theorem 3.3] Consider the Hamiltonian matrix $H=\begin{bmatrix}A & -S\\ -Q & -A^T\end{bmatrix}$, where $A,Q\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $S=\pm BB^T$ and (A,B) is sign-controllable*[2]*. Let $\sigma(H)\cap j\mathbb{R}=\{\mu_1,\mu_2,\ldots,\mu_k\}$ where $\mu_i\in\mathbb{C}$ has multiplicity $2\alpha_i$*

[1] Multiset is a set where the elements also have multiplicities, for example, $\{1,1,2\}$ is a multiset whereas this multiset as a set is merely $\{1,2\}$.

[2] A pair (A,B) is said to be sign-controllable if for each eigenvalue λ_i of A, at least one of $[A-\lambda_iI \;\; B]$ and $[A+\lambda_iI \;\; B]$ is full row rank.

and $\sum_{i=1}^{k}\alpha_i =: w$. Define $\mathscr{R}_\mu := \texttt{ker}\,(\mu I_{2n} - H)^{2n}$. Then the following are equivalent:

(i) ARE has an unmixed Hermitian[3] solution.

(ii) ARE has a Hermitian solution.

(iii) The imaginary eigenvalues of H have even partial multiplicity.

(iv) There exists a c-set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-w}\} \in \sigma(H)$ and an H-invariant subspace[4] $\mathscr{I} \subset \bigoplus_{i=1}^{k}\mathscr{R}_{\lambda_i}$ such that with

$$\texttt{img}\begin{bmatrix} I_n \\ K \end{bmatrix} = \mathscr{I} \oplus \mathscr{R}_{\lambda_1} \oplus \mathscr{R}_{\lambda_2} \oplus \cdots \oplus \mathscr{R}_{\lambda_{n-w}} \quad (4)$$

K is the unique unmixed solution of the ARE $A^TK + KA + Q - KSK = 0$ with $\sigma(A - SK) = \Lambda$.

Note that in Proposition 2.3 the set Λ is a c-set and the solution of the ARE obtained using the root-subspace $\mathscr{R}_\lambda$ of Λ is K. Corresponding to the complementary c-set $-\Lambda$ a graph-subspace of the form in equation (4) can be obtained. Let $\texttt{img}\begin{bmatrix} I_n \\ \widehat{K} \end{bmatrix}$ be the graph-subspace corresponding to the complementary c-set $-\Lambda$. Then, $\widehat{K}$ is also the unique solution of the ARE with $\sigma(A - S\widehat{K}) = -\bar{\Lambda}$. In this paper, we call the $\widehat{K}$ to be the *complementary solution* of the ARE with respect to K. Let $-\bar{\Lambda} = \{\lambda_{n+1-w}, \lambda_{n+2-w}, \ldots, \lambda_{2n-w}\}$. Then it is clear that

$$\begin{aligned} &\texttt{img}\left(\begin{bmatrix} I_n \\ K \end{bmatrix}\right) \cap \texttt{img}\left(\begin{bmatrix} I_n \\ \widehat{K} \end{bmatrix}\right) \\ &= \left(\mathscr{I} \oplus \mathscr{R}_{\lambda_1} \oplus \cdots \oplus \mathscr{R}_{\lambda_{n-w}}\right) \cap \left(\mathscr{I} \oplus \mathscr{R}_{\lambda_{n+1-w}} \oplus \cdots \oplus \mathscr{R}_{\lambda_{2n-w}}\right) \\ &= \mathscr{I}. \end{aligned} \quad (5)$$

In equation (5), we used the fact that, by definition of c-set, $\left(\oplus_{i=1}^{n-w}\mathscr{R}_{\lambda_i}\right) \cap \left(\oplus_{i=n+1-w}^{2n-w}\mathscr{R}_{\lambda_i}\right) = \emptyset$. In this context, it is also customary to call K as a 'stabilizing ARE solution' if the eigenvalues of $(A - SK)$ are Hurwitz, i.e. have strictly negative real part. Further, we call K 'semi-stabilizing ARE solution' if all eigenvalues of $(A - SK)$ have real part nonpositive, and some eigenvalues have zero real part.

Since in this paper we explore the relation between eigenvalues of a Hamiltonian matrix on the imaginary axis and their defectiveness (partial multiplicity), we present a proposition next that would be required to prove one of the main results of this paper (Theorem 3.6).

Proposition 2.4. *[10, Lemma 7.3.3] Let $U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{n\times n}$ be such that $V \geqslant 0$. Define $\mathscr{R}_\lambda(U) := \texttt{ker}(\lambda I_n - U)^n$ and $\mathscr{C}_{U,V} := \texttt{img}\left(\begin{bmatrix} V & UV & \cdots & U^{n-1}V \end{bmatrix}\right)$. Let $W := \begin{bmatrix} U & V \\ 0 & -U^T \end{bmatrix}$. Assume $\mathscr{R}_\lambda(U) \subseteq \mathscr{C}_{U,V}$ for every eigenvalue λ of U. Then, the partial multiplicities of each eigenvalue λ of W on the imaginary axis are all even.*

[3] A Hermitian solution $K \in \mathbb{R}^{n\times n}$ of an ARE is called *unmixed* if $V := \texttt{col}(I_n, K) \in \mathbb{R}^{2n\times n}$ is such that $HV = V\Gamma$, where $H \in \mathbb{R}^{2n\times 2n}$ is the corresponding Hamiltonian matrix and $\sigma(\Gamma)$ is a c-set of H.

[4] The basis for the subspace $\mathscr{I}$ is given by a suitable selection of the eigenvectors and generalized eigenvectors corresponding to the eigenvalues of H on the imaginary axis. The procedure to select such vectors can be found in [5]. Note that the dimension of $\mathscr{I}$ is w.

3. Main results

This section contains the main results of this paper. As noted in the introduction, imaginary eigenvalues of the Hamiltonian matrix correspond to certain stationary trajectories of the system, known as lossless trajectories. These are the trajectories along which there is no dissipation in the system. For lossless systems, the difference between the two extremal ARE solutions $K_{\max} - K_{\min} = 0$. This is because in case of lossless systems, energy supplied is equal to the stored energy along *every* system trajectory. However for dissipative systems, in general, this is not true for all system trajectories. There is possibly a subset of system trajectories for which $x^T(K_{\max} - K_{\min})x$ is zero. Such trajectories are exactly the lossless trajectories present in the system. Thus, dissipative systems have a subset of system trajectories for which the energy supplied or extracted from the system is equal to the energy stored in the system. In [13], the difference between the extremal ARE solutions $(K_{\max} - K_{\min})$ is referred to as the *gap* associated with the ARE and this 'gap' is used to characterize all the solutions of the ARE (see [13, Theorem 6]). The use of this gap to characterize other solutions has also been pursued in [4]. For controllable systems, it is known that the kernel of the gap between extremal ARE solutions is (A, B) invariant [10, Theorem 7.5.3], [3], [12]. We strengthen this result by obtaining the same claim but under a milder assumption, i.e (A, B) is just sign-controllable.

Theorem 3.1. *Consider a system with an i/s/o representation $\dot{x} = Ax + Bu,\ y = Cx + Du$, where $A \in \mathbb{R}^{n\times n},\ B, C^T \in \mathbb{R}^{n\times m},\ D \in \mathbb{R}^{m\times m}$ and $D + D^T > 0$. Suppose (A, B) is sign-controllable and the system is dissipative with respect to the passivity supply rate. Consider the Hamiltonian matrix H as defined in equation (3). Let $K_1, K_2 \in \mathbb{R}^{n\times n}$, where (K_1, K_2) is a pair of complementary solutions. Define $\mathscr{N} := \texttt{ker}(K_1 - K_2)$. Then, $\mathscr{N}$ is (A, B) invariant.*

Proof: Let $\sigma(H) \cap j\mathbb{R} = \{\mu_1, \mu_2, \ldots, \mu_k\}$ where μ_i has algebraic multiplicity $2\alpha_i$ and $\sum_{i=1}^{k}\alpha_i =: w$. Hence the dimension of $\mathscr{I}$ defined in Proposition 2.3 is w (see Footnote 4). Let the c-set corresponding to K_1 and K_2 be Λ and $-\bar{\Lambda}$. From equation (5) it is clear that

$$\texttt{img}\left(\begin{bmatrix} I_n \\ K_1 \end{bmatrix}\right) \cap \texttt{img}\left(\begin{bmatrix} I_n \\ K_2 \end{bmatrix}\right) = \mathscr{I} \quad (6)$$

Therefore, from equation (6) we infer that there exists full-column rank matrices $L_1, L_2 \in \mathbb{R}^{n\times w}$ such that

$$\begin{bmatrix} I_n \\ K_1 \end{bmatrix} L_1 = \begin{bmatrix} I_n \\ K_2 \end{bmatrix} L_2. \quad (7)$$

From equation (7), we have $L_1 = L_2 =: L$. Note that since $\texttt{rank}(L) = w$, the columns of $\begin{bmatrix} I_n \\ K_1 \end{bmatrix} L$ form a basis of $\mathscr{I}$. Note that $\mathscr{I}$ is H-invariant subspace. Hence, we have

$$H\begin{bmatrix} I_n \\ K_1 \end{bmatrix} L = \begin{bmatrix} I_n \\ K_1 \end{bmatrix} L\Gamma, \text{ where } \sigma(\Gamma) = \{\mu_1, \mu_2, \ldots, \mu_k\}. \quad (8)$$

Further, we know that $\texttt{img}\begin{bmatrix} I_n \\ K_1 \end{bmatrix}$ is an H-invariant subspace.

Hence, we have

$$H\begin{bmatrix} I_n \\ K_1 \end{bmatrix} = \begin{bmatrix} I_n \\ K_1 \end{bmatrix} A_{K_1}, \tag{9}$$

where $A_{K_1} \in \mathbb{R}^{n\times n}$, $\sigma(A_{K_1}) = \Lambda \cup \{\mu_1, \mu_2, \ldots, \mu_k\}$ and $A_{K_1} := A - B(D+D^T)^{-1}C + B(D+D^T)^{-1}B^T K_1$. Using equation (8) in equation (9), we have

$$H\begin{bmatrix} I_n \\ K_1 \end{bmatrix} = \begin{bmatrix} I_n \\ K_1 \end{bmatrix} A_{K_1} \Rightarrow H\begin{bmatrix} I_n \\ K_1 \end{bmatrix} L = \begin{bmatrix} I_n \\ K_1 \end{bmatrix} A_{K_1} L,$$

which implies $\begin{bmatrix} I_n \\ K_1 \end{bmatrix} L\Gamma = \begin{bmatrix} I_n \\ K_1 \end{bmatrix} A_{K_1} L$, which in turn implies

$$L\Gamma = \left(A - B((D+D^T)^{-1}C + (D+D^T)^{-1}B^T K_1)\right)L. \tag{10}$$

This proves that $\mathtt{img}(L)$ is (A,B) invariant. Next we show that $\mathtt{img}(L) = \mathscr{N}$. From equation (8) it is clear that $K_1L_1 = K_2L_2 \Rightarrow (K_1 - K_2)L = 0$. Since $L \in \mathbb{R}^{n\times w}$ and $\mathtt{rank}(L) = w$, it is evident that $\mathtt{dim}(\mathscr{N}) \geqslant w$. We prove that $\mathtt{dim}(\mathscr{N}) = w$. Let us assume to the contrary that $\mathtt{dim}(\mathscr{N}) = w+\alpha$, where $\alpha \in \mathbb{N}$ and $0 < \alpha \leqslant (n-w)$. Then, there exists a full-column rank matrix $\hat{L} \in \mathbb{R}^{n\times(w+\alpha)}$ such that $(K_1 - K_2)\hat{L} = 0 \implies \begin{bmatrix} I_n \\ K_1 \end{bmatrix} \hat{L} = \begin{bmatrix} I_n \\ K_2 \end{bmatrix} \hat{L}$. Since $\hat{L}$ has full column rank, this means that the subspace $\mathtt{img}\left(\begin{bmatrix} I_n \\ K_1 \end{bmatrix}\right) \cap \mathtt{img}\left(\begin{bmatrix} I_n \\ K_2 \end{bmatrix}\right) \subseteq \mathbb{R}^{2n\times 2n}$ has a dimension $w+\alpha$. This is a contradiction to equation (6) since dimension of $\mathscr{I}$ is w. Therefore, dimension of $\mathscr{N}$ must be w. This proves that $\mathtt{img}(L) = \mathscr{N}$. Hence, from equation (10) we infer that $\mathscr{N}$ is (A,B)-invariant. □

As motivated in Section 1, defectiveness of the imaginary eigenvalues of the Hamiltonian matrix is essential to characterize the lossless trajectories of a system. A kind-of opposite of 'defectiveness' is the notion of 'normality' of a matrix/eigenvalue: since normal matrices have orthogonal left/right invariant subspaces, unlike defective eigenvalue/eigenvectors. Hence, a natural question is: can a Hamiltonian matrix be normal? We provide a necessary condition[5] for the Hamiltonian matrix to be normal in the next theorem.

Theorem 3.2. *Consider a system $\dot{x} = Ax + Bu$, $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$ and $D + D^T = I_m$. Suppose the Hamiltonian matrix is defined as $H := \begin{bmatrix} A - BC & BB^T \\ -C^TC & -(A-BC)^T \end{bmatrix}$. Let H be a normal matrix and let $B = C^T$. Then, the following statements hold.*

(i) If $A = A^T$ and A is semi-Hurwitz, i.e $\sigma(A) \subset \overline{\mathbb{C}^-}$ then $B = C^T = 0$.
(ii) If $A = -A^T$ then $B = C^T = 0$.

In order to prove Theorem 3.2, we first formulate and prove the following lemma.

Lemma 3.3. *Consider a matrix $A = A^T \in \mathbb{R}^{n\times n}$. If A is semi-Hurwitz (semi-anti-Hurwitz), then $X = 0$ is the maximal (minimal) symmetric solution of $-XA - AX + 2X^2 = 0$.*

[5]Note that in Theorem 3.2 it is assumed that $B = C^T$. For positive real systems, such an assumption is valid since the residue matrix corresponding to the transfer function of such systems is positive-semidefinite. Hence, by Cholesky decomposition (non-unique) we can always find an i/s/o representation of the system where $B = C^T$. We do not dwell on this further.

Proof: The Hamiltonian matrix corresponding to the ARE $-XA - AX + 2X^2 = 0$ is $H := \begin{bmatrix} -A & 2I \\ 0 & A \end{bmatrix}$. Note that $X = 0$ is a symmetric solution of the given ARE. From the Hamiltonian matrix H, it is evident that the graph subspace corresponding to the solution $X = 0$ is given by $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ Note that $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ is the eigenspace corresponding to the eigenvalues of $-A$. If A is semi-Hurwitz then $X = 0$ is the maximal symmetric solution of the given ARE (see [10, Theorem 7.5.1]). Similarly, if A is semi-anti-Hurwitz, then $X = 0$ is the minimal symmetric solution (see [10, Theorem 7.5.1]). □

We now prove Theorem 3.2.

Proof of Theorem 3.2: Since H is normal, we have $HH^T = H^TH$. From the expansion of HH^T and H^TH in terms of (A,B,C), it follows that H is normal if and only if the following two equations are satisfied by (A,B,C):

$$\begin{aligned} A^TA - AA^T + AC^TB^T + BCA^T - A^TBC - C^TB^TA \\ + C^T(CC^T + B^TB)C - B^T(CC^T + B^TB)B^T = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} BB^TA + A^TBB^T + AC^TC + C^TCA^T \\ - B(CC^T + B^TB)C - C^T(CC^T + B^TB)B^T = 0. \end{aligned} \tag{12}$$

It is easy to verify that $A = A^T$ and $B = C^T$ satisfies equation (11). Using $A = A^T$ and $B = C^T$ in equation (12) we have

$$\begin{aligned} BB^TA + A^TBB^T - 2BB^TBB^T = 0 \\ \implies -BB^TA - A^TBB^T + 2BB^TBB^T = 0. \end{aligned} \tag{13}$$

Consider the two statements in Theorem 3.2. (i): Note that since A is semi-Hurwitz, using Lemma 3.3 it is evident that $BB^T = 0$ is the maximal symmetric solution of the ARE (13). Any other solution of the ARE, if it exists, must be negative-semidefinite. Since $BB^T \geqslant 0$, any other nonzero solution of the ARE (13) cannot be decomposed into the form BB^T. Therefore, the only solution of the ARE (13) that can be decomposed into the form BB^T is $BB^T = 0$. Therefore, $B = 0 = C^T$. This proves (i).
(ii): It is easy to verify that $A = -A^T$ and $B = C^T$ satisfies equation (11). Further, substituting $A = -A^T$ and $B = C^T$ in equation (12), we obtain the following: $BB^TBB^T = 0 \Rightarrow BB^T = 0 \Rightarrow B = 0$. Hence, for $A = -A^T$ and $B = C^T$, H is normal only if $B = C^T = 0$. □

In Theorem 3.2, we formulated conditions under which the Hamiltonian matrix H is normal. From Theorem 3.2, we observe that a Hamiltonian matrix H is normal (for a semi-Hurwitz symmetric or a skew symmetric A) only if $B = C^T = 0$. This is a direct consequence of Lemma 3.3. Again using Lemma 3.3, one can construct a non-zero matrix B (and also C^T) for an anti-Hurwitz matrix A such that the Hamiltonian matrix H defined in Theorem 3.2 is normal. One such example is discussed below.

Example 3.4. *Consider the system:*

$$\dot{x} = 2x + \sqrt{2}\ u,\ y = \sqrt{2}x + \frac{1}{2}\ u.$$

The Hamiltonian matrix H corresponding to the passivity supply rate is as defined in equation (3). Hence, $H =$

$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. *Clearly, H is normal. Note that the system is not dissipative with respect to the passivity supply rate.*

Using Lemma 3.3 we show in the next theorem that for a SISO system with a single state, the Hamiltonian matrix H considered in Theorem 3.2 is normal if and only if the system considered is all-pass (after a suitable scaling of the transfer function to have $d = 0.5$).

Theorem 3.5. *Consider a single state, SISO system $\dot{x} = ax + bu$, $y = cx + du$, where $a > 0$, $b = c \neq 0$ and $d = \frac{1}{2}$. Construct the Hamiltonian matrix corresponding to the passivity supply rate:*

$$H := \begin{bmatrix} a - bc & b^2 \\ -c^2 & -a + bc \end{bmatrix} \tag{14}$$

Then, H is normal if and only if the system is all-pass.

Proof: (If): The transfer function of the given system is $G(s) := \frac{1}{2} + \frac{cb}{s-a} = \frac{s+2cb-a}{2(s-a)}$. Since $G(s)$ is all-pass and $\lim_{s\to\infty} G(s) = \frac{1}{2}$, $|G(j\omega)| = \frac{1}{2}$ for all $\omega \in \mathbb{R}$. Thus, we have

$$\frac{\sqrt{\omega^2 + (2cb - a)^2}}{2\sqrt{\omega^2 + a^2}} = \frac{1}{2} \Rightarrow 2cb - a = \pm a \Rightarrow 2b^2 = a \pm a.$$

Since $b = c \neq 0$, $b = \pm\sqrt{a}$. Using these values of b, c and $d = \frac{1}{2}$ in equation (14), we have $H = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$. Thus, $HH^T = H^T H$ and hence H is normal.

(Only if): For H to be normal, i.e, $H^T H = HH^T$ the variables a, b and c must satisfy the equation $ab^2 = b^4$. Using Theorem 3.2, it follows that if $a \leqslant 0$, then H is normal if $b = c = 0$ but $b = c \neq 0$. Hence, a must be a positive real number, i.e., $a > 0$. Thus $ab^2 = b^4 \Rightarrow b = \pm\sqrt{a} = c$. Hence, the transfer function of the system becomes $G(s) = c(s-a)^{-1}b + d = \frac{a}{s-a} + \frac{1}{2} = \frac{s+a}{2(s-a)}$. Since $\frac{1}{4} - G(s)G(-s) = 0$, the system is all-pass. □

Now that we have a necessary condition for the Hamiltonian matrix to be normal, the next natural question is: under what conditions do the Hamiltonian matrix not have non-defective imaginary eigenvalues? As noted in Section 1 Hamiltonian matrix will have defective eigenvalues if the given system is controllable. Hence, in order to find stronger conditions for H to have defective eigenvalues, we assume that the given system is uncontrollable. We also assume that the system is dissipative with respect to the passivity supply rate which ensures existence of a solution to the corresponding ARE.

Theorem 3.6. *Consider a system $\dot{x} = Ax + Bu$, $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, $D = D^T \in \mathbb{R}^{m\times m}$ with (A, B) possibly uncontrollable and $(D + D^T) > 0$. Assume the system has no uncontrollable imaginary axis eigenvalues. Let the system be dissipative with respect to the passivity supply rate. The Hamiltonian matrix H corresponding to the passivity supply rate is as defined in equation (3). Let $\pm j\omega \in \sigma(H)$. If $G(s) := D + C(sI - A)^{-1}B$ has either no poles or no zeros on the imaginary axis, then $j\omega$ is defective in H.*

Proof: First we prove the theorem for the case when $G(s)$ has no poles on the imaginary axis. Since the system is dissipative with respect to the passivity supply rate, there exists a hermitian matrix $X \in \mathbb{R}^{n\times n}$ such that $A^T X + XA + (XB - C^T)(D + D^T)^{-1}(B^T X - C) = 0$. Define $S := \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix}$. Note that S is invertible and $S^{-1} = \begin{bmatrix} I_n & 0 \\ -X & I_n \end{bmatrix}$. Clearly,

$$S^{-1}HS = \begin{bmatrix} I_{n,n} & 0 \\ -X & I_{n,n} \end{bmatrix} H \begin{bmatrix} I_{n,n} & 0 \\ X & I_{n,n} \end{bmatrix} = \begin{bmatrix} A_F & B(D + D^T)^{-1}B^T \\ 0 & -A_F^T \end{bmatrix} \tag{15}$$

where $A_F = A - B(D + D^T)^{-1}C + B(D + D^T)^{-1}B^T X$. Without loss of generality, we can write the matrices A, B and C as:

$$A = \begin{bmatrix} A_{\mathtt{c}} & \widehat{A} \\ 0 & A_{\mathtt{uc}} \end{bmatrix}, B = \begin{bmatrix} B_{\mathtt{c}} \\ 0 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

where $A_{\mathtt{c}} \in \mathbb{R}^{n_c\times n_c}$ and $A_{\mathtt{uc}} \in \mathbb{R}^{(n-n_c)\times(n-n_c)}$. We partition X such that $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$ where $X_{11} \in \mathbb{R}^{n_c\times n_c}$, $X_{12} \in \mathbb{R}^{n_c\times(n-n_c)}$ and $X_{22} \in \mathbb{R}^{(n-n_c)\times(n-n_c)}$. Equation (15) becomes

$$S^{-1}HS = \begin{bmatrix} A_{f1} & A_{f2} & B_{\mathtt{c}}(D + D^T)^{-1}B_{\mathtt{c}}^T & 0 \\ 0 & A_{\mathtt{uc}} & 0 & 0 \\ 0 & 0 & -A_{f1}^T & 0 \\ 0 & 0 & -A_{f2}^T & -A_{\mathtt{uc}}^T \end{bmatrix}, \tag{16}$$

where $A_{f1} = A_{\mathtt{c}} - B_{\mathtt{c}}(D + D^T)^{-1}C_1 + B_{\mathtt{c}}(D + D^T)^{-1}B_{\mathtt{c}}^T X_{11}$ and $A_{f2} = \widehat{A} - B_{\mathtt{c}}(D + D^T)^{-1}C_2 + B_{\mathtt{c}}(D + D^T)^{-1}B_{\mathtt{c}}^T X_{12}$. Note that since H and $S^{-1}HS$ are similar, we have

$$\sigma(H) = \sigma(S^{-1}HS) = \sigma\left(\begin{bmatrix} A_{f1} & A_{f2} \\ 0 & A_{\mathtt{uc}} \end{bmatrix}\right) \cup \sigma\left(\begin{bmatrix} -A_{f1}^T & 0 \\ -A_{f2}^T & -A_{\mathtt{uc}}^T \end{bmatrix}\right)$$
$$= \sigma(A_{f1}) \cup \sigma(-A_{f1}) \cup \sigma(A_{\mathtt{uc}}) \cup \sigma(-A_{\mathtt{uc}}).$$

Since the system has no uncontrollable imaginary axis eigenvalue $\sigma(A_{\mathtt{uc}}) \cap j\mathbb{R} = \emptyset$. Hence, $j\omega \in \sigma(A_{f1}) \cup \sigma(-A_{f1})$. Since $A_{f1} \in \mathbb{R}^{n\times n}$, $j\omega \in \sigma(A_{f1}) \cap \sigma(-A_{f1}) \Rightarrow j\omega \in \sigma(A_{f1})$.

Define $B_{\mathtt{c}}(D + D^T)^{-1}B_{\mathtt{c}}^T =: \widehat{B_{\mathtt{c}}}$. Since the system is (A, B) controllable $\Rightarrow$ the system is $(A_{\mathtt{c}}, B_{\mathtt{c}})$ controllable $\Rightarrow$ the system is $(A_{f1}, \widehat{B_{\mathtt{c}}})$ is controllable. Therefore, $\mathscr{R}_{j\omega}(H) \subset \mathscr{C}_{A_{f1},\widehat{B_{\mathtt{c}}}}$. Using Proposition 2.4, we hence conclude that the partial multiplicity of $j\omega$ is even. This proves that $j\omega$ is defective. Similarly, $-j\omega$ is defective as well.

Next we prove the theorem for the case when $G(s)$ has no zeros on the imaginary axis. Since $G(s)$ has no zeros on the imaginary axis, therefore $G(s)^{-1}$ has no poles on the imaginary axis. Using [11, Lemma 4.3] it is clear that an i/s/o representation of $G(s)^{-1}$ is given by $\dot{p} = \widetilde{A}p + \widetilde{B}f$ and $e = \widetilde{C}p + \widetilde{D}f$, where $\widetilde{A} = A - BD^{-1}C$, $\widetilde{B} = BD^{-1}$, $\widetilde{C} = -D^{-1}C$, and $\widetilde{D} = D^{-1}$. Using [11, Theorem 4.1], we infer that there exists a symmetric matrix $\tilde{X} \in \mathbb{R}^{n\times n}$ that satisfies the ARE:

$$\widetilde{A}^T\tilde{X} + \tilde{X}\widetilde{A} + (\tilde{X}\widetilde{B} - \widetilde{C}^T)(\tilde{D} + \tilde{D}^T)^{-1}(\widetilde{B}^T\tilde{X} - \widetilde{C}) = 0.$$

The corresponding Hamiltonian matrix is

$$\tilde{H} := \begin{bmatrix} \widetilde{A} - \widetilde{B}(\tilde{D} + \tilde{D}^T)^{-1}\widetilde{C} & \widetilde{B}(\tilde{D} + \tilde{D}^T)^{-1}\widetilde{B}^T \\ -\widetilde{C}^T(\tilde{D} + \tilde{D}^T)^{-1}\widetilde{C} & -(\widetilde{A} - \widetilde{B}(\tilde{D} + \tilde{D}^T)^{-1}\widetilde{C})^T \end{bmatrix}$$

Using the fact that $D = D^T$ and $\widetilde{A} = A - BD^{-1}C$, $\widetilde{B} = BD^{-1}$,

$\widetilde{C} = -D^{-1}C$ and $\widetilde{D} = D^{-1}$, we have

$$\begin{cases} \widetilde{A} - \widetilde{B}\left(\tilde{D}+\tilde{D}^T\right)^{-1}\widetilde{C} = A - BD^{-1}C + BD^{-1}\left(D^{-1}+D^{-T}\right)^{-1}D^{-1}C \\ \qquad\qquad = A - BD^{-1}C + \frac{1}{2}BD^{-1}DD^{-1}C \\ \qquad\qquad = A - B(D+D^T)^{-1}C \\ \widetilde{B}(\widetilde{D}+\widetilde{D}^T)^{-1}\widetilde{B}^T \quad = BD^{-1}(D^{-1}+D^{-T})^{-1}D^{-T}B^T \\ \qquad\qquad = B(D+D^T)^{-1}B^T \\ \widetilde{C}^T(\widetilde{D}+\widetilde{D}^T)^{-1}\widetilde{C} \quad = C^TD^{-T}(D^{-1}+D^{-T})^{-1}D^{-1}C \\ \qquad\qquad = C^T(D+D^T)^{-1}C \end{cases} \tag{17}$$

From equation (17) it is evident that $\tilde{H} = H$. Therefore, $j\omega \in \sigma(\tilde{H})$. Using the same line of argument as given for the case when $G(s)$ has no poles on the imaginary axis, we infer that $j\omega$ is defective. □

Thus we infer from Theorem 3.6 that if a purely imaginary eigenvalue of the Hamiltonian matrix is non-defective, the particular imaginary eigenvalue must be an eigenvalue of A. The next natural question therefore is: when does a Hamiltonian matrix admit non-defective imaginary eigenvalues? In the following theorem, we formulate a sufficient condition for a Hamiltonian matrix to admit non-defective imaginary eigenvalues. For formulating this condition, we assume that matrix A has distinct eigenvalues.

Theorem 3.7. *Consider a system $\dot{x} = Ax+Bu$, $y = Cx+Du$, where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, $D = D^T \in \mathbb{R}^{m\times m}$. Let the eigenvalues of matrix A be distinct, and let the Hamiltonian matrix H, possibly with repeated eigenvalues on $j\mathbb{R}$, be as defined in equation* (3)*. Let $\pm j\omega \in \sigma(H)$. If $\pm j\omega$ is uncontrollable and unobservable in A, then $\pm j\omega$ is non-defective in H.*

Proof: Since all eigenvalues of A are distinct, without loss of generality, the matrices $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^{p\times n}$ have the following structure:

$$A = \begin{bmatrix} A' & 0 \\ 0 & A_{\texttt{uco}} \end{bmatrix},\ B = \begin{bmatrix} B' \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} C' & 0 \end{bmatrix}$$

where $A' \in \mathbb{R}^{k\times k}, B', C'^T \in \mathbb{R}^{k\times n}$ and $A_{\texttt{uco}} \in \mathbb{R}^{(n-k)\times(n-k)}$ contains eigenvalues that are both uncontrollable and unobservable. The Hamiltonian matrix can be written as:

$$H = \begin{bmatrix} A' - B'(D+D^T)^{-1}C' & 0 & B'(D+D^T)^{-1}B'^T & 0 \\ 0 & A_{\texttt{uco}} & 0 & 0 \\ -C'^T(D+D^T)^{-1}C' & 0 & -(A'-B'(D+D^T)^{-1}C')^T & 0 \\ 0 & 0 & 0 & -A_{\texttt{uco}}^T \end{bmatrix}$$

Thus, if $\pm j\omega$ are non-defective in $A_{\texttt{uco}}$ (due to distinctiveness of eigenvalues of A), $\pm j\omega$ are non-defective in H. □

4. Conclusions

Existence of an imaginary eigenvalue in a Hamiltonian matrix causes problems in standard ARE solvers. In most of the cases, the imaginary eigenvalues of the Hamiltonian matrix are defective (see Figure 1). However, as observed in the circuit of Figure 2, if there exist uncontrollable states in the system, certain Jordan blocks of size one can appear in the Hamiltonian matrix. Hence, in this paper, we formulated conditions for defectiveness of Hamiltonian matrix imaginary eigenvalues. In summary, the contribution of this paper are:

(i) We formulated conditions under which a Hamiltonian matrix is normal (Theorem 3.2). Normality of the Hamiltonian matrix ensures that all the eigenvalues of the matrix are non-defective. Assuming matrix A to be semi-Hurwitz, symmetric (or skew-symmetric) and $B = C^T$, we proved that the Hamiltonian matrix is normal only if $B = C^T = 0$. Further, we also showed that for a single state SISO system the Hamiltonian matrix corresponding to the passivity supply rate is normal if and only if the system is all-pass (Theorem 3.5).

(ii) In Theorem 3.6, we formulated a sufficiency condition for the imaginary eigenvalues of the Hamiltonian matrix to be defective. We showed that for a system that is dissipative with respect to passivity supply rate and has no uncontrollable eigenvalues on the imaginary axis, the imaginary eigenvalues of the corresponding Hamiltonian matrix are defective if none of the open-loop poles (or zeros) of the system are on the imaginary axis.

(iii) In Theorem 3.7, we provided a sufficiency condition for the imaginary eigenvalues of the Hamiltonian matrix to be non-defective. We showed that for a system that is dissipative with respect to passivity supply rate, an imaginary eigenvalue of the Hamiltonian matrix is non-defective if the imaginary eigenvalue is an uncontrollable and unobservable eigenvalue of the system matrix.

References

[1] M.N. Belur and C. Praagman, An efficient algorithm for computing the $\mathscr{H}_\infty$ -norm, *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pages 1656-1660, 2011.

[2] C. Bhawal, D. Pal and M.N. Belur, On the link between storage functions of allpass systems and Gramians, *Proceedings of the 53rd IEEE Conference on Decision and Control (CDC)*, Australia, pages 4175-4180, Dec 2017.

[3] W.A. Coppel, Matrix quadratic equations, *Bulletin of the Australian Mathematical Society*, vol. 10, no. 3, pages 377-401, 1974.

[4] A.S.A. Dilip and H.K. Pillai, Yet another characterization of solutions of the Algebraic Riccati Equation, *Linear Algebra and its Applications*, vol. 481, pages 1-35, 2015.

[5] G. Freiling, On the existence of Hermitian solutions of general algebraic Riccati equations, *Proceedings of the 14th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Perpignan, France, 2000.

[6] G. Freiling, V. Mehrmann, H. Xu, Existence, uniqueness, and parametrization of Lagrangian invariant subspaces, *SIAM Journal on Matrix Analysis & Applications*, vol. 23, pages 1045-1069, 2002.

[7] S.C. Jugade, D. Pal, R.K. Kalaimani and M.N. Belur, Stationary trajectories, singular Hamiltonian systems and ill-Posed interconnection, *Proceedings of the IEEE European Control Conference (ECC)*, Zurich, Switzerland, pages 1740-1745, 17-19 July, 2013.

[8] R.K. Kalaimani, M.N. Belur and D. Chakraborty, Singular LQR control, optimal PD controller and inadmissible initial conditions, *IEEE Transactions on Automatic Control*, vol. 58, no. 10, pages 2603-2608, 2013.

[9] A. Kothyari, C. Praagman and M.N. Belur, Lossless systems storage function: new results and numerically-stable and non-iterative computational methods, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 2018.

[10] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, 1995.

[11] R.U. Chavan, V.P. Samuel, K. Mallick, M.N. Belur, Optimal charging/discharging and commutativity properties of ARE solutions for RLC circuits, *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Groningen, Netherlands, 2014.

[12] M.A. Shayman, Geometry of the Algebraic Riccati Equation, Part I, *SIAM Journal of Control and Optimization*, vol. 21, no. 3, pages 375-394, 1983.

[13] J.C. Willems, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Transactions on Automatic Control*, vol. 16,no. 6,pages 621-634, 1971.