# Generalized Riccati theory: A Hamiltonian system approach

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by

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This thesis is dedicated to my parents, brother, sister-in-law, and fiancée.

# Thesis Approval

The thesis entitled

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# Declaration

I, Chayan Bhawal, declare that this written submission represents my ideas in my own words and where others ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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# Abstract

The primary objective of this thesis is to put forth a generalized Riccati theory that is applicable not only to problems/systems that admit algebraic Riccati equation (ARE) but also to problems/systems that do not admit AREs due to singularity of certain matrices. To achieve this we use a more fundamental object than the AREs. We use the linear matrix inequalities (LMIs) from which AREs are known to arise; we call these LMIs the dissipation LMIs. The primary reason for using these LMIs is the fact that their existence do not depend on the nonsingularity of any matrices. Primarily, we deal with two typical applications in this thesis where AREs do not exist but the dissipation LMIs do, viz., a singular linear quadratic regulator (LQR) problem with the underlying system having a single-input and a passive SISO system with a strictly proper transfer function. We call the dissipation LMI corresponding to a singular LQR problem the LQR LMI and the one corresponding to a passive SISO system the KYP LMI. In order to achieve our objective, we first show that the maximal and rank-minimizing solutions of the LQR and KYP LMI, respectively can be computed by an extension of a conventional Hamiltonian based method used to solve these LMIs for the case when they admit AREs. This extension comes in the form of compensating the eigenspaces of a suitable matrix pencil by adding new basis vectors coming from a subspace of the strongly reachable space corresponding to the underlying Hamiltonian system. This straightaway leads to interesting system-theoretic interpretations in terms of the dissipation LMI solutions. Using the method to compute the maximal solution of an LQR LMI, we not only show that almost every (made precise in a suitable topology) singular LQR problem can be solved using a proportional-derivative (PD) state-feedback controller, but also provide a method to design such controllers. To this end, we also characterize the optimal trajectories of a singular LQR problem corresponding to an arbitrary initial condition. We show that, similar to the singular LQR case, a passive SISO system with proper transfer function can be confined to its lossless trajectories using PD state-feedback controllers. Apart from these, we also present algorithms to compute the solutions of KYP LMIs admitted by a special and very familiar class of passive systems called lossless systems (ARE does not exist for these too). These algorithms are designed using different notions of control theory and network theory like states and costates of a system, Foster-Cauer network synthesis methods, two-dimensional discrete Fourier transform, observability and controllability Gramian.

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# List of symbols

## Fields, sets, rings and spaces

| | | |
|---|---|---|
| $\mathbb{R}$ | : | The field of real numbers. |
| $\mathbb{C}$ | : | The field of complex numbers. |
| $\mathbb{N}$ | : | The set of natural numbers. |
| $\mathbb{Z}$ | : | The set of integers. |
| $\mathbb{R}_+$ | : | The set of positive-real numbers. |
| $\mathbb{C}_+$ | : | The set of complex numbers in open-right half $\mathbb{C}$ plane. |
| $\mathbb{C}_-$ | : | The set of complex numbers in open-left half $\mathbb{C}$ plane. |
| $\mathbb{R}[s]$ | : | The set of polynomial functions in one-variable $s$ with coefficients from $\mathbb{R}$. |
| $\mathbb{R}(s)$ | : | The set of rational functions in one-variable $s$ with coefficients from $\mathbb{R}$. |
| $\mathbb{C}[s]$ | : | The set of polynomials functions in one-variable $s$ with coefficients from $\mathbb{C}$. |
| $\mathbb{R}[x,y]$ | : | The set of polynomials functions in two-variable $x,y$ with coefficients from $\mathbb{R}$. |
| $\mathbb{C}[x,y]$ | : | The set of polynomials functions in two-variable $x,y$ with coefficients from $\mathbb{C}$. |
| $\mathbb{R}^{\mathtt{n}}$ | : | The space of column vectors with $\mathtt{n}$ elements from $\mathbb{R}$. |
| $\mathbb{C}^{\mathtt{n}}$ | : | The space of column vectors with $\mathtt{n}$ elements from $\mathbb{C}$. |
| $\mathbb{R}^{1\times\mathtt{n}}$ | : | The space of row vectors with $\mathtt{n}$ elements from $\mathbb{R}$. |
| $\mathbb{C}^{1\times\mathtt{n}}$ | : | The space of row vectors with $\mathtt{n}$ elements from $\mathbb{C}$. |
| $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ | : | The set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}$. |
| $\mathbb{C}^{\mathtt{n}\times\mathtt{p}}$ | : | The set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{C}$. |
| $\mathbb{R}[s]^{\mathtt{n}\times\mathtt{p}}$ | : | The set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}[s]$. |
| $\mathbb{R}(s)^{\mathtt{n}\times\mathtt{p}}$ | : | The set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}(s)$. |
| $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})$ | : | The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$. |
| $\mathfrak{C}^\infty$ | : | The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. |
| $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})\vert_{\mathbb{R}_+}$ | : | The space of all functions from $\mathbb{R}_+$ to $\mathbb{R}^{\mathtt{n}}$ that are restrictions of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$. |
| $\mathfrak{C}^{\mathtt{n}}_{\mathtt{imp}}$ | : | The space of impulsive smooth distributions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ (See Definition 2.12) |

## Fields, sets, rings and spaces

| | | |
|---|---|---|
| $\mathtt{img}\,(A)$ | : | Subspace spanned by the columns of a matrix $A$. |
| $\mathtt{ker}\,(A)$ | : | Subspace spanned by the kernel of a matrix $A$. |
| $\mathfrak{R}$ | : | A commutative ring with multiplicative identity 1. |
| $\langle p(x), q(y) \rangle$ | : | A ideal generated by polynomials $p(x)$ and $q(y)$. (Used in Section 7.7.3) |
| $\mathfrak{R}/\mathbb{I}$ | : | The quotient ring of an ideal $\mathbb{I}$. (See Section 7.2.3) |
| $\sqrt{\mathbb{J}}$ | : | Radical of an ideal $\mathbb{J}$. (See Definition 7.3) |
| $\mathbb{V}(\mathbb{I})$ | : | Denotes variety of an ideal $\mathbb{I}$. |
| $\mathscr{I}(\mathbb{V}(\mathbb{I}))$ | : | Ideal consisting of polynomials whose roots are in the variety of an ideal $\mathbb{I}$. |
| $\{0\}$ | : | The zero subspace. |
| $\mathfrak{I}(A,B)$ | : | The set of $(A,B)$-invariant subspaces. (See Definition 2.9) |
| $\mathfrak{I}(A,B;\mathtt{ker}\,C)$ | : | The set of $(A,B)$-invariant subspaces inside $\mathtt{ker}\,C$. (See Section 2.2.4) |
| $\sup\mathfrak{I}(A,B;\mathtt{ker}\,C)$ | : | Supremal element of the set $\mathfrak{I}(A,B;\mathtt{ker}\,C)$. (See Section 2.2.4) |
| $\mathfrak{C}(A,B)$ | : | The set of controllability subspaces of $(A,B)$. (See Definition 2.11) |
| $\mathfrak{C}(A,B;\mathtt{ker}\,C)$ | : | The set of controllability subspaces of $(A,B)$ inside $\mathtt{ker}\,C$. |
| $\sup\mathfrak{C}(A,B;\mathtt{ker}\,C)$ | : | Supremal element of the set $\mathfrak{C}(A,B;\mathtt{ker}\,C)$. (See Section 2.2.4) |
| $\mathscr{O}_w$ | : | Weakly unobservable subspace of a system. (See Definition 2.13) |
| $\mathscr{R}_s$ | : | Strongly reachable subspace of a system. (See Definition 2.15) |
| $\mathscr{O}_{wg}$ | : | Good slow subspace of a system. (See Section 2.2.4) |
| $\mathscr{O}_{wb}$ | : | Bad slow subspace of a system. (See Section 2.2.4) |
| $\mathbf{F}(\mathscr{R})$ | : | The set of friends of an $(A,B)$-invariant subspace $\mathscr{R}$. (See Section 2.2.4) |
| $\mathtt{roots}\,(p(s))$ | : | The set of the roots of a polynomial $p(s)$, where an element is included in the set as many times as it appears as a root. |
| $\sigma(A)$ | : | $\mathtt{roots}\,(\mathtt{det}(sI-A))$. |
| $\sigma(E,A)$ | : | $\mathtt{roots}\,(\mathtt{det}(sE-A))$. |
| $\Lambda$ | : | Lambda-set of $\mathtt{det}(sE-H)$. (See Definition 2.18) |

## Miscellaneous

| | | |
|---|---|---|
| $\mathtt{det}(A)$ | : | Determinant of the matrix $A$. |
| $A\|_{\mathscr{S}}$ | : | Restriction of a matrix $A$ to its invariant subspace $\mathscr{S}$. |
| $\mathtt{deg}(p(s))$ | : | Degree of a polynomial $p(s)$. |
| $\mathtt{degdet}(P(s))$ | : | Degree of the determinant of a polynomial matrix $P(s)$. |
| $A \uplus B$ | : | Disjoint union of sets $A$ and $B$. |
| $A \odot B$ | : | Elementwise multiplication of $A$ and $B$, where $A, B$ are matrices of same size (also called Hadamard product). |
| $A \oslash B$ | : | Elementwise division of $A$ and $B$, where $A, B$ are matrices of same size. |
| $\mathtt{rank}(A)$ | : | Rank of a matrix $A$. |

## Miscellaneous

| | | |
|---|---|---|
| $\texttt{nrank}(G(s))$ | : | Normal rank of a rational function matrix $G(s)$. |
| $\texttt{num}(g(s))$ | : | Numerator of $g(s)$, where $g(s) \in \mathbb{R}(s)$. |
| $\texttt{rootnum}(g(s))$ | : | Roots of $\texttt{num}(g(s))$, where $g(s) \in \mathbb{R}(s)$. |
| $\texttt{degnum}(g(s))$ | : | Degree of $\texttt{num}(g(s))$, where $g(s) \in \mathbb{R}(s)$. |
| $\mathscr{F}(A)$ | : | 2D-DFT matrix corresponding to a matrix $A$. |
| $\mathscr{F}^{-1}(A)$ | : | Inverse 2D-DFT matrix corresponding to a matrix $A$. |
| $\sqrt{A}$ | : | Square-root of a matrix $A$. |
| $\\|A\\|_2$ | : | Induced 2-norm of a matrix $A$. |
| $A^\dagger$ | : | Pseudo-inverse of a matrix $A$. |
| $A^{-T}$ | : | Transpose of the inverse of a matrix $A$, i.e., $(A^{-1})^T$. |
| $f(x,y)\vert_{p,q}$ | : | Evaluation of the function $f(x)$ at $x = p$ and $y = q$. |
| $\texttt{adj}(A)$ | : | Adjugate of a matrix $A$. |
| $A \geqslant B$ | : | $(A-B)$ is positive-semidefinite, where $A$ and $B$ are symmetric. |
| $e_i$ | : | Denotes a vector with 1 in the $i$-th position and zero elsewhere. |
| $A(1:\texttt{n}, 1:\texttt{m})$ | : | An $\texttt{n} \times \texttt{m}$ submatrix of a matrix $A$ with $1^{st}$ to $\texttt{n}^{th}$ rows and $1^{st}$ to $\texttt{m}^{th}$ columns of $A$. |
| $A = [a_{pq}]_{p,q=1,2,\ldots\texttt{n}}$ | : | $A \in \mathbb{R}^{\texttt{n}\times\texttt{n}}$ with element $a_{pq}$ in the $p$-th row and $q$-th column. |
| $A = [A_j]$ | : | $A_j$ is the $j$-th column/row of $A$. |
| $I_\texttt{n} \in \mathbb{R}^{\texttt{n}\times\texttt{n}}$ | : | $\texttt{n} \times \texttt{n}$ identity matrix. |
| $0_{\texttt{n},\texttt{m}}$ | : | A zero matrix with $\texttt{n}$ rows and $\texttt{m}$ columns. |
| $\texttt{diag}(G_1, \ldots, G_\texttt{m})$ | : | Block diagonal matrix where $G_1, \ldots, G_\texttt{m}$ are square matrices. |
| $\texttt{col}(B_1, B_2, \ldots, B_\texttt{n})$ | : | $\begin{bmatrix} B_1^T & B_2^T & \cdots & B_\texttt{n}^T \end{bmatrix}^T$. |
| $A \otimes B$ | : | Kronecker product of matrices $A$ and $B$. |
| $\mathscr{V} \cong \mathscr{W}$ | : | The vector-space $\mathscr{V}$ is isomorphic to vector-space $\mathscr{W}$. |
| $\mathscr{V} \perp \mathscr{W}$ | : | The vector-space $\mathscr{V}$ is orthogonal to vector-space $\mathscr{W}$. |
| $\mathscr{V} \oplus \mathscr{W}$ | : | Direct-sum of subspaces $\mathscr{V}$ and $\mathscr{W}$. |
| $\texttt{dim}(\mathscr{V})$ | : | Dimension of a vector-space $\mathscr{V}$. |
| $\delta(t)$ | : | Dirac delta impulse function. For brevity $\delta$ is used as well. |
| $\delta^{(i)}$ | : | $i$-th distributional derivative of $\delta$ with respect to $t$. |
| $\binom{\texttt{n}}{k}$ | : | Number of $k$ combinations from a set of $\texttt{n}$ elements. |
| $\bar{\lambda}$ | : | Complex-conjugate of $\lambda \in \mathbb{C}$. |
| $\texttt{Re}(p)$ | : | Real-part of a vector $p \in \mathbb{C}^\texttt{n}$. |
| $\texttt{Im}(p)$ | : | Imaginary-part of a vector $p \in \mathbb{C}^\texttt{n}$. |
| $\bullet$ | : | Used when a dimension need not be specified; for example $\mathbb{R}^{\texttt{n}\times\bullet}$ denotes the set of real constant matrices having $\texttt{n}$ rows. |

# List of abbreviations

| | | |
|---|---|---|
| ARE | : | Algebraic Riccati Equation. |
| ARI | : | Algebraic Riccati Inequality. |
| BIBO | : | Bounded-input Bounded-output. |
| CGCARE | : | Constrained Generalized Continuous ARE. |
| CVX | : | Matlab Software for Disciplined Convex Programming. |
| DAE | : | Differential Algebraic Equation. |
| i/s/o | : | Input-State-Output. |
| KYP | : | Kalman-Yakubovich-Popov. |
| LAPACK | : | Linear Algebra Package. |
| LM | : | Leading monomial. |
| LME | : | Linear Matrix Equality. |
| LMI | : | Linear Matrix Inequality. |
| LQR | : | Linear Quadratic Regulator. |
| LTS | : | Long Term Support. |
| MATLAB | : | Matrix Laboratory. |
| MIMO | : | Multi-Input Multi-Output. |
| MPB | : | Minimal Polynomial Basis. |
| PD | : | Proportional-derivative. |
| RLC | : | Resistor, Inductor and Capacitor. |
| SCILAB | : | Scientific Laboratory. |
| SDP | : | Semi-Definite Programming. |
| SFT | : | Spectral Factorization Technique. |
| SISO | : | Single-Input Single-Output. |
| SVD | : | Singular Value Decomposition. |
| YALMIP | : | Yet Another LMI Parser. |

# Chapter 1

# Introduction

The emergence of algebraic Riccati equation (ARE) in quadratic optimal control and dissipativity theory has been one of the cornerstones in control theory [Kal60], [Wil71], [TW91]. The elegant theoretical framework of ARE combined with numerically stable algorithms for computation of ARE solutions are perhaps the primary reasons for the widespread application of ARE in control and system theory [LR95], [KTK99], [BS13]. From the literature on AREs, it is known that an ARE always arises from a linear matrix inequality (LMI); we call these LMIs the *dissipation LMIs* for ease of reference [LR95]. Nonsingularity of certain matrices, depending on the application (e.g. $D+D^T$ in case of passive systems), is crucial for the reduction of these dissipation LMIs to their corresponding AREs. We call such matrices the *feed-through terms* and the condition of nonsingularity of these matrices the *feed-through regularity condition*. Interestingly, unlike AREs, existence of a dissipation LMI does not depend on the feed-through regularity condition. Hence, there are systems/problems where an ARE does not exist, due to non-satisfaction of the feed-through regularity condition, but the dissipation LMI does. Thus, the fundamental object in any analysis that involves an ARE is not the ARE itself but the dissipation LMI from which such an ARE arises. Since the theory developed for AREs crucially hinges on the feed-through regularity conditions, the application of AREs are limited to systems/problems that satisfy these conditions. Hence, there is a natural need for a common theoretical framework that generalizes the theory of AREs to the dissipation LMIs such that the generalized theory no longer has to depend on the feed-through regularity condition. In this thesis, we bridge this gap between the ARE literature and the dissipation LMIs. Typical examples of systems/problems where the AREs do not exist, but the dissipation LMIs do, are the singular linear quadratic regulator (LQR) problems and passive single-input single-output (SISO) systems that admit strictly proper transfer functions. We divide the thesis into two parts:

I. Infinite-horizon singular LQR problems,

II. Passive systems.

One of the salient features of the solutions of an ARE is the fact that such solutions have elegant system-theoretic interpretations. For example, in an infinite-horizon LQR problem, it is known that the *maximal* solution of the corresponding ARE helps in characterizing the optimal trajectories of the system [Kir04]. Further, such a solution also leads to the design of the state-feedback controller that solves the corresponding LQR problem. Similarly, in passive systems, the solutions of the corresponding ARE is related to the notion of optimal-charging and optimal-discharging of the system [WT98]. Hence, while bridging the gap between the ARE literature and the dissipation LMIs it is important that we generalize these system-theoretic interpretations in terms of the dissipation LMI solutions. To this end we not only put forth a generalized Riccati theory but also provide methods to design feedback-controllers to solve infinite-horizon singular LQR problems and confine passive systems to their optimal charging/discharging trajectories.

## 1.1 A brief literature survey

In this section we present a brief literature survey of the problems we are dealing with in this thesis. A more detailed literature survey is done in the beginning of each chapter of the thesis based on the objective of each chapter.

### 1.1.1 Infinite-horizon singular LQR problems

The objective of an infinite-horizon singular LQR problem is as follows:

**Problem 1.1. (Singular LQR problem)** *Consider a system $\Sigma$ with state-space dynamics $\frac{d}{dt}x = Ax + Bu$, where $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $B \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$. Then, for every initial condition $x_0 \in \mathbb{R}^{\mathrm{n}}$, find an input $u$ that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt, \tag{1.1}$$

*where* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$ *and $R$ is singular.*

Since in this thesis we deal with infinite-horizon singular LQR problems only, we drop the term infinite-horizon in the sequel. A typical example of a singular LQR problem is the minimization of energy associated with a damped spring-mass system.

**Example 1.2.** *Consider a damped spring-mass system as in Figure 1.1 with m, q, c, k, and u being the mass, displacement of mass, coefficient of viscous friction, spring constant, and applied force, respectively. On using $(p_1, p_2)$ as states, where $p_1 := q$ and $p_2 := \dot{q}$, the dynamics of the system is given by the following state-space equation:*

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_2 \\ -\frac{c}{m}p_2 - \frac{k}{m}p_1 + \frac{u}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

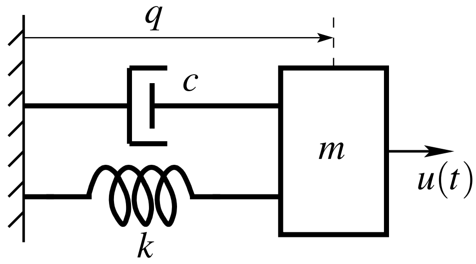


Figure 1.1: A damped spring-mass system

*Then, for every initial condition $x_0 \in \mathbb{R}^n$, find an input u that minimizes the total energy of the system, i.e, find u that minimizes the functional*

$$J(x_0,u) = \int_0^\infty \left(\frac{1}{2}kp_1^2 + \frac{1}{2}mp_2^2\right) dt = \frac{1}{2}\int_0^\infty \left( \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \right) dt. \tag{1.2}$$

*Note that there is no cost associated with the applied force ($R = 0$) in equation* (1.2).

Another area in which singular LQR problems naturally arise is that of *cheap control* problems, i.e., problems where the cost of the control $u$ is cheap relative to that of the state $x$. In such problems the cost functional is of the form:

$$J(x_0,u) := \int_0^\infty \left(x^T Qx + \varepsilon^2 u^T Ru\right) dt,$$

where $Q \geqslant 0$, $R \geqslant 0$ and $\varepsilon$ is a small positive parameter. Evidently, singular LQR problems are a limiting case ($\varepsilon \to 0$) of cheap LQR problems [HS83, Comment 2.12], [SS87]. The singular LQR problem, therefore, becomes relevant in any design problem that uses cheap control, in order to predict its limiting behavior. Such design problems can be pole-positioning problems ([AM71] [KS72]), inverse-regulator problems ([MA73]), differential games ([Pet86]) among other control applications.

It is noteworthy that for the case when the LQR Problem 1.1 has $R > 0$, called the *regular LQR problem*, an ARE of the form $A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0$ exists and a suitable solution of this ARE is used to design static state-feedback controllers to solve the problem. However, in [HS83] the authors showed that for a singular LQR problem the inputs that minimize equation (1.1), called *optimal inputs*, are impulsive in nature and hence cannot be implemented by a static state-feedback control law. Following this work, the authors in [WKS86] provided a method, based on Morse's canonical form, to compute the optimal inputs for the singular LQR Problem 1.1 and alluded to the fact that such inputs can be implemented using high-gain feedback controllers. Another interesting work in [Sch83] established a link between the optimal cost of a singular LQR problem and the maximal solution, among all rank-minimizing solutions, of the corresponding dissipation LMI. In this thesis, we call such a solution the *maximal rank-minimizing* solution of the corresponding dissipation LMI. Some other areas in which work related to the singular case has been done in the past are singular spectral-factorization in [CF89], singular $H_2$ control in [Sto92], singular $H_2$ and $H_\infty$ control in [CS92], etc.. There has been interesting work in this area in the recent years, as well. In [KBC13] the authors showed that singular LQR problems, under suitable assumptions, can be solved by proportional-derivative controllers. On the other hand, in [FN14], [FN16] and [FN18] the authors established that some singular LQR problems can be solved using static state-feedback. However, the results present in the literature, to the best of our knowledge, neither provide a method to solve a singular LQR problem using state-feedback in general nor links the solutions of the dissipation LMI that arises in such a problem to the optimal input that solves the problem.

### 1.1.2 Passive systems

A passive system with a minimal input-state-output (i/s/o) representation of the form $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + Du$ is known to admit solutions to the LMI arising out of the Kalman-Yakubovich-Popov (KYP) lemma [Kal63], [Yak62], [Pop64]:

$$\begin{bmatrix} A^TK + KA & KB - C^T \\ B^TK - C & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{1.3}$$

These solutions are known in the literature as the *storage functions* of the system due to their link to stored energy of the system [WT98]. We call the inequality (1.3) the *KYP LMI* for the ease of reference. Those passive systems that satisfy $D + D^T > 0$, the feed-through regularity condition here, therefore admit an ARE: $A^TK + KA + (KB - C^T)(D + D^T)^{-1}(B^TK - C) = 0$. However, there is a large class of passive systems that do not admit such an ARE but does admit an KYP LMI of the form in inequality (1.3). A typical example of such a system is an RLC network.

**Example 1.3.** *Consider the RLC network given in Figure 1.2. On using $(v_c, i_L)$ as states, where $v_c$ is the capacitor voltage and $i_L$ is the inductor current, the state-space dynamics of the system is given by:*

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$i = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

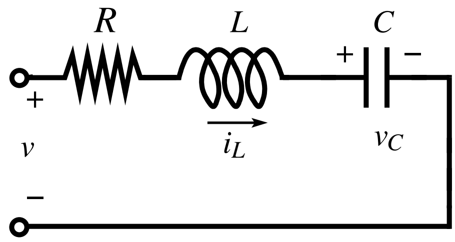



Figure 1.2: An RLC circuit

*Note that the RLC network in Figure 1.2 does not admit a feedthrough term, i.e, $D = 0$ and hence, it does not admit an ARE.*

Recently, it has been shown in [Rei11] that using the notion of *deflating subspaces* on a suitable matrix pencil it is possible to compute special solutions of the LMI (1.3). In [RRV15] the authors further generalized this idea to differential-algebraic systems, as well. It is noteworthy that the idea of using deflating subspaces to compute solutions of dissipation LMIs was introduced in [vD81]. Importantly, the idea of deflating subspaces provide a generalized framework for solving dissipation LMIs of the form (1.3) and the ones arising in singular LQR problems. However, there is no literature available, to the best of our knowledge, as to how these deflating subspaces can be linked to trajectories of the system or design state-feedback strategies to solve a problem like the one in Problem 1.1.

## 1.2 Contributions and outline of the thesis

Based on the application that we are dealing with, the entire thesis is divided into two parts. The first part is dedicated to singular LQR problems and the second to storage functions of passive SISO systems that do not admit AREs. Although there are two parts to the thesis there is common underlying theory that we develop throughout the thesis. We string it all together in the final chapter, Chapter 8, of the thesis to arrive at a generalized Riccati theory. Most of the results in this thesis are for single-input (in the singular LQR case) or single-input single-output (in the passivity case) systems, unless mentioned otherwise.

Now that we have a clear idea about the organization of the thesis, we present the main objectives and contributions of each of the chapters next. Part-I of the thesis consists of Chapters 2 - 4 and Chapters 5 - 7 form Part-II of the thesis.

**Chapter 2:** Computation of the optimal cost of an LQR problem is known to depend on the maximal rank-minimizing solution of the corresponding LMI. Hence, our objective is to provide a method to compute the maximal rank-minimizing solution of the LMI arising in a singular LQR problem.

<u>Contribution:</u> We present a method to compute the maximal rank-minimizing solution of the dissipation LMI that arises in a singular LQR problem. We show that one of the methods, based on *Hamiltonian systems*, to compute the maximal rank-minimizing solution of a dissipation LMI that admits ARE can indeed be extended to work for the singular case. We achieve this by substituting the role of the eigenspace involved in the computation of the maximal rank-minimizing solution of an LQR LMI by certain subspaces, namely weakly unobservable (slow) and strongly reachable (fast) subspaces, of the Hamiltonian system. To this end we present a novel characterization of the slow and fast subspaces of a SISO system in terms of certain matrix pencil. The theory developed in this chapter lays the foundation of the theoretical framework that generalizes the theory of AREs to dissipation LMIs.

**Chapter 3:** Solution of a regular LQR problem using static state-feedback is known to be possible. However, a state-feedback control law to solve a singular LQR problem is not known, in general. Hence, our objective is to find a state-feedback control law that solves the singular LQR problem.

<u>Contribution:</u> Using the theory developed in Chapter 2, we establish that almost every infinite-horizon LQR control problem with single-input admits an optimal solution in the form of a

feedback that is a suitable constant linear combination of the state and its first derivative, a *PD (proportional plus derivative) state-feedback*. The only assumption that we make is that a suitable matrix pair does not admit any eigenvalues on the imaginary axis. The theory developed in this chapter provides a system-theoretic interpretation to the maximal rank-minimizing solution of the dissipation LMI that arises in a singular LQR problem.

**Chapter 4:** It has been shown in the literature that solvability of the constrained generalized continuous algebraic Riccati equation (CGCARE) is a necessary and sufficient condition for a singular LQR problem to admit a solution that is implementable as a static state-feedback control law. Hence, our objective is to find conditions for the solvability of CGCARE.
<u>Contribution:</u> We provide a set of necessary and sufficient conditions for the solvability of a CGCARE. Using these conditions we show that a CGCARE generically does not admit solutions. This further leads to the conclusion that a singular LQR problem generically disallows solution by a static state-feedback law. The theory developed in this chapter shows that almost all singular LQR problems cannot be solved using static state-feedback controllers. Hence, in order to solve such problems we need to use PD-controllers that we designed in Chapter 3.

**Chapter 5:** The objective of this chapter is to provide an algorithm to compute the rank-minimizing solutions of a KYP LMI corresponding to a passive SISO system, for the case when the system does not admit an ARE. Such solutions of the KYP LMI are also known as the storage functions of the system. We call passive SISO systems that do not admit AREs and have no poles and zeros on the imaginary axis *singularly passive SISO systems*.
<u>Contribution:</u> Using the notions of weakly unobservable subspace and strongly reachable subspace we propose an algorithm to compute the rank-minimizing solutions of the KYP LMI (1.3) corresponding to a singularly passive SISO system. The theory developed in this chapter is analogous to the one developed in Chapter 2.

**Chapter 6:** Passive systems that admit ARE are known to admit *extremal storage functions* and *lossless trajectories*. Extremal storage functions are the maximal and minimal solutions of an ARE. On the other hand, lossless trajectories of a passive system are special trajectories related to the notion of optimal-charging and optimal-discharging of RLC circuits. Both the notions of extremal storage functions and lossless trajectories are known to be interlinked for passive systems that admit AREs. Hence, the objective of this chapter is to generalize the notion of extremal storage functions, lossless trajectories and the link between them for singularly passive SISO systems.

<u>Contribution:</u> We show that the set of solutions of the KYP LMI for singularly passive systems can be partially ordered with two extremal solutions with one being a maximum and the other being a minimum. This result is derived from a system-theoretic result that shows that the confinement of the initial conditions of a singularly passive SISO system over a suitably chosen set results in smooth *lossless* trajectories. All these results finally lead to a characterization of the lossless trajectories of a singularly passive SISO system. Further, we also introduce the notion of *fast lossless trajectories* of a singularly passive SISO system in this chapter. The results in this chapter are analogous to the ones developed in Chapter 3.

**Chapter 7:** In Chapter 7 we look into a special and very familiar class of passive systems called lossless systems. These systems do not satisfy the feed-through regularity condition and admit the KYP LMI (1.3) with equality. Lossless systems being special passive systems exhibit certain characteristics that other passive systems do not exhibit. Hence, the objective of this chapter is to propose methods to compute the storage functions of lossless systems utilizing the special characteristic properties of lossless systems.
Contribution: In this chapter we propose new results and algorithms to compute the storage function of a lossless system. The results in this chapter do not share the same theoretical framework as is developed in Chapters 2 - 6. We use five different techniques to compute the storage function of a lossless system. The first method is based on inversion of a controllability matrix, the second method is LC realization based (Foster, Cauer and their combinations) and the third is based on the Bezoutian of two polynomials. The notion of controllability/observability Gramians is used for the fourth, while the last method is based on the algebraic relations between the states and costates of a lossless system. A comparative study among the five methods shows that the Bezoutian method is one of the best in computational time and accuracy. Three different methods to compute the Bezoutian is also reported in the chapter: Euclidean long division, Pseudo-inverse method and the two dimensional discrete Fourier transform.

**Chapter 8:** In this chapter we draw parallels between the results in Part-I and Part-II of the thesis. Finally, irrespective of whether ARE exists or not, we arrive at a generalized theory applicable to singular LQR problems corresponding to a single-input, singularly passive SISO systems, and singular case of bounded-real SISO systems, as well.

# Part I

# Singular LQR problems

# Chapter 2

# Maximal rank-minimizing solution of an LQR LMI: single-input case

## 2.1 Introduction

Singular linear quadratic regulator (LQR) problem is an important problem in optimal control with a long history [KS72], [Fra79], [HS83], [Sch83], [SS87], [WKS86], [HSW00]. This problem still continues to be an active area of research [PNM08], [KBC13], [FN14], [FN16], [FN18]. In order to motivate the results in this chapter, we first state the infinite-horizon LQR problem [Kal60].

**Problem 2.1.** **(Infinite-horizon LQR problem)** *Consider a controllable system $\Sigma$ with minimal state-space dynamics $\frac{d}{dt}x = Ax + Bu$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition $x_0 \in \mathbb{R}^{\mathtt{n}}$, find an input $u$ that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt, \tag{2.1}$$

*where* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$ *and* $R \geqslant 0$.

A typical example of an infinite-horizon LQR problem is as follows:

**Example 2.2.** *Consider a system with state-space dynamics*

$$\dot{x}_1 = x_1 + x_3, \quad \dot{x}_2 = x_1 + x_3 + u, \quad \dot{x}_3 = x_1 + x_2$$

*For every initial condition $x_0$, find an input $u$ that minimizes the functional $\int_0^\infty x_3^2 \, dt$.*

Problem 2.1 with singular $R$ is known in the literature as the *singular LQR problem* and with $R > 0$ it is known as the *regular LQR problem*. Evidently, Example 2.2 is a singular LQR problem. The input $u$ that solves the LQR Problem 2.1 is known as the *optimal input* and the corresponding states $x$ are known as the *optimal state-trajectories* of the system. Further, the minimizers $\texttt{col}(x,u)$ of $J(x_0,u)$ in equation (2.1) are also called the *optimal trajectories* of the system $\Sigma$. Interestingly, it is known in the literature that the regular LQR problem, under suitable assumptions, is solvable using a static state-feedback law of the form $u(t) = Fx(t)$, where $F := -R^{-1}(B^T K_{\texttt{max}} + S^T)$ and $K_{\texttt{max}}$ is the maximal solution of the algebraic Riccati equation (ARE):

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0. \tag{2.2}$$

In other words, for regular LQR problems the state-feedback law $u(t) = Fx(t)$ confines the set of state-trajectories of the system $\Sigma$ to the optimal ones. However, it is known that for the singular LQR case such a confinement, using the feedback law $u(t) = Fx(t)$, is not possible [HS83], [WKS86]. For one, a feedback matrix $F$, as defined above, does not exist because $R$ is non-invertible for the singular LQR case. Moreover, the ARE itself does not exist either. However, all LQR problems, irrespective of regular or singular, admit LMIs of the form:

$$\begin{bmatrix} A^T K + KA + Q & KB + S \\ B^T K + S^T & R \end{bmatrix} \geqslant 0. \tag{2.3}$$

We call this the *LQR LMI*. Notably, it has been established in [Sch83] that for any LQR problem, the optimal cost is given by $x_0^T K_{\texttt{max}} x_0$, where $K_{\texttt{max}}$ is the maximal among all the rank-minimizing solutions of the LQR LMI (2.3). For ease of reference, we call such a solution the *maximal rank-minimizing solution* of the LQR LMI. Hence, in order to compute the optimal cost of an LQR problem, it is imperative that the maximal rank-minimizing solution of the LQR LMI (2.3) be computed. For a regular LQR problem, the maximal rank-minimizing solution of the LQR LMI is given by the maximal solution of the corresponding ARE. There are numerous methods to compute the maximal solution of an ARE. However, these methods cannot be used to compute the maximal rank-minimizing solution of an LQR LMI for the singular case. In this chapter, we show that for single-input systems, one of the methods to compute the maximal rank-minimizing solution of an LQR LMI for the regular case (Proposition 2.19) can be extended to the singular case. This method, for the regular case, is based on computing a suitable eigenspace of the corresponding *Hamiltonian system* [IOW99, Chapter 5]. A direct extension of this method to the singular case fails, since the dimension of the suitable eigenspace of the Hamiltonian system in such a case is less than what is required to compute the maximal rank-minimizing solution of the LQR LMI (see Example 2.20). We show in this chapter that the Hamiltonian system based method for the regular case can indeed be extended to the singular case by substituting the role of eigenspace of the Hamiltonian system in the regular case by the subspaces namely *weakly unobservable (slow)* and *strongly reachable (fast) subspaces* of the Hamiltonian system.

The idea of strongly reachable and weakly unobservable subspaces have been known to be crucial in singular LQR problem (see [HS83], [WKS86], [HSW00]). In these works, the strongly reachable and weakly unobservable subspaces of a system, on which the singular LQR problem is posed, have been characterized. Recursive algorithms, to compute such subspaces for a system, have also been provided in these works. We, however, apply these notions not to the system itself, but to the corresponding Hamiltonian system that one may obtain directly by applying Pontryagin's maximum principle (PMP) to the problem (notwithstanding the fact that the impulsive nature of the optimal control for singular problems makes application of PMP inappropriate). The singularity of $R$ (and hence of the LQR problem) manifests itself in causing the Hamiltonian system to be given by a system of differential algebraic equations (DAEs), as opposed to a system of differential equations in state-space form for the regular case. The DAEs of the Hamiltonian system naturally give rise to its weakly unobservable and strongly reachable subspaces. These subspaces ultimately lead us to a method to construct maximal rank-minimizing solution of the LQR LMI for a single-input system (Theorem 2.30).

In order to arrive at this method, we first use the recursive algorithms to characterize the weakly unobservable and strongly reachable subspaces of a single-input single-output (SISO) system in terms of a suitable matrix pencil known as the *Rosenbrock system matrix*. These are the first two main results of this chapter that we develop in Section 2.3 (Theorem 2.24 and Theorem 2.25). The primary take away from the results in Section 2.3 is the relation between the relative degree of the transfer function of a system and the dimensions of its weakly unobservable and strongly reachable subspaces. We exploit this relation and the fact that for autonomous systems the weakly unobservable and strongly reachable subspaces are the direct summands of the state-space to develop a method to compute the maximal rank-minimizing solution of the LQR LMI for the singular case. This is the third main result of this chapter (Theorem 2.30), which we present in Section 2.4. Another result that leads to Theorem 2.30 is the *disconjugacy* property of a certain eigenspace of a suitable matrix pencil called the Hamiltonian matrix pencil. This is the fourth main result of this chapter (Theorem 2.32) presented in Section 2.4.

## 2.2 Preliminaries

In this section we review some of the preliminary notions required to develop the results in this chapter.

### 2.2.1 Regular and singular matrix pencils

The notion of regular and singular matrix pencils are crucially used throughout the thesis and these are defined as follows:

**Definition 2.3.** [Dai89, Definition 1-2.1] *A matrix pencil* $U(s) := sU_1 - U_2 \in \mathbb{R}[s]^{\mathtt{n}\times\mathtt{n}}$ *is said to be* regular *if there exist a* $\lambda \in \mathbb{C}$ *such that* $\mathtt{det}(\lambda U_1 - U_2) \neq 0$. *In other words,* $U(s)$ *is regular if* $\mathtt{det}(sU_1 - U_2) \neq 0$. *On the other hand, the matrix pencil* $U(s)$ *is* singular *if* $\mathtt{det}(sU_1 - U_2) = 0$.

For the sake of brevity, we call the matrix pair $(U_1, U_2)$ regular (singular) if its corresponding matrix pencil $(sU_1 - U_2)$ is regular (singular). Another concept that is used throughout this thesis is the notion of eigenvalues and eigenvectors corresponding to a linear matrix pencil. We define them next.

**Definition 2.4.** [Dua10, Section 3.6] *Consider a regular matrix pencil* $(sU_1 - U_2)$ *with* $\lambda \in \texttt{roots}(\texttt{det}(sU_1 - U_2))$. *Then* $\lambda$ *is called an* eigenvalue *of* $(U_1, U_2)$ *and every nonzero vector* $v \in \texttt{ker}\,(\lambda U_1 - U_2)$ *is called an* eigenvector *of the matrix pair* $(U_1, U_2)$ *corresponding to the eigenvalue* $\lambda$. *Further, every nonzero vector* $\tilde{v} \in \texttt{ker}\,(\lambda U_1 - U_2)^k$, *where* $k \in \{2, 3, \ldots\}$, *is called a* generalized eigenvector *of the matrix pair* $(U_1, U_2)$ *corresponding to the eigenvalue* $\lambda$.

The number of times $\lambda \in \mathbb{C}$ appears as a root of $\texttt{det}(sU_1 - U_2)$ is called the algebraic multiplicity of the eigenvalue $\lambda$. We use the symbol $\sigma(U_1, U_2)$ to denote the set of eigenvalues of $(U_1, U_2)$ (with $\lambda \in \sigma(U_1, U_2)$ included in the set as many times as its algebraic multiplicity).

### 2.2.2 Output-nulling representation and Rosenbrock system matrix

Next we define the notion of Rosenbrock system matrix that has been extensively used in this thesis.

**Definition 2.5.** [Ros67] *Consider a system with an input-state-output (i/s/o) representation of the form*

$$\frac{d}{dt}x = Ax + Bu, \text{ and } y = Cx + Du, \text{ where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, \text{ and } D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}.$$

*Then, the matrix* $\begin{bmatrix} sI_{\mathtt{n}} - A & -B \\ -C & -D \end{bmatrix}$ *is called the* Rosenbrock system matrix *and the matrix pair* $\left(\begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0_{\mathtt{p},\mathtt{p}} \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ *is called the corresponding* Rosenbrock matrix pair.

Among the different ways of representing a system, the *output-nulling representation* of a system is of importance to us in this thesis and hence, we define this next.

**Definition 2.6.** [WT02] *A system is said to be in its* output-nulling representation *if it admits an i/s/o dynamics of the following form:*

$$\frac{d}{dt}x = Ax + Bu, \text{ and } 0 = Cx + Du, \text{ where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, \text{ and } D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}.$$

### 2.2.3 Canonical form of singular descriptor systems

In this thesis, we extensively use one of the canonical forms of a regular matrix pencil (see [Dai89] for more on different canonical forms). We review the result that leads to such a canonical form next.

**Proposition 2.7.** [Dai89, Lemma 1-2.2] *A matrix pair* $(U_1, U_2)$ *is* regular *if and only if there exist nonsingular matrices* $Z_1$ *and* $Z_2$ *such that* $Z_1U_1Z_2 = \mathtt{diag}(I_{\mathtt{n}_1}, Y)$ *and* $Z_1U_2Z_2 = \mathtt{diag}(U, I_{\mathtt{n}_2})$, *where* $\mathtt{n}_1 + \mathtt{n}_2 = \mathtt{n}, U \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_1}$, *and* $Y \in \mathbb{R}^{\mathtt{n}_2 \times \mathtt{n}_2}$ *is nilpotent*[1].

A matrix pair $(U_1, U_2)$ in the form $\left( \begin{bmatrix} I_{\mathtt{n}_1} & \\ & Y \end{bmatrix}, \begin{bmatrix} U & \\ & I_{\mathtt{n}_2} \end{bmatrix} \right)$ is said to be in a canonical form. Further, note that $\mathtt{det}(sU_1 - U_2) = k \times \mathtt{det}(sI_{\mathtt{n}_1} - U)$, where $k \in \mathbb{R} \setminus \{0\}$. Hence, $\mathtt{roots}\,(\mathtt{det}(sU_1 - U_2)) = \mathtt{roots}\,(\mathtt{det}(sI_{\mathtt{n}_1} - U))$. In other words, the eigenvalues of $U$ are the finite eigenvalues of the matrix pair $(U_1, U_2)$. This canonical form of linear matrix pencils is extensively used in singular descriptor system literature to decompose a singular descriptor system into two subsystems, namely the *slow and fast subsystems*. The next proposition sheds light into such a decomposition: see [Dai89] for more on such decompositions.

**Proposition 2.8.** [Dai89, Section 1-4] *Consider a singular descriptor system* $\Sigma_{\mathtt{sing}}$ *with a state-space dynamics* $U_1 \frac{d}{dt} x = U_2 x$, *where* $\mathtt{det}(sU_1 - U_2) \neq 0$, $U_1, U_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ *and* $U_1$ *is singular. Then, there exists nonsingular matrices* $Z_1, Z_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ *such that*

$$\frac{d}{dt} x_1 = U x_1 \text{ and } Y x_2 = x_2 \tag{2.4}$$

*with the coordinate transformation* $\mathtt{col}(x_1, x_2) = Z_2^{-1} x$, $Z_1U_1Z_2 = \mathtt{diag}(I_{\mathtt{n}_1}, Y)$, *and* $Z_1U_2Z_2 = \mathtt{diag}(U, I_{\mathtt{n}_1})$, *where* $\mathtt{n}_1 + \mathtt{n}_2 = \mathtt{n}$ *and* $Y$ *is nilpotent with a nilpotency index* $\mathtt{h}$.
*Further, the unique states of the system due to an initial condition* $x_0$ *are given by the following:*

$$x(t) = Z_2 \begin{bmatrix} I_{\mathtt{n}_1} \\ 0 \end{bmatrix} e^{Ut} \begin{bmatrix} I_{\mathtt{n}_1} & 0 \end{bmatrix} Z_2^{-1} x_0 - Z_2 \begin{bmatrix} 0 \\ I_{\mathtt{n}_2} \end{bmatrix} \sum_{i=1}^{\mathtt{h}-1} \delta^{(i-1)} Y^i \begin{bmatrix} 0 & I_{\mathtt{n}_2} \end{bmatrix} Z_2^{-1} x_0. \tag{2.5}$$

The system $\begin{bmatrix} I_{\mathtt{n}_1} & \\ & Y \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} U & \\ & I_{\mathtt{n}_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is said to be a *canonical form* of the system $\Sigma_{\mathtt{sing}}$. From equation (2.5) it is evident that the subspace spanned by the first $\mathtt{n}_1$ columns of $Z_2$ corresponds to the slow (exponential) states of the system $\Sigma_{\mathtt{sing}}$. Hence, we call it the *slow subsystem* of the system $\Sigma_{\mathtt{sing}}$. Further, the subspace spanned by the last $\mathtt{n}_2$ columns of $Z_2$ corresponds to the fast (impulsive) states of the system $\Sigma_{\mathtt{sing}}$ and hence we call it the *fast subsystem* of the system $\Sigma_{\mathtt{sing}}$.

### 2.2.4 $(A,B)$-invariant subspace and controllability subspace

We briefly review the notions of $(A,B)$-invariant subspace and controllability subspace next (see [Won85, Chapters 4 and 5] for more on these subspaces).

**Definition 2.9.** [Won85, Section 4.2] $A \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ *and* $B \in \mathbb{R}^{\mathtt{n} \times \mathtt{m}}$. *A subspace* $\mathscr{S} \subseteq \mathbb{R}^{\mathtt{n}}$ *is said to be* $(A,B)$-invariant *if there exists a matrix* $F \in \mathbb{R}^{\mathtt{m} \times \mathtt{n}}$ *such that* $(A + BF)\mathscr{S} \subseteq \mathscr{S}$.

[1] A nilpotent matrix $Y$ is a square matrix such that $Y^{\mathtt{h}} = 0$ for some positive integer $\mathtt{h}$. The smallest positive integer $\mathtt{h}$ for which $Y^{\mathtt{h}} = 0$ is called the nilpotency index of a nilpotent matrix $Y$.

Following the notation in [Won85], we use the symbol $\mathfrak{I}(A,B)$ for the family of $(A,B)$-invariant subspaces. The notation $\mathbf{F}(\mathscr{S})$ is used for the collection of matrices $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A+BF)\mathscr{S} \subseteq \mathscr{S}$. Such a matrix $F$ is called a *friend* of $\mathscr{S}$. The next proposition provides a test for determining whether a given subspace is $(A,B)$-invariant [Won85, Lemma 4.2]. We use this test throughout this thesis.

**Proposition 2.10.** [Won85, Lemma 4.2] *A subspace $\mathscr{S} \subseteq \mathbb{R}^{\mathtt{n}}$ is $(A,B)$-invariant if and only if $A\mathscr{S} \subseteq \mathscr{S} + \mathtt{img}\, B$.*

The notation $\mathfrak{I}(A,B;\mathtt{ker}\,C)$ denotes the family of $(A,B)$-invariant subspaces that are contained in $\mathtt{ker}\,C$, where $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$. It is known in the literature that the set $\mathfrak{I}(A,B;\mathtt{ker}\,C)$ admits a unique supremal element [Won85, Lemma 4.4]. We use the symbol $\mathtt{sup}\,\mathfrak{I}(A,B;\mathtt{ker}\,C)$ to represent the supremal element. This implies that for all $\mathscr{S} \in \mathfrak{I}(A,B;\mathtt{ker}\,C)$, we must have $\mathscr{S} \subseteq \mathtt{sup}\,\mathfrak{I}(A,B;\mathtt{ker}\,C)$ .

**Definition 2.11.** [Won85, Section 5.1] *Consider $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. A subspace $\mathscr{R} \subseteq \mathbb{R}^{\mathtt{n}}$ is a* controllability subspace *of the pair $(A,B)$ if there exist $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ and $G \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, such that $\mathscr{R}$ is the reachable subspace of the pair $(A+BF,BG)$, i.e.*

$$\mathscr{R} = \mathtt{img}\begin{bmatrix} BG & (A+BF)BG & (A+BF)^2BG & \cdots & (A+BF)^{\mathtt{n}-1}BG \end{bmatrix}.$$

We use the symbol $\mathfrak{C}(A,B)$ for the family of controllability subspaces of $(A,B)$. The notation $\mathfrak{C}(A,B;\mathtt{ker}\,C)$ denotes the family of controllability subspaces that are contained in $\mathtt{ker}\,C$. Similar to $\mathfrak{I}(A,B;\mathtt{ker}\,C)$, the set $\mathfrak{C}(A,B;\mathtt{ker}\,C)$ also admits a unique supremal element that we represent by $\mathtt{sup}\,\mathfrak{C}(A,B;\mathtt{ker}\,C)$ [Won85, Theorem 5.4].

Using the notation $(A+BF)|_{\mathscr{S}}$ to represent the restriction of $(A+BF)$ to the $(A,B)$ invariant subspace $\mathscr{S}$, we define the set

$$\mathscr{B} := \{\mathscr{S} \in \mathfrak{I}(A,B,\mathtt{ker}\,C) \mid \text{ there exists } F \in \mathbf{F}(\mathscr{S}) \text{ such that } \sigma\left((A+BF)|_{\mathscr{S}}\right) \subsetneq \mathbb{C}_-\}.$$

We call any subspace in $\mathscr{B}$ a *good $(A,B)$-invariant subspace inside* $\mathtt{ker}\,C$. As shown in [Won85, Lemma 5.8], the set $\mathscr{B}$ admits a supremal element defined as $\mathscr{S}^*_{\mathtt{g}} := \mathtt{sup}\,\mathscr{B}$, i.e., for all elements $\mathscr{S} \in \mathscr{B}, \mathscr{S} \subseteq \mathscr{S}^*_{\mathtt{g}}$. Hence, $\mathscr{S}^*_{\mathtt{g}}$ is called the *largest good $(A,B)$-invariant subspace inside* $\mathtt{ker}\,C$. On the other hand, if $\sigma\left((A+BF)|_{\mathscr{S}}\right) \subsetneq \mathbb{C}_+$ in the definition of the set $\mathscr{B}$, then we call any subspace in $\mathscr{B}$ a *bad $(A,B)$-invariant subspace inside* $\mathtt{ker}\,C$ and the corresponding supremal element the *largest bad $(A,B)$-invariant subspace inside* $\mathtt{ker}\,C$.

Let $\mathscr{S}^* := \mathtt{sup}\,\mathfrak{I}(A,B;\mathtt{ker}\,C)$ and $\mathscr{R}^* := \mathtt{sup}\,\mathfrak{C}(A,B;\mathtt{ker}\,C)$. Further, let $F \in \mathbf{F}(\mathscr{S}^*)$. Clearly, $\mathscr{R}^* \subseteq \mathscr{S}^*$. Since $\mathscr{R}^*$ is $(A,B)$-invariant hence the space $\mathscr{S}^*$ can be factored as $\mathscr{S}^* = \mathscr{R}^* + \mathscr{S}^*/\mathscr{R}^*$. Let $\overline{(A+BF)|_{\mathscr{S}^*}}$ denote the map induced by $(A+BF)|_{\mathscr{S}^*}$ on the factor space $\mathscr{S}^*/\mathscr{R}^*$. Then, it is known that the set of eigenvalues $\sigma\left(\overline{(A+BF)|_{\mathscr{S}^*}}\right)$ remains invariant for all $F \in \mathbf{F}(\mathscr{S}^*)$. For a system with an i/s/o representation $\frac{d}{dt}x = Ax+Bu$ and $y = Cx$, the complex numbers $\sigma\left(\overline{(A+BF)|_{\mathscr{S}^*}}\right)$ are known as the *transmission zeros* of the system. Note importantly that, for a single-input controllable system, we have $\mathscr{R}^* = \{0\}$. Consequently, $\mathscr{S}^*/\mathscr{R}^* = \mathscr{S}^*$, and $\overline{(A+BF)|_{\mathscr{S}^*}} = (A+BF)|_{\mathscr{S}^*}$. This means that for single-input systems, $\sigma((A+BF)|_{\mathscr{S}^*})$ is the set of the transmission zeros. In other words, the set $\sigma((A+BF)|_{\mathscr{S}^*})$ remains invariant for all $F \in \mathbf{F}(\mathscr{S}^*)$. Further, it can also be shown that for a controllable and observable SISO system, the set $\sigma((A+BF)|_{\mathscr{S}^*})$ is equal to the set of the roots of the numerator of $G(s)$ (elements included in the set with multiplicity), where $G(s) = C(sI_{\mathtt{n}} - A)^{-1}B \in \mathbb{R}(s)$ ([Won85, Section 5.5]). Using the symbol $\mathtt{rootnum}(p(s))$ to denote the roots of the numerator of a rational function $p(s) \in \mathbb{R}(s)$, we can therefore infer that $\sigma((A+BF)|_{\mathscr{S}^*}) = \mathtt{rootnum}(G(s))$. This property of single-input systems is essential for the development of the theory in Section 2.3 and Section 2.4.

### 2.2.5 Weakly unobservable and strongly reachable subspaces

Consider the system $\Sigma$ with an i/s/o representation $\frac{d}{dt}x = Ax + Bu$ and $0 = Cx$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$ and $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$. Associated with such a system are two important subspaces called the weakly unobservable subspace and the strongly reachable subspace. We briefly review the properties of these subspaces next (see [HS83] for more on these spaces). Before we delve into the definitions of these subspaces, we need to define the space of impulsive-smooth distributions $\mathfrak{C}^{\mathtt{w}}_{\mathtt{imp}}$ (see [HS83], [WKS86]). In the sequel, we use the symbol $\delta$ and $\delta^{(i)}$ to denote Dirac delta impulse function supported at zero and the $i$-th distributional derivative of $\delta$ with respect to $t$, respectively. We also use the symbol $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{n}})|_{\mathbb{R}_+}$ to denote the space of all functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are restrictions of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$.

**Definition 2.12.** [HS83, Definition 3.1] *The set of impulsive-smooth distributions $\mathfrak{C}^{\mathtt{w}}_{\mathtt{imp}}$ is defined as:*

$$\mathfrak{C}^{\mathtt{w}}_{\mathtt{imp}} := \left\{ f = f_{\mathtt{reg}} + f_{\mathtt{imp}} \,|\, f_{\mathtt{reg}} \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})|_{\mathbb{R}_+} \text{ and } f_{\mathtt{imp}} = \sum_{i=0}^{k} a_i \delta^{(i)}, \text{ with } a_i \in \mathbb{R}^{\mathtt{w}}, k \in \mathbb{N} \right\}.$$

In what follows, we denote the state-trajectory $x(t)$ and output-trajectory $y(t)$ of the system $\Sigma$ corresponding to initial condition $x_0$ and input $u(t)$ using the symbols $x(t;x_0,u)$ and $y(t;x_0,u)$, respectively. The symbol $x(0_+;x_0,u)$ denotes the state-trajectory that can be reached from $x_0$ instantaneously on application of the input $u(t)$ at $t = 0$.

**Definition 2.13.** [HS83, Definition 3.8] *A state $x_0 \in \mathbb{R}^{\mathtt{n}}$ is called* weakly unobservable *if there exists a regular input $u(t) \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}$ such that $y(t;x_0,u) \equiv 0$ for all $t \geqslant 0$. The collection of all such weakly unobservable states is called a* weakly unobservable subspace *of the state-space and is denoted by $\mathscr{O}_w$.*

Next we review one of the properties of weakly unobservable subspace that is crucially used in this thesis.

**Proposition 2.14.** [HS83, Theorem 3.10] *The weakly unobservable subspace $\mathscr{O}_w$ is the largest $(A,B)$-invariant subspace inside the kernel of $C$, i.e., $\mathscr{O}_w = \mathtt{sup}\,\mathfrak{I}(A,B;\mathtt{ker}\,C)$.*

The other space that we are interested in, is the space of strongly reachable states.

**Definition 2.15.** [HS83, Definition 3.13] *A state $x_1 \in \mathbb{R}^{\mathtt{n}}$ is called* strongly reachable *(from the origin) if there exists an input $u(t) \in \mathfrak{C}^{\mathtt{m}}_{\mathtt{imp}}$ such that $x(0_+;0,u) \equiv x_1$ and $y(t;0,u) \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{p}})|_{\mathbb{R}_+}$. The collection of all such strongly reachable states is called the* strongly reachable subspace *of the state-space and is denoted by $\mathscr{R}_s$.*

A method to compute the space $\mathscr{R}_s$ is given by the following recursion (see [HS83] for more on the algorithm)

$$\mathscr{R}_0 := \{0\} \subsetneq \mathbb{R}^{\mathtt{n}}, \text{ and } \mathscr{R}_{i+1} := \begin{bmatrix} A & B \end{bmatrix} \left\{ (\mathscr{W}_i \oplus \mathscr{P}) \cap \mathtt{ker} \begin{bmatrix} C & 0_{\mathtt{p,m}} \end{bmatrix} \right\} \subseteq \mathscr{R}_s, \tag{2.6}$$

where $\mathscr{W}_i := \{ \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right] \in \mathbb{R}^{\mathtt{n+m}} \,|\, w \in \mathscr{R}_i \}$ and $\mathscr{P} := \left\{ \left[\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}\right] \in \mathbb{R}^{\mathtt{n+m}} \,|\, \alpha \in \mathbb{R}^{\mathtt{m}} \right\}$. In Section 2.3.1 we use this recursive algorithm to characterize the strongly reachable subspace of a single-input system in terms of the Rosenbrock system matrix.

Since the space $\mathscr{O}_w$ deals with infinitely differentiable inputs, we call $\mathscr{O}_w$ the *slow subspace* of a system. Further, note that since $\mathscr{O}_w$ is the largest $(A,B)$-invariant subspace inside the kernel of $C$, such a subspace also admits largest good and largest bad $(A,B)$-invariant subspace inside the kernel of $C$. We call such a space the *good slow subspace* and the *bad slow subspace* of the system, respectively and denote them with the symbols $\mathscr{O}_{wg}$ and $\mathscr{O}_{wb}$, respectively. On the other hand, since the space $\mathscr{R}_s$ admits impulsive inputs, we call $\mathscr{R}_s$ the *fast subspace* of the system.

In the sequel, we use the notion of autonomy of a system and its relation with the spaces $\mathscr{O}_w$ and $\mathscr{R}_s$. Hence, we define autonomy of a system first and then review the result [HSW00, Lemma 3.3] that establishes a noteworthy property of $\mathscr{O}_w$ and $\mathscr{R}_s$ for autonomous systems.

**Definition 2.16.** [HSW00] *A system with an output-nulling representation $\frac{d}{dt}x = Ax + Bu$ and $0 = Cx$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, and $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, is called* autonomous *if for every initial condition $x_0 \in \mathscr{O}_w$ the system has a unique solution $\mathtt{col}(x,u)$.*

**Proposition 2.17.** [HSW00, Lemma 3.3] *Consider the system $\frac{d}{dt}x = Ax + Bu$ and $0 = Cx$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, and $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$. Then the following are equivalent:*

*(1) The system is autonomous.*

*(2) $G(s) := C(sI_{\mathtt{n}} - A)^{-1}B$ is invertible as a rational matrix.*

*(3) $\mathscr{O}_w \oplus \mathscr{R}_s = \mathbb{R}^{\mathtt{n}}$ and $\mathtt{ker} \begin{bmatrix} B \\ 0_{\mathtt{p,m}} \end{bmatrix} = \{0\}$.*

Since we are deal with single-input systems in this thesis, we consider the matrix $B$ to be of full column-rank without loss of generality. Hence, the second part of Statement *(3)* in the proposition is always true.

### 2.2.6 ARE and Hamiltonian systems

One of the widely used methods to compute the maximal solution of the ARE (2.2) is to use the basis of a suitable eigenspace of the matrix pair $(E,H)$, where

$$E := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{p},\mathtt{p}} \end{bmatrix}, \text{ and } H := \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & R \end{bmatrix}. \tag{2.7}$$

We call the matrix pair $(E,H)$ the *Hamiltonian matrix pair* and the matrix pencil $(sE-H)$ the *Hamiltonian pencil*. The suitable eigenspace used to compute the maximal rank-minimizing solution of the ARE (2.2) correspond to a certain choice of eigenvalues of $(E,H)$. In order to understand this choice of eigenvalues the notion of Lambda-sets is essential and hence we define Lambda-sets next.

**Definition 2.18.** [Kuč91, PB08] *Let $p(s)$ be an even-degree polynomial with $\mathtt{roots}\,(p(s)) \cap j\mathbb{R} = \emptyset$. A set of complex numbers $\Lambda \subsetneq \mathtt{roots}\,(p(s))$ is called a Lambda-set of $p(s)$ if it satisfies the following properties:*

*(1)* $\Lambda = \bar{\Lambda}$, *i.e., if $\lambda \in \Lambda$ then, $\bar{\lambda} \in \Lambda$. (complex conjugacy)*

*(2)* $\Lambda \cap (-\Lambda) = \emptyset$, *i.e., if $\lambda \in \Lambda$ then, $-\lambda \notin \Lambda$. (unmixing)*

*(3)* $\Lambda \cup (-\Lambda) = \mathtt{roots}\,(p(s))$ *(counted with multiplicity).*

Now that we have the definition for Lambda-sets, we review the method to compute the maximal solution of the ARE (2.2) (see [IOW99] for more). Recall that the maximal solution of an ARE is the maximal rank-minimizing solution of the corresponding LMI (2.3).

**Proposition 2.19.** *Consider the LQR Problem 2.1 with $R > 0$. Let the corresponding Hamiltonian matrix pair $(E,H)$ be as defined in equation* (2.7)*. Assume $\sigma(E,H) \cap j\mathbb{R} = \emptyset$. Let $\Lambda$ be a Lambda-set of $\mathtt{det}(sE-H)$ with cardinality $\mathtt{n}$ and $\Lambda \subsetneq \mathbb{C}_-$. Let $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $V_{3\Lambda} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ be such that the columns of $V_{\mathtt{e}\Lambda} = \mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ form a basis of the $\mathtt{n}$-dimensional eigenspace of $(E,H)$ corresponding to the eigenvalues of $(E,H)$ in $\Lambda$. Then, the following statements hold.*

*(1)* $V_{1\Lambda}$ *is invertible.*

*(2)* $K_{\mathtt{max}} := V_{2\Lambda} V_{1\Lambda}^{-1}$ *is symmetric.*

*(3)* $K_{\mathtt{max}}$ *is the maximal solution of the ARE* (2.2).

*(4)* $K_{\mathtt{max}}$ *is the maximal rank-minimizing solution of the corresponding LQR LMI* (2.3).

*(5)* $K_{\mathtt{max}} \geqslant 0$.

Although Proposition 2.19 does not explicitly use invertibility of $R$ while finding the maximal rank-minimizing solution of the LQR LMI, yet the proposition cannot be used to compute such a solution for the singular LQR LMI. We motivate the reason for this using Example 2.2 stated at the beginning of this chapter.

**Example 2.20.** *From Example 2.2, we know that the state-space dynamics is:*

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u.$$

*Further, the functional to be minimized can be rewritten as*

$$\int_0^\infty \left(x^T Q x\right) dt, \text{ where } Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } x := \mathtt{col}(x_1, x_2, x_3).$$

*On construction of the Hamiltonian pencil pair $(E,H)$ using $A,B,Q$ in Example 2.20, it can be verified that $\mathtt{det}(sE-H) = 1-s^2$. Hence, $\Lambda = \{-1\}$. The eigenvector of $(E,H)$ corresponding to $-1$, is $\begin{bmatrix} 1 & 1 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}^T$. Therefore, $V_{1\Lambda} = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}^T$ and $V_{2\Lambda} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T$. But $V_{1\Lambda}$ is not a square matrix. Thus, Proposition 2.19 cannot be used to solve singular LQR problems.*

From Example 2.2, it is clear that Proposition 2.19 fails in case of singular LQR problems because the degree of $\mathtt{det}(sE-H)$ is strictly less than $2\mathtt{n}$. This fall in the degree causes a deficit in the cardinality of possible Lambda-sets of $\mathtt{det}(sE-H)$. Indeed, a Lambda set of $\mathtt{det}(sE-H)$ can now have cardinality strictly less than $\mathtt{n}$; we define it as $\mathtt{n_s} < \mathtt{n}$. Consequently, the eigenspace of $(E,H)$ corresponding to such a Lambda-set would also show a deficit in its dimension from being $\mathtt{n}$. This deficit in the dimension of the eigenspace is required to be compensated by $(\mathtt{n}-\mathtt{n_s})$ suitable vectors. These suitable vectors must be the basis of a space complementary to the eigenspace that supplies the $\mathtt{n_s}$ vectors. Of course, this compensation cannot cannot be done by the basis vectors of any arbitrary complementary space, since we would not get a solution of the LQR LMI then. Our main result, Theorem 2.30, shows exactly what this complementary space needs to be for getting the maximal rank-minimizing solution of the LQR LMI.

Since we deal with the singular LQR problem for single-input systems, we rewrite the LQR Problem 2.1 for the single-input case as follows:

**Problem 2.21. (Single-input singular LQR problem)** *Consider a controllable system $\Sigma$ with state-space dynamics $\frac{d}{dt}x = Ax + bu$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $b \in \mathbb{R}^{\mathtt{n}}$. Then, for every initial condition $x_0 \in \mathbb{R}^{\mathtt{n}}$, find an* admissible *input $u$ that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \left(x^T Q x\right) dt, \text{ where } Q \geqslant 0. \tag{2.8}$$

In the formulation of the singular LQR problem above, we have not explicitly defined the space from which the inputs $u$ need to be chosen. Since in this chapter we are primarily concerned with the maximal rank-minimizing solution of an LQR LMI and do not deal with the trajectory level interpretations of the LQR problem, we delay the definition of admissible inputs to Chapter 3 (see Definition 3.4).

Note that the LQR LMI (2.3) with respect to Problem 2.21 takes the following form:

$$\begin{bmatrix} A^TK+KA+Q & Kb \\ b^TK & 0 \end{bmatrix} \geqslant 0 \Leftrightarrow \begin{cases} A^TK+KA+Q \geqslant 0, \\ Kb = 0. \end{cases} \tag{2.9}$$

Further, for single-input singular LQR problems as defined in LQR Problem 2.21, the Hamiltonian matrix pair in equation (2.7) takes the following form:

$$E := \begin{bmatrix} I_{\rm n} & 0 & 0 \\ 0 & I_{\rm n} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } H := \begin{bmatrix} A & 0 & b \\ -Q & -A^T & 0 \\ 0 & b^T & 0 \end{bmatrix}. \tag{2.10}$$

Interestingly, the Hamiltonian matrix pencil $(E,H)$ in equation (2.10) can be associated with a differential algebraic system as given below:

$$\underbrace{\begin{bmatrix} I_{\rm n} & 0 & 0 \\ 0 & I_{\rm n} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt}\begin{bmatrix} x \\ z \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & b \\ -Q & -A^T & 0 \\ 0 & b^T & 0 \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u \end{bmatrix}. \tag{2.11}$$

The system represented by this first order representation (2.11) is called the *Hamiltonian system*; we use $\Sigma_{\rm Ham}$ to denote this system (see [IOW99] for more on Hamiltonian systems). Further, the Hamiltonian system in equation (2.11) can be written in an *output-nulling representation* as given below:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \widehat{A}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{b}u, \quad 0 = \widehat{c}\begin{bmatrix} x \\ z \end{bmatrix}, \tag{2.12}$$

where $\widehat{A} := \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}$, $\widehat{b} := \begin{bmatrix} b \\ 0 \end{bmatrix}$ and $\widehat{c} := \begin{bmatrix} 0 & b^T \end{bmatrix}$. Note that the Hamiltonian matrix pair $(E,H)$ in equation (2.11) is indeed the Rosenbrock matrix pair for the Hamiltonian system $\Sigma_{\rm Ham}$ in equation (2.12).

In what follows, we shall need the notion of disconjugacy of an eigenspace of the Hamiltonian matrix pair. We review this next.

**Definition 2.22.** [IOW99, Definition 6.1.5] *Let $\mathscr{E}$ be an eigenspace of $(E,H)$, where $(E,H)$ is as defined in equation* (2.7)*. Assume the columns of a matrix $V_e$ to be the basis of $\mathscr{E}$. Conforming to the partition of $H$, let $V_e := \texttt{col}(V_1,V_2,V_3)$. Then, $\mathscr{E}$ is called* disconjugate *if $V_1$ is full column-rank.*

## 2.3 Characterization of slow and fast subspaces in terms of Rosenbrock system matrix

Consider $\Sigma_{\mathrm{p}}$ to be a system with an output-nulling representation of the form:

$$\frac{d}{dt}x = Px + Lu, \text{ and } 0 = Mx, \text{ where } P \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}, L, M^T \in \mathbb{R}^{\mathtt{N}} \setminus \{0\}. \tag{2.13}$$

Define the matrix pair

$$U_1 := \begin{bmatrix} I_{\mathtt{N}} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(\mathtt{N}+1)\times(\mathtt{N}+1)} \text{ and } U_2 := \begin{bmatrix} P & L \\ M & 0 \end{bmatrix} \in \mathbb{R}^{(\mathtt{N}+1)\times(\mathtt{N}+1)}. \tag{2.14}$$

Note that $(sU_1 - U_2)$ is the Rosenbrock system matrix for the system $\Sigma_{\mathrm{p}}$ in equation (2.13) and $(U_1, U_2)$ is the corresponding Rosenbrock matrix pair. In this section we characterize the slow subspace $\mathscr{O}_w$ (weakly unobservable) and fast subspace $\mathscr{R}_s$ (strongly reachable) of the system $\Sigma_{\mathrm{p}}$ in terms of the matrix pencil $(U_1, U_2)$. Further, we also characterize the good slow subspace of $\Sigma_{\mathrm{p}}$ in terms of the eigenspace of $(U_1, U_2)$. Hence, we have divided this section into three subsections; the first being characterization of the fast subspace of $\Sigma_{\mathrm{p}}$. In the second and third subsection we characterize the slow and good slow subspaces of $\Sigma_{\mathrm{p}}$, respectively in terms of the eigenspace of the Rosenbrock matrix pair $(U_1, U_2)$.

### 2.3.1 Characterization of the fast subspace

In order to characterize the fast subspace, we need certain identities related to the Markov parameters of the system $\Sigma_{\mathrm{p}}$. We present this in the next lemma and follow it up with a result that characterizes the fast subspace of the system $\Sigma_{\mathrm{p}}$ in terms of the matrix pair $(U_1, U_2)$. In the sequel, we use the symbol $\mathtt{degdet}(p(s))$ to denote the degree of a polynomial $p(s) \in \mathbb{R}[s]$.

Properties of the Markov parameters of a SISO system

**Lemma 2.23.** *Consider the system $\Sigma_{\mathrm{p}}$ as defined in equation* (2.13). *Let the corresponding Rosenbrock matrix pair $(U_1, U_2)$ be as defined in equation* (2.14). *Assume* $\mathtt{det}(sU_1 - U_2) \neq 0$. *Define* $\mathtt{degdet}(sU_1 - U_2) =: \mathtt{N_s}$ *and* $\mathtt{N_f} := \mathtt{N} - \mathtt{N_s}$. *Then,*

$$MP^kL = 0, \text{ for } k \in \{0, 1, \ldots, \mathtt{N_f} - 2\} \text{ and } MP^{\mathtt{N_f}-1}L \neq 0. \tag{2.15}$$

*Proof:* Define $G(s) := M(sI_{\mathtt{N}} - P)^{-1}L \in \mathbb{R}(s)$. Using the notion of Schur complement, we have

$$\mathtt{det}(sU_1 - U_2) = \mathtt{det}\begin{bmatrix} sI_{\mathtt{N}} - P & -L \\ -M & 0 \end{bmatrix} = -M(sI_{\mathtt{N}} - P)^{-1}L \times \mathtt{det}(sI_{\mathtt{N}} - P) \Rightarrow G(s) = -\frac{\mathtt{det}(sU_1 - U_2)}{\mathtt{det}(sI_{\mathtt{N}} - P)}.$$

Since $\mathtt{degdet}(sU_1 - U_2) =: \mathtt{N_s}$ and $\mathtt{degdet}(sI_{\mathtt{N}} - P) = \mathtt{N}$, the relative degree of $G(s)$ must be $\mathtt{N} - \mathtt{N_s} = \mathtt{N_f}$. Now on expanding $(sI_{\mathtt{N}} - P)^{-1}$ in a Taylor series about $s = \infty$, we have

$$G(s) = M(sI_{\mathtt{N}} - P)^{-1}L = \frac{1}{s}M\left(I_{\mathtt{N}} + \frac{P}{s} + \frac{P^2}{s^2} + \cdots\right)L = \frac{ML}{s} + \frac{MPL}{s^2} + \frac{MP^2L}{s^3} + \cdots.$$

Since the relative degree of the rational polynomial $G(s)$ is $\mathtt{N_f}$. Hence, we can infer from the Taylor expansion of $G(s)$ that

$$\lim_{s\to\infty} s^{k+1}G(s) = 0 = MP^kL \text{ for } k \in \{0,1,\ldots,\mathtt{N_f}-2\}.$$

Further, since relative degree of $G(s)$ is $\mathtt{N_f}$, $\lim_{s\to\infty} s^{\mathtt{N_f}}G(s) \neq 0$. Hence, $MP^{\mathtt{N_f}-1}L \neq 0$. ■

Now using Lemma 2.23 we characterize the fast subspace of a SISO system.

Characterization of the fast subspace of a SISO system

**Theorem 2.24.** *Consider the system* $\Sigma_{\mathrm{p}}$ *as defined in equation* (2.13). *Let the corresponding matrix pencil pair* $(U_1,U_2)$ *be as defined in equation* (2.14). *Assume* $\mathtt{det}(sU_1-U_2)\neq 0$. *Define* $\mathtt{degdet}(sU_1-U_2) =: \mathtt{N_s}$ *and* $\mathtt{N_f} := \mathtt{N}-\mathtt{N_s}$. *Let* $\mathscr{R}_s$ *be the fast subspace of* $\Sigma_{\mathrm{p}}$. *Then, the following statements are true:*

*(1)* $\mathscr{R}_s = \mathtt{img}\begin{bmatrix} L & PL & \cdots & P^{\mathtt{N_f}-1}L\end{bmatrix}$.

*(2)* $\mathtt{dim}(\mathscr{R}_s) = \mathtt{N_f}$.

*Proof:* *(1)*: From equation (2.6) in Section 2.2.5, the recursive algorithm to compute the fast subspace of $\Sigma_{\mathrm{p}}$ is given by:

$$\begin{aligned}\mathscr{R}_0 := \{0\} \subsetneq \mathbb{R}^{\mathtt{N}} \text{ and } \mathscr{R}_{i+1} &:= \begin{bmatrix} P & L\end{bmatrix}\left\{(\mathscr{W}_i \oplus \mathscr{P})\cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix}\right\} \subseteq \mathscr{R}_s,\\ &= \begin{bmatrix} P & L\end{bmatrix}\left\{\left(\mathscr{W}_i \cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix}\right)\oplus\left(\mathscr{P}\cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix}\right)\right\} \subseteq \mathscr{R}_s.\end{aligned} \tag{2.16}$$

where $\mathscr{W}_i := \left\{\left[\begin{smallmatrix} w\\ 0\end{smallmatrix}\right] \in \mathbb{R}^{\mathtt{N}+1} \,|\, w \in \mathscr{R}_i\right\}$ and $\mathscr{P} := \left\{\left[\begin{smallmatrix} 0\\ \alpha\end{smallmatrix}\right] \in \mathbb{R}^{\mathtt{N}+1} \,|\, \alpha \in \mathbb{R}\right\}$. Note that since $\mathscr{P} \cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix} = \mathscr{P}$, the recursion in equation (2.16) can be rewritten as

$$\mathscr{R}_0 = \{0\} \subsetneq \mathbb{R}^{\mathtt{N}} \text{ and } \mathscr{R}_{i+1} = \begin{bmatrix} P & L\end{bmatrix}\left\{\left(\mathscr{W}_i \cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix}\right)\oplus \mathscr{P}\right\} \subseteq \mathscr{R}_s. \tag{2.17}$$

Now, we claim that $\mathscr{R}_k = \mathtt{img}\, L + \mathtt{img}\,(PL) + \cdots + \mathtt{img}\,(P^{k-1}L)$ for $k \in \{1,2,3,\ldots,\mathtt{N_f}\}$. To prove this we use mathematical induction along with Lemma 2.23.

*Base case:* $(k=1)$ Since $\mathscr{R}_0 = \{0\}$, we have $\mathscr{W}_0 = \{0\}$. Therefore, we have $\left(\mathscr{W}_0 \cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix}\right) = \{0\} \subsetneq \mathbb{R}^{\mathtt{N}+1}$. Then, using equation (2.17), we have

$$\mathscr{R}_1 = \begin{bmatrix} P & L\end{bmatrix}\left\{\left(\mathscr{W}_0 \cap \mathtt{ker}\begin{bmatrix} M & 0\end{bmatrix}\right)\oplus \mathscr{P}\right\} = \begin{bmatrix} P & L\end{bmatrix}\{\{0\}\oplus\mathscr{P}\} = \mathtt{img}\, L.$$

*Induction step:* Assume $\mathscr{R}_k = \mathtt{img}\, L + \mathtt{img}\,(PL) + \cdots + \mathtt{img}\,(P^{k-1}L)$ for $k < \mathtt{N_f}$. We prove that $\mathscr{R}_{k+1} = \mathtt{img}\, L + \mathtt{img}\,(PL) + \cdots + \mathtt{img}\,(P^{k}L)$.

From equation (2.17), we have

$$\begin{aligned}\mathscr{R}_{k+1} &= \begin{bmatrix} P & L \end{bmatrix} \left\{ \left( \mathscr{W}_k \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\} \\ &= \begin{bmatrix} P & L \end{bmatrix} \left\{ \left( \left( \sum_{i=0}^{k-1} \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \right) \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\} \\ &= \begin{bmatrix} P & L \end{bmatrix} \left\{ \sum_{i=0}^{k-1} \left( \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\}. \end{aligned} \tag{2.18}$$

Since $\begin{bmatrix} M & 0 \end{bmatrix} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} = MP^iL = 0$ for $i < \mathtt{N_f} - 1$ (from Lemma 2.23), we must have

$$\sum_{i=0}^{k-1} \left( \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) = \sum_{i=0}^{k-1} \left( \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \right).$$

Thus, from equation (2.18) we have

$$\mathscr{R}_{k+1} = \begin{bmatrix} P & L \end{bmatrix} \left\{ \sum_{i=0}^{k-1} \left( \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\} = \mathtt{img}\, L + \mathtt{img}\,(PL) + \cdots + \mathtt{img}\,(P^k L).$$

By the principle of mathematical induction, we conclude that

$$\mathscr{R}_k = \mathtt{img}\, L + \mathtt{img}\,(PL) + \cdots + \mathtt{img}\,(P^{k-1} L) \text{ for } k \in \{1, 2, 3, \ldots, \mathtt{N_f}\}. \tag{2.19}$$

This proves our claim.

Next we claim that $\mathscr{R}_{\mathtt{N_f}+1} = \mathscr{R}_{\mathtt{N_f}}$. From equation (2.17) and equation (2.19), we have

$$\begin{aligned}\mathscr{R}_{\mathtt{N_f}+1} &= \begin{bmatrix} P & L \end{bmatrix} \left\{ \left( \mathscr{W}_{\mathtt{N_f}} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\} \\ &= \begin{bmatrix} P & L \end{bmatrix} \left\{ \sum_{i=0}^{\mathtt{N_f}-1} \left( \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\} \\ &= \begin{bmatrix} P & L \end{bmatrix} \left\{ \sum_{i=0}^{\mathtt{N_f}-2} \left( \mathtt{img} \begin{bmatrix} P^i L \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) + \left( \mathtt{img} \begin{bmatrix} P^{\mathtt{N_f}-1} L \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} \right) \oplus \mathscr{P} \right\}. \end{aligned} \tag{2.20}$$

From Lemma 2.23, we know that $MP^{\mathtt{N_f}-1}L \neq 0$. Hence, $\mathtt{img} \begin{bmatrix} P^{\mathtt{N_f}-1} L \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} M & 0 \end{bmatrix} = 0$. Hence, from equation (2.19) and equation (2.20) we have $\mathscr{R}_{\mathtt{N_f}+1} = \mathscr{R}_{\mathtt{N_f}}$. Thus, from [HS83] (see discussion after equation 3.22), we infer that $\mathscr{R}_{\mathtt{N_f}}$ characterized in equation (2.19) is the fast subspace $\mathscr{R}_s$ of $\Sigma_{\mathrm{p}}$, i.e., $\mathscr{R}_{\mathtt{N_f}} = \mathscr{R}_s$. From equation (2.19), Statement *(1)* of the lemma directly follows.

*(2):* Define $W := \begin{bmatrix} L & PL & \cdots P^{\mathtt{N_f}-1} L \end{bmatrix}$. To the contrary, let us assume that there exists a nontrivial vector $w \in \mathbb{R}^{\mathtt{N_f}}$ such that $Ww = 0$. Conforming to the partition of $W$ let $w := \mathtt{col}(w_0, w_1, \ldots, w_{\mathtt{N_f}-1})$.

Now, we pre-multiply $W$ with $M$ in the equation $Ww = 0$ and use the fact that $MP^kL = 0$ for $k \in \{0, 1, \ldots, \mathtt{N_f} - 2\}$ from Lemma 2.23:

$$\begin{bmatrix} ML & MPL & \cdots & MP^{\mathtt{N_f}-1}L \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{\mathtt{N_f}-1} \end{bmatrix} = 0 \Rightarrow MP^{\mathtt{N_f}-1}Lw_{\mathtt{N_f}-1} = 0$$

$$\Rightarrow w_{\mathtt{N_f}-1} = 0 \,(\text{since } MP^{\mathtt{N_f}-1}L \neq 0).$$

Next, we pre-multiply $W$ with $MP$ in the equation $Ww = 0$ and use Lemma 2.23 with the fact that $w_{\mathtt{N_f}-1} = 0$:

$$\begin{bmatrix} MPL & MP^2L & \cdots MP^{\mathtt{N_f}-1}L & MP^{\mathtt{N_f}}L \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{\mathtt{N_f}-2} \\ 0 \end{bmatrix} = 0 \Rightarrow MP^{\mathtt{N_f}-1}Lw_{\mathtt{N_f}-2} = 0 \Rightarrow w_{\mathtt{N_f}-2} = 0.$$

Continuing in the same manner, it is evident that $w_i = 0$ for $i \in \{0, 1, \ldots, \mathtt{N_f} - 1\}$. However, this is a contradiction since we assume $w$ to be nonzero. Therefore, there exists no nontrivial vector in the kernel of $W$, i.e., $W$ is full column-rank. Hence, from Statement *(1)* of the lemma, it directly follows that $\mathtt{dim}(\mathscr{R}_s) = \mathtt{N_f}$. ∎

Thus, from Theorem 2.24 we establish that for a SISO system the fast subspace is the space spanned by the columns of a truncated controllability matrix. This is expected because it is known in the literature that for a SISO system the strongly reachable subspace is spanned by a truncated controllability matrix [Wil81]. However, the main contribution of Theorem 2.24 is Statement *(2)* which shows that the dimension of the fast subspace depends on the relative degree of the transfer function of the system. An important point to note here is that the relative degree of a system remains invariant irrespective of the i/s/o representation of the system being minimal or non-minimal. Hence, the dimension of the fast subspace is a system property. Another salient feature of the fast subspace of $\Sigma_\mathrm{p}$ is that it is a $\mathtt{N_f}$-dimensional subspace inside the controllable subspace of the system $\Sigma_\mathrm{p}$.

### 2.3.2 Characterization of the slow subspace

As motivated in Section 2.2.5, let $\mathscr{O}_w$ be the slow subspace of the system $\Sigma_\mathrm{p}$ defined in equation (2.13). In the next lemma we establish that $\mathscr{O}_w$ can be characterized by the eigenvectors of the Rosenbrock system matrix $(U_1, U_2)$.

Characterization of the slow subspace of a SISO system

**Theorem 2.25.** *Consider the system $\Sigma_{\mathrm{p}}$ as defined in equation (2.13) and the corresponding Rosenbrock matrix pair $(U_1,U_2)$ as defined in equation (2.14). Assume $\mathtt{det}(sU_1-U_2)\neq 0$ and $\mathtt{degdet}(sU_1-U_2)=:\mathtt{N_s}$. Consider $\mathscr{O}_w$ to be the slow subspace of $\Sigma_{\mathrm{p}}$. Let $\widehat{V}_1\in\mathbb{R}^{\mathtt{N}\times\mathtt{N_s}}$ and $\widehat{V}_2\in\mathbb{R}^{1\times\mathtt{N_s}}$ be such that*

$$\underbrace{\begin{bmatrix} P & L\\ M & 0\end{bmatrix}}_{U_2}\begin{bmatrix}\widehat{V}_1\\ \widehat{V}_2\end{bmatrix}=\underbrace{\begin{bmatrix} I_{\mathtt{N}} & 0\\ 0 & 0\end{bmatrix}}_{U_1}\begin{bmatrix}\widehat{V}_1\\ \widehat{V}_2\end{bmatrix}J,\ \text{where } J\in\mathbb{R}^{\mathtt{N_s}\times\mathtt{N_s}} \text{ and } \sigma(J)=\mathtt{roots}\left(\mathtt{det}(sU_1-U_2)\right). \quad (2.21)$$

*Then, the following statements are true:*
*(1) $\mathscr{O}_w=\mathtt{img}\,\widehat{V}_1$.* *(2) $\mathtt{dim}(\mathscr{O}_w)=\mathtt{N_s}$.* *(3) $\widehat{V}_1$ is full column-rank.*

*Proof: (1):* From equation (2.21), it is clear that $P\widehat{V}_1+L\widehat{V}_2=\widehat{V}_1J$. Hence, by Proposition 2.10, $\mathtt{img}\,\widehat{V}_1$ is a $(P,L)$-invariant subspace. Further, from equation (2.21), $M\widehat{V}_1=0$. Therefore, $\mathtt{img}\,\widehat{V}_1\in\mathfrak{I}(P,L;\mathtt{ker}\,M)$. We claim that $\mathtt{img}\,\widehat{V}_1=\sup\mathfrak{I}(P,L;\mathtt{ker}\,M)$, i.e, $\mathtt{img}\,\widehat{V}_1=\mathscr{O}_w$ (by Proposition 2.14).

Let us assume to the contrary that $\mathtt{img}\,\widehat{V}_1$ is not the largest $(P,L)$-invariant subspace inside $\mathtt{ker}\,M$. Then, there exists a nontrivial subspace $\mathscr{V}_e$ such that the space $\mathtt{img}\,\widehat{V}_1\oplus\mathscr{V}_e=\mathscr{O}_w$, where $\mathtt{dim}(\mathscr{V}_e)=:\ell$. Let $\mathscr{V}_e=\mathtt{img}\,\widehat{V}_e$, where $\widehat{V}_e\in\mathbb{R}^{\mathtt{N}\times\ell}$ is a full column-rank matrix. Since $\mathtt{img}\,\widehat{V}_1\oplus\mathscr{V}_e=\mathscr{O}_w$ and $\mathscr{O}_w$ is $(P,L)$-invariant inside $\mathtt{ker}\,M$, we must have by Proposition 2.10

$$P\mathscr{O}_w\subseteq\mathscr{O}_w+\mathtt{img}\,L\Rightarrow P(\mathtt{img}\,V_{1\wedge}\oplus\mathscr{V}_e)\subseteq\mathscr{O}_w+\mathtt{img}\,L\Rightarrow P\mathscr{V}_e\subseteq\mathscr{O}_w+\mathtt{img}\,L \text{ and } M\mathscr{V}_e=\{0\}.$$

Therefore, there exist $T_1\in\mathbb{R}^{1\times\ell}$, $T_2\in\mathbb{R}^{\mathtt{N_s}\times\ell}$, and $T_3\in\mathbb{R}^{\ell\times\ell}$ such that

$$P\widehat{V}_e=LT_1+\begin{bmatrix}\widehat{V}_1 & \widehat{V}_e\end{bmatrix}\begin{bmatrix}T_2\\ T_3\end{bmatrix} \text{ and } M\widehat{V}_e=0. \quad (2.22)$$

Therefore, writing equation (2.21) and equation (2.22) together we have

$$\underbrace{\begin{bmatrix} P & L\\ M & 0\end{bmatrix}}_{U_1}\begin{bmatrix}\widehat{V}_1 & \widehat{V}_e\\ \widehat{V}_2 & -T_1\end{bmatrix}=\underbrace{\begin{bmatrix} I_{\mathtt{N}} & 0\\ 0 & 0\end{bmatrix}}_{U_2}\begin{bmatrix}\widehat{V}_1 & \widehat{V}_e\\ \widehat{V}_2 & -T_1\end{bmatrix}\begin{bmatrix} J & T_2\\ 0 & T_3\end{bmatrix}. \quad (2.23)$$

Since $(sU_1-U_2)$ is a regular matrix pencil, we can rewrite $(U_1,U_2)$ in the canonical form as described in Section 2.2.3. Therefore, there exist nonsingular matrices $Z_1,Z_2\in\mathbb{R}^{(\mathtt{N}+1)\times(\mathtt{N}+1)}$ such that $U_1=Z_1\begin{bmatrix} I & 0\\ 0 & Y\end{bmatrix}Z_2$ and $U_2=Z_1\begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}Z_2$, where $Y\in\mathbb{R}^{(\mathtt{N}+1-\mathtt{N_s})\times(\mathtt{N}+1-\mathtt{N_s})}$ is a nilpotent matrix. Define $\widehat{U}_1:=\begin{bmatrix} I & 0\\ 0 & Y\end{bmatrix}$ and $\widehat{U}_2:=\begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}$. Using this in equation (2.23), we have

$$Z_1\begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}Z_2\begin{bmatrix}\widehat{V}_1 & \widehat{V}_e\\ \widehat{V}_2 & -T_1\end{bmatrix}=Z_1\begin{bmatrix} I & 0\\ 0 & Y\end{bmatrix}Z_2\begin{bmatrix}\widehat{V}_1 & \widehat{V}_e\\ \widehat{V}_2 & -T_1\end{bmatrix}\begin{bmatrix} J & T_2\\ 0 & T_3\end{bmatrix}. \quad (2.24)$$

Let $Z_2 \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} =: \begin{bmatrix} T_{\mathtt{N_s}} \\ \widetilde{T} \end{bmatrix}$, where $T_{\mathtt{N_s}} \in \mathbb{R}^{\mathtt{N_s} \times \mathtt{N_s}}$ and $\widetilde{T} \in \mathbb{R}^{(\mathtt{N}+1-\mathtt{N_s}) \times \mathtt{N_s}}$. From equation (2.24) we have

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} Z_2 \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} Z_2 \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} J \Rightarrow \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{\mathtt{N_s}} \\ \widetilde{T} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} T_{\mathtt{N_s}} \\ \widetilde{T} \end{bmatrix} J \tag{2.25}$$

Therefore, from equation (2.25) we have $\widetilde{T} = Y\widetilde{T}J$. Pre- and post-multiplying this equation by $Y$ and $J$, respectively we have $Y\widetilde{T}J = Y^2\widetilde{T}J^2 \Rightarrow \widetilde{T} = Y^2\widetilde{T}J^2$. Continuing pre- and post-multiplication with $Y$ and $J$, it is clear that $\widetilde{T} = Y^k\widetilde{T}J^k$ for all $k \in \mathbb{N}$. However, since $Y$ is nilpotent matrix, it admits a nilpotency index say $\mathtt{h}$. Thus, we have $\widetilde{T} = Y^{\mathtt{h}}\widetilde{T}J^{\mathtt{h}} = 0$. Therefore, we have $Z_2 \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} = \begin{bmatrix} T_{\mathtt{N_s}} \\ 0 \end{bmatrix}$. Define $Z_2 \begin{bmatrix} \widehat{V}_e \\ -T_1 \end{bmatrix} =: \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}$, $\Upsilon_1 \in \mathbb{R}^{\mathtt{N_s} \times \ell}$ and $\Upsilon_2 \in \mathbb{R}^{(\mathtt{N}+1-\mathtt{N_s}) \times \ell}$. Thus, from equation (2.24) we have

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{\mathtt{N_s}} & \Upsilon_1 \\ 0 & \Upsilon_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} T_{\mathtt{N_s}} & \Upsilon_1 \\ 0 & \Upsilon_2 \end{bmatrix} \begin{bmatrix} J & T_2 \\ 0 & T_3 \end{bmatrix}. \tag{2.26}$$

Thus, we have $\Upsilon_2 = Y\Upsilon_2T_3 \Rightarrow Y\Upsilon_2T_3 = Y^2\Upsilon_2T_3^2 = \Upsilon_2$. Using this line of reasoning, it is evident that $Y^k\Upsilon_2T_3^k = \Upsilon_2$ for all $k \in \mathbb{N}$. Since $Y$ is a nilpotent matrix, it admits a nilpotency index $\mathtt{h} \in \mathbb{N}$ and therefore, $Y^{\mathtt{h}} = 0$. Thus, we must have $\Upsilon_2 = 0$. Since $T_{\mathtt{N_s}}$ is a nonsingular matrix, $\mathtt{img}\,\Upsilon_1 \subsetneq T_{\mathtt{N_s}}$. Thus, we have

$$\begin{aligned} \mathtt{img} \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} = \mathtt{img} \begin{bmatrix} \Upsilon_1 \\ 0 \end{bmatrix} \subsetneq \mathtt{img} \begin{bmatrix} T_{\mathtt{N_s}} \\ 0 \end{bmatrix} &\Rightarrow \mathtt{img} \left( Z_2^{-1} \begin{bmatrix} \Upsilon_1 \\ 0 \end{bmatrix} \right) \subsetneq \mathtt{img} \left( Z_2^{-1} \begin{bmatrix} T_{\mathtt{N_s}} \\ 0 \end{bmatrix} \right) \\ &\Rightarrow \mathtt{img} \begin{bmatrix} \widehat{V}_e \\ -T_1 \end{bmatrix} \subsetneq \mathtt{img} \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} \Rightarrow \mathtt{img}\,\widehat{V}_e \subsetneq \mathtt{img}\,\widehat{V}_1. \end{aligned}$$

Therefore, there does not exist any nontrivial subspace $\mathscr{V}_e$ such that $\mathtt{img}\,\widehat{V}_1 \oplus \mathscr{V}_e = \mathscr{O}_w$. This is a contradiction to the assumption that $\mathtt{img}\,\widehat{V}_1 \neq \mathtt{sup}\,\mathfrak{I}(P,L;\ker M)$. Hence, $\mathtt{img}\,\widehat{V}_1 = \mathscr{O}_w$.

*(2):* Define $G(s) := M(sI_{\mathtt{N}} - P)^{-1}L$. Now computing $\mathtt{det}(sU_1 - U_2)$ using the notion of Schur complement with respect to $(sI_{\mathtt{N}} - P)$, we have

$$\mathtt{det}(sU_1 - U_2) = \mathtt{det} \begin{bmatrix} sI_{\mathtt{N}} - P & -L \\ -M & 0 \end{bmatrix} = (-M(sI_{\mathtt{N}} - P)^{-1}L) \times \mathtt{det}(sI_{\mathtt{N}} - P). \tag{2.27}$$

Since $\mathtt{det}(sU_1 - U_2) \neq 0$, we must have $M(sI_{\mathtt{N}} - P)^{-1}L = G(s) \neq 0$. Hence, $G(s)$ is nonzero rational polynomial. Therefore, from Proposition 2.17 we have $\mathscr{O}_w \oplus \mathscr{R}_s = \mathbb{R}^{\mathtt{N}}$. From Statement (2) of Lemma 2.24, we know that $\mathtt{dim}(\mathscr{R}_s) = \mathtt{N} - \mathtt{N_s}$. Therefore, $\mathtt{dim}(\mathscr{O}_w) = \mathtt{N_s}$.

*(3)*: From Statements *(1)* and *(2)* of this theorem, it follows that $\mathtt{dim}(\mathscr{O}_w) = \mathtt{dim}\left(\mathtt{img}\,\widehat{V}_1\right) = \mathtt{N_s}$. Therefore, $\widehat{V}_1$ is full column-rank. ■

Thus, the dimension of the slow subspace of a SISO system is equal to $\mathtt{N} - \mathtt{N_f}$. For a SISO system that is both controllable and observable, the dimension of the slow subspace is equal to the degree of the numerator of the system's transfer function. On the other hand, for a non-minimal system the dimension of the slow subspace = degree of numerator of the system's transfer function (after pole-zero cancellation) + number of uncontrollable/unobservable (or both) eigenvalues of the system.

Next we characterize the good slow subspace of the system $\Sigma_{\mathtt{p}}$ in terms of the eigenspace of the Rosenbrock matrix pair $(U_1, U_2)$. From Theorem 2.25 it is clear that the columns of $\widehat{V}_1$ is the basis of $\mathscr{O}_w$. Further, from equation (2.21) we know that

$$\begin{bmatrix} P & L \\ M & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} = \begin{bmatrix} I_{\mathtt{N}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} J. \tag{2.28}$$

Assuming that $\sigma(J) \cap j\mathbb{R} = \emptyset$, it is clear that $\sigma(J)$ can be partitioned as $\sigma(J) = \sigma_{\mathtt{g}}(J) \uplus \sigma_{\mathtt{b}}(J)$, where $\sigma_{\mathtt{g}}(J) \subsetneq \mathbb{C}_-$, $\sigma_{\mathtt{b}}(J) \subsetneq \mathbb{C}^+$. Therefore, there exists a nonsingular matrix $T$ such that $T^{-1}JT = \begin{bmatrix} J_{\mathtt{g}} & 0 \\ 0 & J_{\mathtt{b}} \end{bmatrix}$, where $\sigma(J_{\mathtt{g}}) = \sigma_{\mathtt{g}}(J)$ and $\sigma(J_{\mathtt{b}}) = \sigma_{\mathtt{b}}(J)$. Define $\begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} T = \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{1\mathtt{b}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{b}} \end{bmatrix}$ where the partitioning is done conforming to the partition in $T^{-1}JT$. Then, equation (2.28) takes the following form:

$$\begin{aligned} \begin{bmatrix} P & L \\ M & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} T &= \begin{bmatrix} I_{\mathtt{N}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} TT^{-1}JT \\ \Rightarrow \begin{bmatrix} P & L \\ M & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{1\mathtt{b}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{b}} \end{bmatrix} &= \begin{bmatrix} I_{\mathtt{N}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{1\mathtt{b}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{b}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & 0 \\ 0 & J_{\mathtt{b}} \end{bmatrix}. \end{aligned} \tag{2.29}$$

We claim in the next lemma that the good slow subspace of the system $\Sigma_{\mathtt{p}}$ is given by $\mathtt{img}\,\widehat{V}_{1\mathtt{g}}$.

A basis for the good slow subspace of a SISO system

**Lemma 2.26.** *Consider the system $\Sigma_{\mathtt{p}}$ as defined in equation* (2.13) *and the corresponding Rosenbrock matrix pair $(U_1, U_2)$ as defined in equation* (2.14). *Assume $\mathtt{det}(sU_1 - U_2) \neq 0$ and $\sigma(U_1, U_2) \cap j\mathbb{R} = \emptyset$. Define the family of subspaces:*

$$\mathscr{B} := \{\mathscr{S} \in \mathfrak{I}(P, L, \mathtt{ker}\, M) \mid \text{there exists } F \in \mathbf{F}(\mathscr{S}) \text{ such that } \sigma\left((P + LF)|_{\mathscr{S}}\right) \subsetneq \mathbb{C}_-\}.$$

*Let $\mathscr{O}_{wg} := \mathtt{sup}\,\mathscr{B}$. Consider $\widehat{V}_{1\mathtt{g}}$ to be as defined in equation* (2.29). *Then,*

$$\mathtt{img}\,\widehat{V}_{1\mathtt{g}} = \mathscr{O}_{wg}.$$

*Proof:* Since $\widehat{V}_1$ is full column-rank (by Theorem 2.25), $\widehat{V}_{1\mathtt{g}}$ is full column-rank, as well. Let us assume to the contrary that $\mathtt{img}\,\widehat{V}_{1\mathtt{g}} \subsetneq \mathscr{O}_{wg}$. Then there exists a nontrivial subspace $\widetilde{\mathscr{V}}$ such that $\mathtt{img}\,\widehat{V}_{1\mathtt{g}} \oplus \widetilde{\mathscr{V}} = \mathscr{O}_{wg}$. Define $\mathtt{dim}(\mathtt{img}\,\widehat{V}_{1\mathtt{g}}) =: \mathtt{N}_{\mathtt{g}}$ and $\mathtt{dim}(\widetilde{\mathscr{V}}) =: \mathtt{N}_{\ell}$. Let $\widetilde{\mathscr{V}} =: \mathtt{img}\,\widetilde{V}$, where $\widetilde{V} \in \mathbb{R}^{\mathtt{N} \times \mathtt{N}_{\ell}}$ is full column-rank. Following the same line of argument as in the proof of Statement *(1)* of Theorem 2.25, there exist $\widehat{T}_1 \in \mathbb{R}^{1 \times \mathtt{N}_{\ell}}, \widehat{T}_2 \in \mathbb{R}^{\mathtt{N}_{\mathtt{g}} \times \mathtt{N}_{\ell}}$ and $\widehat{T}_3 \in \mathbb{R}^{\mathtt{N}_{\ell} \times \mathtt{N}_{\ell}}$ such that

$$P\widetilde{V} = L\widehat{T}_1 + \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widetilde{V} \end{bmatrix} \begin{bmatrix} \widehat{T}_2 \\ \widehat{T}_3 \end{bmatrix}, M\widetilde{V} = 0 \text{ and } \sigma(\widehat{T}_3) \subsetneq \mathbb{C}_-. \tag{2.30}$$

Therefore, from equation (2.29) and equation (2.30) we have

$$\underbrace{\begin{bmatrix} P & L \\ M & 0 \end{bmatrix}}_{U_2} \begin{bmatrix} \widehat{V}_{1\mathrm{g}} & \widetilde{V} \\ \widehat{V}_{2\mathrm{g}} & -\widehat{T}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} I_{\mathtt{N}} & 0 \\ 0 & 0 \end{bmatrix}}_{U_1} \begin{bmatrix} \widehat{V}_{1\mathrm{g}} & \widetilde{V} \\ \widehat{V}_{2\mathrm{g}} & -\widehat{T}_1 \end{bmatrix} \begin{bmatrix} J_{\mathrm{g}} & \widehat{T}_2 \\ 0 & \widehat{T}_3 \end{bmatrix} \text{ and } \sigma(\widehat{T}_3) \cup \sigma(J_{\mathrm{g}}) \subsetneq \mathbb{C}_-. \tag{2.31}$$

Now there exist nonsingular matrices $Z_1, Z_2 \in \mathbb{R}^{(\mathtt{N}+1)\times(\mathtt{N}+1)}$ such that $U_1 = Z_1 \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} Z_2$ and $U_2 = Z_1 \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} Z_2$. Therefore, equation (2.31) takes the following form:

$$Z_1 \underbrace{\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}}_{\widehat{U}_2} Z_2 \begin{bmatrix} \widehat{V}_{1\mathrm{g}} & \widetilde{V} \\ \widehat{V}_{2\mathrm{g}} & -\widehat{T}_1 \end{bmatrix} = Z_1 \underbrace{\begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}}_{\widehat{U}_1} Z_2 \begin{bmatrix} \widehat{V}_{1\mathrm{g}} & \widetilde{V} \\ \widehat{V}_{2\mathrm{g}} & -\widehat{T}_1 \end{bmatrix} \begin{bmatrix} J_{\mathrm{g}} & \widehat{T}_2 \\ 0 & \widehat{T}_3 \end{bmatrix}. \tag{2.32}$$

From equation (2.32) it is clear that $\mathtt{img}\left(Z_2 \begin{bmatrix} \widehat{V}_{1\mathrm{g}} \\ \widehat{V}_{2\mathrm{g}} \end{bmatrix}\right)$ is a subspace of the eigenspace of the matrix pair $(\widehat{U}_1, \widehat{U}_2)$. Note that any eigenvector (or generalized eigenvector) of the matrix pair $(\widehat{U}_1, \widehat{U}_2)$ will be of the form $\mathtt{col}(w, 0) \in \mathbb{R}^{(\mathtt{N}+1)}$, where $w \in \mathbb{R}^{\mathtt{N_s}}$ is an eigenvector (or generalized eigenvector) of $J_{\mathrm{g}}$. Thus, there exists a full column-rank matrix $T_{\mathtt{N_g}} \in \mathbb{R}^{\mathtt{N_s}\times\mathtt{N_g}}$ such that $Z_2 \begin{bmatrix} \widehat{V}_{1\mathrm{g}} \\ \widehat{V}_{2\mathrm{g}} \end{bmatrix} = \begin{bmatrix} T_{\mathtt{N_g}} \\ 0 \end{bmatrix} \in \mathbb{R}^{(\mathtt{N}+1)\times\mathtt{N_g}}$. Define $Z_2 \begin{bmatrix} \widetilde{V} \\ -\widehat{T}_1 \end{bmatrix} =: \begin{bmatrix} \widehat{\Upsilon}_1 \\ \widehat{\Upsilon}_2 \end{bmatrix}$, where $\widehat{\Upsilon}_1 \in \mathbb{R}^{\mathtt{N_s}\times\mathtt{N_g}}$ and $\widehat{\Upsilon}_2 \in \mathbb{R}^{(\mathtt{N}+1-\mathtt{N_s})\times\mathtt{N_g}}$. Thus, from equation (2.32) we have

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{\mathtt{N_g}} & \widehat{\Upsilon}_1 \\ 0 & \widehat{\Upsilon}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} T_{\mathtt{N_g}} & \widehat{\Upsilon}_1 \\ 0 & \widehat{\Upsilon}_2 \end{bmatrix} \begin{bmatrix} J_{\mathrm{g}} & \widehat{T}_2 \\ 0 & \widehat{T}_3 \end{bmatrix}. \tag{2.33}$$

From equation (2.33), we have $\widehat{\Upsilon}_2 = Y\widehat{\Upsilon}_2\widehat{T}_3$. Since $Y$ is nilpotent, similar to the proof of Statement *(1)* of Theorem 2.25, we must have $\widehat{\Upsilon}_2 = 0$. Hence, equation (2.33) becomes

$$J \begin{bmatrix} T_{\mathtt{N_g}} & \widehat{\Upsilon}_1 \end{bmatrix} = \begin{bmatrix} T_{\mathtt{N_g}} & \widehat{\Upsilon}_1 \end{bmatrix} \begin{bmatrix} J_{\mathrm{g}} & \widehat{T}_2 \\ 0 & \widehat{T}_3 \end{bmatrix}. \tag{2.34}$$

Since $\sigma(J_{\mathrm{g}}) \cup \sigma(\widehat{T}_3) \subseteq \sigma(J)$, $\sigma(J) \cap \mathbb{C}_- = \sigma(J_{\mathrm{g}})$, and $\sigma(J) \cap j\mathbb{R} = \emptyset$, we must have $\sigma(\widehat{T}_3) \subsetneq \mathbb{C}^+$. However this is a contradiction to the fact that $\sigma(\widehat{T}_3) \subsetneq \mathbb{C}_-$ (see equation (2.30)). Therefore, there exists no nontrivial subspace $\widetilde{\mathscr{V}}$ such that $\mathtt{img}\,\widehat{V}_{1\mathrm{g}} \oplus \widetilde{\mathscr{V}} = \mathscr{O}_{wg}$. Hence, $\widehat{V}_{1\mathrm{g}} = \mathscr{O}_{wg}$. ■

From Lemma 2.26 it is evident that for a controllable and observable SISO system, the dimension of the good slow subspace is equal to number of zeros of the system that have negative real parts. On the other hand, for a non-minimal SISO case (uncontrollable/unobservable or both), the dimension of the good slow subspace = number of zeros of the system that have negative real parts + number of uncontrollable/unobservable (or both) eigenvalues of the system with negative real part. Since we are dealing with SISO systems, in terms of transmission zeros, the dimension of the good slow subspace of the system is equal to the transmission zeros of the system with negative real parts.

For a SISO system $\Sigma_{\mathrm{p}}$ with $\mathtt{det}(sU_1 - U_2) \neq 0$ and $\sigma(U_1, U_2) \cap j\mathbb{R} = \emptyset$, the state-space admits a direct-sum decomposition of the following form.

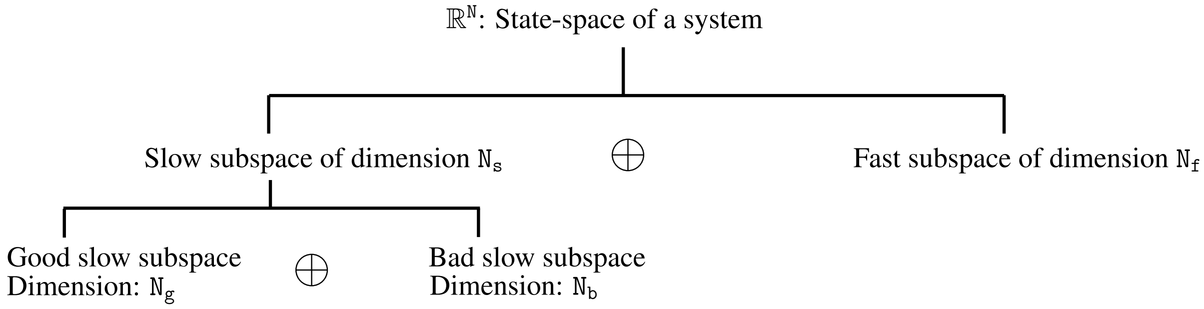



Figure 2.1: A direct-sum decomposition of the state-space $\mathbb{R}^{\mathtt{N}}$

In the next section we illustrate the results of this section with examples.

# Illustrative examples

The first example we consider is that of a controllable and observable system.

**Example 2.27.** *Consider a system $\Sigma_{\mathrm{p}}$ with the following i/s/o representation:*

$$\frac{d}{dt}x = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix}}_{P} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{L} u, \quad 0 = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}}_{M} x \tag{2.35}$$

*The transfer function for this system is: $G(s) = \dfrac{s+1}{s^4+4s^3+3s^2+2s+1}$. Here $\mathtt{N} = 4$ and the relative degree is $\mathtt{N_f} = 3$. Hence, $\mathtt{N_s} = \mathtt{N} - \mathtt{N_f} = 1$. The Rosenbrock matrix pair corresponding to this system is:*

$$U_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -2 & -3 & -4 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \textit{ and } U_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*By simple multiplication it can be verified that $\mathtt{det}(sU_1 - U_2) = -(s+1)$. Hence $\sigma(U_1, U_2) = -1$. The eigenvector of $(U_1, U_2)$ corresponding to $-1$ eigenvalue is $\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}$. Therefore, we have $\widehat{V}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$ and $\mathtt{img}\,\widehat{V}_1$ is the slow subspace of the system.*

*On the other hand, by Lemma 2.24, the fast subspace of the system is given by*

$$\mathscr{R}_s = \texttt{img}\begin{bmatrix} L & PL & P^2L \end{bmatrix} = \texttt{img}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}.$$

*It is evident that*

$$\mathbb{R}^4 = \texttt{img}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \oplus \texttt{img}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}.$$

*Further, note that since $\sigma(U_1,U_2) \subsetneq \mathbb{C}_-$, therefore $\texttt{img}\,\widehat{V}_1$ is indeed the only good slow subspace of the system.*

Note that in Example 2.27 the system is in a minimal i/s/o representation. Hence, the dimension of the slow subspace is equal to the degree of the numerator of $G(s)$. However, as explained above this is not the case for non-minimal state-space representations. We illustrate this with the help of another example where the i/s/o representation is not minimal.

**Example 2.28.** *Consider a system $\Sigma_{\mathtt{p}}$ with the following i/s/o representation:*

$$\frac{d}{dt}x = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{bmatrix}}_{P} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{L} u, \quad 0 = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}}_{M} x \tag{2.36}$$

*The transfer function for this system is:*

$$G(s) = \frac{1}{s^3+9s^2+26s+24}.$$

*Here $\mathtt{N} = 4$ and the relative degree $\mathtt{N_f} = 3$. Therefore $\mathtt{N_s} = 1$. The Rosenbrock matrix pair corresponding to this system is:*

$$U_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -24 & -50 & -35 & -10 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } U_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*By simple multiplication it can be verified that $\texttt{det}(sU_1 - U_2) = -(s+1)$. Hence $\sigma(U_1,U_2) = -1$. The eigenvector of $(U_1,U_2)$ corresponding to $-1$ eigenvalue is $\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}$. Therefore, we have $\widehat{V}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$ and $\texttt{img}\,\widehat{V}_1$ is the slow subspace of the system.*

*On the other hand, by Lemma 2.24, the fast subspace of the system is given by*

$$\mathscr{R}_s = \mathtt{img}\begin{bmatrix} L & PL & P^2L \end{bmatrix} = \mathtt{img}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -10 \\ 1 & -10 & 65 \end{bmatrix}.$$

*It is evident that*

$$\mathbb{R}^4 = \mathtt{img}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \oplus \mathtt{img}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -10 \\ 1 & -10 & 65 \end{bmatrix}.$$

*Further, note that since* $\sigma(U_1, U_2) \subsetneq \mathbb{C}_-$, *therefore* $\mathtt{img}\,\widehat{V}_1$ *is indeed the only good slow subspace of the system.*

Note that in Example 2.28 the degree of the numerator of $G(s)$ is zero. However, we have $\mathtt{N_s} = 1$. This is because the system in the example is unobservable with $-1$ as the unobservable eigenvalue. Hence, the dimension of the slow subspace = degree of the numerator of $G(s)$ + number of uncontrollable/unobservable (or both) of the system = $0+1 = 1$.

Now that we have characterized the slow and fast subspaces of a system in terms of Rosenbrock system matrix, we use these subspaces to present a method to compute the maximal rank-minimizing solution of an LQR LMI for single-input systems.

## 2.4 Maximal rank-minimizing solution of LQR LMI: single-input case

Before we present the first main result of this section, we show the determinant of the Hamiltonian pencil admits a Lambda-set. Note that this result is known in the literature, however we reproduce it next as a lemma for the sake of completeness.

A Hamiltonian matrix pair admits Lambda-sets

**Lemma 2.29.** *Consider the singular LQR Problem 2.21 with a Hamiltonian matrix pair* $(E,H)$ *as defined in equation* (2.10) *and* $\sigma(E,H) \cap j\mathbb{R} = \emptyset$. *Let* $C \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$ *be a full row-rank matrix such that* $Q =: C^T C$. *Define* $G(s) =: C(sI_{\mathtt{n}} - A)^{-1}b$. *Assume let* $G(s) =: \frac{N(s)}{d(s)}$, *where* $N(s) \in \mathbb{R}[s]^{\mathtt{r}}$ *and* $d(s) = \mathtt{det}(sI_{\mathtt{n}} - A)$. *Then, the following statements are true:*

*(1)* $\mathtt{det}(sE - H) = N(-s)^T N(s)$.

*(2) If* $\mathtt{det}(sE - H) \notin \mathbb{R}$, *then* $\mathtt{det}(sE - H)$ *admits a Lambda-set.*

*Proof: (1):* On computing $\mathtt{det}(sE - H)$ using Schur-complement with respect to $(sI_{2\mathtt{n}} - \widehat{A})$, we have

$$\mathtt{det}(sE-H)=\mathtt{det}\begin{bmatrix} sI_{2\mathtt{n}}-\widehat{A} & -\widehat{b} \\ -\widehat{c} & 0 \end{bmatrix} = -\widehat{c}(sI_{2\mathtt{n}}-\widehat{A})^{-1}\widehat{b}\times \mathtt{det}(sI_{2\mathtt{n}}-\widehat{A}) \tag{2.37}$$

Now using the fact that $Q=C^TC$, we have

$$\begin{aligned}
\widehat{c}(sI_{2\mathtt{n}}-\widehat{A})^{-1}\widehat{b} &= \begin{bmatrix} 0 & b^T \end{bmatrix} \begin{bmatrix} sI_{\mathtt{n}}-A & 0 \\ Q & sI_{\mathtt{n}}+A^T \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & b^T \end{bmatrix} \begin{bmatrix} (sI_{\mathtt{n}}-A)^{-1} & 0 \\ -(sI_{\mathtt{n}}+A^T)^{-1}Q(sI_{\mathtt{n}}-A)^{-1} & (sI_{\mathtt{n}}+A^T)^{-1} \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & b^T \end{bmatrix} \begin{bmatrix} (sI_{\mathtt{n}}-A)^{-1} & 0 \\ -(sI_{\mathtt{n}}+A^T)^{-1}C^TC(sI_{\mathtt{n}}-A)^{-1} & (sI_{\mathtt{n}}+A^T)^{-1} \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \\
&= -b^T(sI_{\mathtt{n}}+A^T)^{-1}C^TC(sI_{\mathtt{n}}-A)^{-1}b \\
&= G(-s)^TG(s) = \frac{N(-s)^TN(s)}{d(-s)d(s)}.
\end{aligned} \tag{2.38}$$

Using equation (2.38) and the fact that $\mathtt{det}(sI_{2\mathtt{n}}-\widehat{A}) = d(-s)d(s)$ in equation (2.37), we have

$$\mathtt{det}(sE-H) = \frac{N(-s)^TN(s)}{d(-s)d(s)} \times \mathtt{det}(sI_{2\mathtt{n}}-\widehat{A}) = N(-s)^TN(s).$$

*(2):* From Statement *(1)* of this lemma it is clear that $\sigma(E,H) = \mathtt{roots}\left(N(-s)^TN(s)\right)$. Note that

$$\lambda \in \mathtt{roots}\left(N(-s)^TN(s)\right) \Rightarrow -\lambda \in \mathtt{roots}\left(N(-s)^TN(s)\right).$$

Further, since $N(-s)^TN(s) \in \mathbb{R}[s]$, we must have

$$\lambda \in \mathtt{roots}\left(N(-s)^TN(s)\right) \Rightarrow \bar{\lambda} \in \mathtt{roots}\left(N(-s)^TN(s)\right).$$

Thus, the roots of $N(-s)^TN(s)$ are symmetric about the real and imaginary-axis of the $\mathbb{C}$-plane. Therefore, $N(-s)^TN(s) = \mathtt{det}(sE-H)$ is a even-degree polynomial. Let $\mathtt{degdet}(sE-H) =: 2\mathtt{n_s}$. Since $\sigma(E,H)\cap j\mathbb{R} = \emptyset \Rightarrow \mathtt{roots}\left(N(-s)^TN(s)\right) = \emptyset$, we must have $\mathtt{n_s}$ roots of $\mathtt{det}(sE-H)$ in $\mathbb{C}_-$ and the rest $\mathtt{n_s}$ in $\mathbb{C}_+$. By the definition of Lambda-sets in Definition 2.18, the collection of roots of $\mathtt{det}(sE-H)$ in $\mathbb{C}_-$ (or $\mathbb{C}_+$) is a Lambda-set of $\mathtt{det}(sE-H)$. ∎

Using the fact that the determinant of a Hamiltonian pencil is a even-degree polynomial that admits a Lambda-set, we present the first main result of this section that provides a method to compute the maximal rank-minimizing solution of an LQR LMI.

A method to compute the maximal rank-minimizing solution of an LQR LMI

**Theorem 2.30.** *Consider Problem 2.21 with the corresponding Hamiltonian matrix pair* $(E,H)$ *as defined in equation* (2.11). *Assume* $\sigma(E,H)\cap j\mathbb{R}=\emptyset$ *and* $\det(sE-H)\neq 0$. *Define* $\mathrm{degdet}(sE-H)=:2\mathtt{n_s}$. *Let* $\Lambda$ *be a Lambda-set of* $\det(sE-H)$ *with cardinality* $\mathtt{n_s}<\mathtt{n}$ *such that* $\Lambda\subsetneq\mathbb{C}_-$. *Let* $V_{1\Lambda}$, $V_{2\Lambda}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ *and* $V_{3\Lambda}\in\mathbb{R}^{1\times\mathtt{n_s}}$ *be such that the columns of* $V_{\mathrm{e}\Lambda}=\mathrm{col}(V_{1\Lambda},V_{2\Lambda},V_{3\Lambda})$ *form a basis of the* $\mathtt{n_s}$*-dimensional eigenspace of* $(E,H)$ *corresponding to the eigenvalues of* $(E,H)$ *in* $\Lambda$*:*

$$\begin{bmatrix} A & 0 & b\\ -Q & -A^T & 0\\ 0 & b^T & 0\end{bmatrix}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\\ V_{3\Lambda}\end{bmatrix}=\begin{bmatrix} I_{\mathtt{n}} & 0 & 0\\ 0 & I_{\mathtt{n}} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\\ V_{3\Lambda}\end{bmatrix}\Gamma, \tag{2.39}$$

*where* $\sigma(\Gamma)=\Lambda$. *Construct* $V_\Lambda:=\mathrm{col}\,(V_{1\Lambda},V_{2\Lambda})$ *and assume* $\mathtt{n_f}:=\mathtt{n}-\mathtt{n_s}$. *Define* $W:=\begin{bmatrix}\widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b}\end{bmatrix}\in\mathbb{R}^{2\mathtt{n}\times\mathtt{n_f}}$, *where* $\widehat{A}$ *and* $\widehat{b}$ *are as defined in equation* (2.12). *Let* $X_\Lambda:=\begin{bmatrix}V_\Lambda & W\end{bmatrix}=:\begin{bmatrix}X_{1\Lambda}\\ X_{2\Lambda}\end{bmatrix}$. *Then, the following statements hold.*

*(1)* $X_{1\Lambda}$ *is invertible.*

*(2)* $K_{\mathtt{max}}:=X_{2\Lambda}X_{1\Lambda}^{-1}$ *is symmetric.*

*(3)* $K_{\mathtt{max}}$ *is a rank-minimizing solution of LMI* (2.9).

*(4) For any other solution* $K$ *of LMI* (2.9), $K\leqslant K_{\mathtt{max}}$.

*(5)* $K_{\mathtt{max}}\geqslant 0$.

We defer the proof of this theorem till the development of a few auxiliary results. Note the close parallel between Proposition 2.19 and Theorem 2.30. For the case when $\mathtt{n_f}=0$, i.e. the regular LQR case, Theorem 2.30 is indeed equivalent to Proposition 2.19. Thus, Theorem 2.30 is a generalization to Proposition 2.19.

Now we relate the results in Section 2.3 with the Hamiltonian system $\Sigma_{\mathrm{Ham}}$ defined in Section 2.2.6. Using the parallel between the output-nulling representations of $\Sigma_{\mathrm{p}}$ (in equation (2.13)) and $\Sigma_{\mathrm{Ham}}$ (in equation (2.12)), we define $P:=\widehat{A}$, $L:=\widehat{b}$, $M:=\widehat{c}$, $U_1:=E$, and $U_2:=H$. Further, we have $\mathrm{degdet}(sE-H)=2\mathtt{n_s}$. Therefore, $\mathtt{N_s}=2\mathtt{n_s}$ and $\mathtt{N_f}=\mathtt{N}-\mathtt{N_s}=2\mathtt{n}-2\mathtt{n_s}=2\mathtt{n_f}$. Thus, the dimension of the slow and fast subspace of the Hamiltonian system $\Sigma_{\mathrm{Ham}}$ corresponding a singularly passive SISO system is $2\mathtt{n_s}$ and $2\mathtt{n_f}$, respectively. Hence, Theorem 2.24, Theorem 2.25, and Lemma 2.26 can be directly applied to the system $\Sigma_{\mathrm{Ham}}$. From Lemma 2.26 it is evident that $\mathrm{img}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}$ is the largest good $(\widehat{A},\widehat{b})$-invariant subspace inside the kernel of $\widehat{c}$. Hence, the good slow subspace of $\Sigma_{\mathrm{Ham}}$ is given by $\mathscr{O}_{wg}=\mathrm{img}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}$. Further, using Theorem 2.24, it is also evident that $\mathrm{img}\,W\subsetneq\mathscr{R}_s$, where $W$ is as defined in Theorem 2.30 and $\mathscr{R}_s$ is the fast subspace of $\Sigma_{\mathrm{Ham}}$.

Before we start developing the results required for the proof of Theorem 2.30, we review a result that establishes the relation between the basis vectors of the left- and right-eigenspaces of Hamiltonian matrix pair (see [IOW99] for more on these properties).

**Proposition 2.31.** [IOW99, Proposition 6.18] *Let the columns of $V_{\mathtt{e}\Lambda} = \mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ span the eigenspace of $(E,H)$ corresponding to the eigenvalues in $\Lambda$, where $E, H, V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda}, \Lambda$ are as defined in Theorem 2.30. Then, the following statements are true:*

*(1) Rows of $\begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T & V_{3\Lambda}^T \end{bmatrix}$ are the basis of the left eigenspace of $(E,H)$ corresponding to eigenvalues in $-\Lambda$.*

*(2) $V_{1\Lambda}^T V_{2\Lambda} = V_{2\Lambda}^T V_{1\Lambda}$.*

These properties of the eigenspaces of $(E,H)$ is crucially used in the sequel. Now we develop the results required for the proof of Theorem 2.30. The first step in the proof of Theorem 2.30 is the following theorem:

Disconjugacy of an eigenspace of the Hamiltonian matrix pair

**Theorem 2.32.** *Let $V_{1\Lambda}$ be as defined in Theorem 2.30. Then, $V_{1\Lambda}$ is full-column rank.*

Since $V_{\mathtt{e}\Lambda} = \mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ is a basis of the eigenspace of $(E,H)$, in terms of Definition 2.22, Theorem 2.32 establishes that the subspace $\mathtt{img}\, V_{\mathtt{e}\Lambda}$ is disconjugate. We develop the proof for the disconjugacy of $\mathtt{img}\, V_{\mathtt{e}\Lambda}$ in the next section.

### 2.4.1 Disconjugacy of $\mathtt{img}\, V_{\mathtt{e}\Lambda}$

In this section we prove Theorem 2.32 using a few auxiliary results. The main result that helps us to prove Theorem 2.32 is the claim that the good slow subspace of $\Sigma_{\text{Ham}}$, i.e., $\mathscr{O}_{wg} = \mathtt{img} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}$ can be decomposed into two subspaces. Such a decomposition not only helps us in proving Theorem 2.32 but also provides significant insight into the computation of the optimal cost of a singular LQR problem. One of the subspaces obtained during such a decomposition is linked with the good slow subspace of the system $\Sigma$ itself. Hence, we first reveal the link between the good slow subspace of the system $\Sigma$ and the good slow subspace of $\Sigma_{\text{Ham}}$ in Lemma 2.34 using a well-known proposition from [Won85] next.

**Proposition 2.33.** [Won85, Lemma 5.8] *Define the family*

$$\mathscr{B}_\Sigma := \left\{ \mathscr{V} \subsetneq \mathbb{R}^{\mathtt{n}} \,|\, \exists F \in \mathbb{R}^{1\times \mathtt{n}} \text{ such that } (A+bF)\mathscr{V} \subseteq \mathscr{V}, Q\mathscr{V} = 0, \sigma((A+bF)|_{\mathscr{V}} \subsetneq \mathbb{C}_- \right\}.$$

*Then, $\mathscr{B}_\Sigma$ has a unique supremal element.*

Note that the unique supremal element of $\mathscr{B}_\Sigma$ is indeed the largest good $(A,b)$-invariant subspace in the kernel of $Q$. In the next lemma we establish the relation between this subspace and the subspace $\mathscr{O}_{wg} = \mathtt{img} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}$ of $\Sigma_{\text{Ham}}$.

Relation between the supremal element of $\mathscr{B}_\Sigma$ and the good slow subspace of $\Sigma_{\text{Ham}}$

**Lemma 2.34.** *Let $\mathscr{V}_\text{g} := \sup \mathscr{B}_\Sigma$. Suppose $V_\text{g} \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$ be such that $V_\text{g}$ is full column-rank and $\text{img}\, V_\text{g} = \mathscr{V}_\text{g}$. Define $\mathscr{V}_{\text{g}Ham} := \text{img}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix}$. Let $V_{1\Lambda}, V_{2\Lambda}$ be as defined in Theorem 2.30. Then, $\mathscr{V}_{\text{g}Ham} \subseteq \text{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$.*

*Proof:* Recall $\mathscr{O}_{wg} = \text{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$. Since $\mathscr{V}_\text{g} = \text{img}\, V_\text{g} \in \mathscr{B}_\Sigma$, there exists $F \in \mathbf{F}(\mathscr{V}_\text{g})$ such that $(A+bF)V_\text{g} = V_\text{g}J_\text{g}$, where $J_\text{g} = (A+bF)|_{\mathscr{V}_\text{g}}$ and $\sigma(J_\text{g}) \subsetneq \mathbb{C}_-$. Define $V_{3\text{g}} := FV_\text{g}$. Then,

$$\begin{bmatrix} A & 0 & b \\ -Q & -A^T & 0 \\ 0 & b^T & 0 \end{bmatrix}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3\text{g}}\end{bmatrix} = \begin{bmatrix} I_\mathtt{n} & 0 & 0 \\ 0 & I_\mathtt{n} & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3\text{g}}\end{bmatrix} J_\text{g} \Rightarrow \begin{bmatrix} \widehat{A} & \widehat{b} \\ \widehat{c} & 0 \end{bmatrix}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3\text{g}}\end{bmatrix} = \begin{bmatrix} I_{2\mathtt{n}} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3\text{g}}\end{bmatrix} J_\text{g}. \quad (2.40)$$

Thus, $\sigma(J_\text{g}) \subsetneq \sigma(E,H)$. Using Proposition 2.10 in equation (2.40) we can infer that $\mathscr{V}_{\text{g}Ham} = \text{img}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix}$ is an $(\widehat{A},\widehat{b})$-invariant subspace. Further, using the fact that $\widehat{c}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} = 0$ in equation (2.40), it is evident that $\mathscr{V}_{\text{g}Ham}$ is an $(\widehat{A},\widehat{b})$-invariant subspace inside $\ker \widehat{c}$ with $\sigma(J_\text{g}) \subsetneq \mathbb{C}_-$. Since $\mathscr{O}_{wg}$ is the largest good $(\widehat{A},\widehat{b})$-invariant subspace inside $\ker \widehat{c}$, we have $\mathscr{V}_{\text{g}Ham} \subseteq \mathscr{O}_{wg}$. ∎

Since $\text{img}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix} \subseteq \text{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$ and $\text{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$ is the largest good $(\widehat{A},\widehat{b})$-invariant subspace inside $\ker \widehat{c}$, it is evident that $\text{img}\begin{bmatrix} V_\text{g} \\ 0_{\mathtt{n},\mathtt{g}}\end{bmatrix}$ can be extended to $\text{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$. The next lemma deals with such an extension.

Decomposition of the good slow subspace of the Hamiltonian system $\Sigma_{\text{Ham}}$

**Lemma 2.35.** *Let $V_{1\mathtt{e}}, V_{2\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n}-\mathtt{g})}$ be such that $\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix}$ is full column-rank and $\text{img}\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix} = \mathscr{O}_{wg}$, where $\mathscr{O}_{wg} = \text{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$ with $V_{1\Lambda}, V_{2\Lambda}$ as defined in Theorem 2.30 and $V_\text{g}$ is as defined in Lemma 2.34. Then, the following statements are true*

*(1) $V_{2\mathtt{e}}$ is full column-rank.*

*(2) $\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}}\end{bmatrix}$ is full column-rank.*

*Proof: (1):* Since $\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}}\end{bmatrix}$ is $(\widehat{A},\widehat{b})$-invariant inside $\ker \widehat{c}$, by Proposition 2.10 we have

$$\widehat{A}\left(\text{img}\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}}\end{bmatrix}\right) \subsetneq \text{img}\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}}\end{bmatrix} + \text{img}\, \widehat{b} \quad \text{and} \quad \widehat{c}\left(\text{img}\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}}\end{bmatrix}\right) = \{0\}$$

$$\Rightarrow \widehat{A}\left(\text{img}\begin{bmatrix} V_{1\mathtt{e}} \\ V_{2\mathtt{e}}\end{bmatrix}\right) \subsetneq \text{img}\begin{bmatrix} V_\text{g} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}}\end{bmatrix} + \text{img}\, \widehat{b} \quad \text{and} \quad \widehat{c}\left(\text{img}\begin{bmatrix} V_{1\mathtt{e}} \\ V_{2\mathtt{e}}\end{bmatrix}\right) = \{0\}.$$

Hence, there must exist $V_{3\mathtt{e}} \in \mathbb{R}^{1\times(\mathtt{n}-\mathtt{g})}, \Gamma_{12} \in \mathbb{R}^{\mathtt{g}\times(\mathtt{n_s}-\mathtt{g})}$ and $\Gamma_{22} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times(\mathtt{n_s}-\mathtt{g})}$ such that

$$\widehat{A}\begin{bmatrix} V_{1\mathtt{e}} \\ V_{2\mathtt{e}}\end{bmatrix} = \begin{bmatrix} V_\text{g} \\ 0\end{bmatrix}\Gamma_{12} + \begin{bmatrix} V_{1\mathtt{e}} \\ V_{2\mathtt{e}}\end{bmatrix}\Gamma_{22} - \widehat{b}V_{3\mathtt{e}} \text{ and } \widehat{c}\begin{bmatrix} V_{1\mathtt{e}} \\ V_{2\mathtt{e}}\end{bmatrix} = 0. \quad (2.41)$$

Now writing equation (2.41) and equation (2.40) together, we have

$$\begin{bmatrix} A & 0 & b \\ -Q & -A^T & 0 \\ 0 & b^T & 0 \end{bmatrix} \begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \\ 0 & V_{2\mathrm{e}} \\ V_{3\mathrm{g}} & V_{3\mathrm{e}} \end{bmatrix} = \begin{bmatrix} I_{\mathrm{n}} & 0 & 0 \\ 0 & I_{\mathrm{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \\ 0 & V_{2\mathrm{e}} \\ V_{3\mathrm{g}} & V_{3\mathrm{e}} \end{bmatrix} \begin{bmatrix} J_{\mathrm{g}} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix}. \tag{2.42}$$

Since $\mathtt{img}\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \\ 0_{\mathrm{n,g}} & V_{2\mathrm{e}} \end{bmatrix} = \mathscr{O}_{wg}$, from equation (2.42) we have $\sigma\left(\begin{bmatrix} J_{\mathrm{g}} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix}\right) \subsetneq \mathbb{C}_{-} \Rightarrow \sigma(\Gamma_{22}) \subsetneq \mathbb{C}_{-}$. From equation (2.42) we have the following equations:

$$AV_{1\mathrm{e}} + bV_{3\mathrm{e}} = V_{\mathrm{g}}\Gamma_{12} + V_{1\mathrm{e}}\Gamma_{22}, \tag{2.43}$$

$$-QV_{1\mathrm{e}} - A^T V_{2\mathrm{e}} = V_{2\mathrm{e}}\Gamma_{22}, \tag{2.44}$$

$$b^T V_{2\mathrm{e}} = 0. \tag{2.45}$$

From Statement *(2)* of Proposition 2.31, we can infer that

$$\begin{bmatrix} 0 \\ V_{2\mathrm{e}}^T \end{bmatrix} \begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \end{bmatrix} = \begin{bmatrix} V_{\mathrm{g}}^T \\ V_{1\mathrm{e}}^T \end{bmatrix} \begin{bmatrix} 0 & V_{2\mathrm{e}} \end{bmatrix} \Rightarrow \begin{cases} V_{2\mathrm{e}}^T V_{\mathrm{g}} = 0, \\ V_{2\mathrm{e}}^T V_{1\mathrm{e}} = V_{1\mathrm{e}}^T V_{2\mathrm{e}}. \end{cases} \tag{2.46}$$

Now pre-multiplying equations (2.43) and equation (2.44) with $V_{2\mathrm{e}}^T$ and $-V_{1\mathrm{e}}^T$, respectively and adding, we get

$$V_{2\mathrm{e}}^T AV_{1\mathrm{e}} + V_{2\mathrm{e}}^T bV_{3\mathrm{e}} + V_{1\mathrm{e}}^T QV_{1\mathrm{e}} + V_{1\mathrm{e}}^T A^T V_{2\mathrm{e}} = V_{2\mathrm{e}}^T V_{\mathrm{g}}\Gamma_{12} + V_{2\mathrm{e}}^T V_{1\mathrm{e}}\Gamma_{22} - V_{1\mathrm{e}}^T V_{2\mathrm{e}}\Gamma_{22}. \tag{2.47}$$

Using equation (2.45), equation (2.46) in equation (2.47) , we have

$$V_{2\mathrm{e}}^T AV_{1\mathrm{e}} + V_{1\mathrm{e}}^T QV_{1\mathrm{e}} + V_{1\mathrm{e}}^T A^T V_{2\mathrm{e}} = 0. \tag{2.48}$$

To the contrary, let us assume $V_{2\mathrm{e}}$ is not full column-rank. Therefore, there exists a nonzero $w \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})}$ such that $V_{2\mathrm{e}}w = 0$. Pre- and post-multiplying equation (2.48) with $w^T$ and $w$, respectively and using $V_{2\mathrm{e}}w = 0$, we get $w^T V_{1\mathrm{e}}^T QV_{1\mathrm{e}}w = 0$. Since $Q \geqslant 0$, we must have

$$QV_{1\mathrm{e}}w = 0 \Rightarrow \ker V_{2\mathrm{e}} \subseteq \ker(QV_{1\mathrm{e}}). \tag{2.49}$$

Post-multiplying equation (2.44) with $w$, we have

$$-QV_{1\mathrm{e}}w - A^T V_{2\mathrm{e}}w = V_{2\mathrm{e}}\Gamma_{22}w \Rightarrow -A^T V_{2\mathrm{e}}w = V_{2\mathrm{e}}\Gamma_{22}w \Rightarrow \ker V_{2\mathrm{e}} \text{ is } \Gamma_{22}\text{-invariant.} \tag{2.50}$$

Therefore, from equation (2.50) it follows that there exists a full column-rank matrix $\widetilde{T} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times \bullet}$ such that $V_{2\mathrm{e}}\widetilde{T} = 0$ and $\Gamma_{22}\widetilde{T} = \widetilde{T}\widetilde{\Gamma}, \sigma(\widetilde{\Gamma}) \subseteq \sigma(\Gamma_{22}) \subsetneq \mathbb{C}_{-}$. Further, from equation (2.49), we must have $QV_{1\mathrm{e}}\widetilde{T} = 0$. Post-multiplying equation (2.43) by $\widetilde{T}$, we get

$$AV_{1\mathrm{e}}\widetilde{T} + bV_{3\mathrm{e}}\widetilde{T} = V_{\mathrm{g}}\Gamma_{12}\widetilde{T} + V_{1\mathrm{e}}\Gamma_{22}\widetilde{T} \Rightarrow AV_{1\mathrm{e}}\widetilde{T} + bV_{3\mathrm{e}}\widetilde{T} = V_{\mathrm{g}}\Gamma_{12}\widetilde{T} + V_{1\mathrm{e}}\widetilde{T}\widetilde{\Gamma}. \tag{2.51}$$

Using Proposition 2.10 combined with the fact that $\mathtt{img}\, V_{\mathrm{g}}$ is a good $(A,b)$-invariant subspace of the system and $\sigma(\widetilde{\Gamma}) \subsetneq \mathbb{C}_{-}$, we infer that $\mathtt{img}\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}}\widetilde{T} \end{bmatrix}$ is also a good $(A,b)$-invariant subspace. Further, $Q\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}}\widetilde{T}\end{bmatrix} = 0$. Thus, $\mathtt{img}\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}}\widetilde{T}\end{bmatrix} \in \mathscr{B}_\Sigma$, where $\mathscr{B}_\Sigma$ is as defined in Proposition (2.33). Since $\mathscr{V}_{\mathrm{g}} = \mathtt{sup}\,\mathscr{B}_\Sigma$ and $\mathtt{img}\,V_{\mathrm{g}} = \mathscr{V}_{\mathrm{g}}$, we must have $\mathtt{img}\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}}\widetilde{T}\end{bmatrix} = \mathscr{V}_{\mathrm{g}}$. Therefore, there exist $\alpha_1 \in \mathbb{R}^{\mathrm{g}}$ and a nonzero $\alpha_2 \in \mathbb{R}^{\bullet}$ such that $V_{\mathrm{g}}\alpha_1 + V_{1\mathrm{e}}\widetilde{T}\alpha_2 = 0$, i.e., $\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \\ 0 & V_{2\mathrm{e}}\end{bmatrix}\begin{bmatrix}\alpha_1 \\ \widetilde{T}\alpha_2\end{bmatrix} = \begin{bmatrix} V_{\mathrm{g}}\alpha_1 + V_{1\mathrm{e}}\widetilde{T}\alpha_2 \\ V_{2\mathrm{e}}\widetilde{T}\alpha_2\end{bmatrix} = 0$. Note that since $\widetilde{T}$ is full column-rank, $\widetilde{T}\alpha_2 \neq 0$. Thus, we have a nonzero vector $\begin{bmatrix}\alpha_1 \\ \widetilde{T}\alpha_2\end{bmatrix}$ inside $\mathtt{ker}\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \\ 0 & V_{2\mathrm{e}}\end{bmatrix}$. This is a contradiction to the fact that $\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}} \\ 0 & V_{2\mathrm{e}}\end{bmatrix}$ is full column-rank. Thus, $V_{2\mathrm{e}}$ must be full column-rank.

*(2):* To the contrary, assume that $\begin{bmatrix} V_{\mathrm{g}} & V_{1\mathrm{e}}\end{bmatrix}$ is not full column-rank. Then, there exist $\beta_1 \in \mathbb{R}^{\mathrm{g}}$ and $\beta_2 \in \mathbb{R}^{(\mathrm{n}-\mathrm{g})}$ such that $\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} \neq 0$ and $V_{\mathrm{g}}\beta_1 + V_{1\mathrm{e}}\beta_2 = 0$.

Now pre-multiplying equation (2.43) with $V_{2\mathrm{e}}^T$ and adding it to the transpose of equation (2.44) post-multiplied with $V_{1\mathrm{e}}$, we have

$$V_{2\mathrm{e}}^T A V_{1\mathrm{e}} + V_{2\mathrm{e}}^T b V_{3\mathrm{e}} - V_{1\mathrm{e}}^T Q V_{1\mathrm{e}} - V_{2\mathrm{e}}^T A V_{1\mathrm{e}} = V_{2\mathrm{e}}^T V_{\mathrm{g}}\Gamma_{12} + V_{2\mathrm{e}}^T V_{1\mathrm{e}}\Gamma_{22} + \Gamma_{22}^T V_{2\mathrm{e}}^T V_{1\mathrm{e}}. \tag{2.52}$$

Using equation (2.45) and equation (2.46) in equation (2.52), we have

$$\Gamma_{22}^T V_{2\mathrm{e}}^T V_{1\mathrm{e}} + V_{2\mathrm{e}}^T V_{1\mathrm{e}}\Gamma_{22} = -V_{1\mathrm{e}}^T Q V_{1\mathrm{e}}. \tag{2.53}$$

Let us assume that there exists a nonzero $y \in \mathtt{ker}\,(V_{2\mathrm{e}}^T V_{1\mathrm{e}})$. Pre- and post-multiplying (2.53) by $y^T$ and $y$, respectively and using equation (2.46) we have

$$y^T\Gamma_{22}^T V_{2\mathrm{e}}^T V_{1\mathrm{e}} y + y^T V_{2\mathrm{e}}^T V_{1\mathrm{e}}\Gamma_{22} y = -y^T V_{1\mathrm{e}}^T Q V_{1\mathrm{e}} y \Rightarrow y^T V_{1\mathrm{e}}^T Q V_{1\mathrm{e}} y = 0 \Rightarrow Q V_{1\mathrm{e}} y = 0. \tag{2.54}$$

Now, post-multiplying equation (2.53) with $y$ and using equation (2.54), we have

$$V_{2\mathrm{e}}^T V_{1\mathrm{e}}\Gamma_{22} y = 0 \Rightarrow \mathtt{ker}\,(V_{2\mathrm{e}}^T V_{1\mathrm{e}}) \text{ is } \Gamma_{22}\text{-invariant}. \tag{2.55}$$

using equation (2.55) and the fact that $\sigma(\Gamma_{22}) \subsetneq \mathbb{C}_-$, we have

$$\exists \text{ a nonzero } q \in \mathbb{C}^{(\mathrm{n}-\mathrm{g})} \text{ such that } \Gamma_{22} q = \mu q \text{ and } V_{2\mathrm{e}}^T V_{1\mathrm{e}} q = 0, \text{ where } \mu \in \mathbb{C}^-. \tag{2.56}$$

Post-multiplying equation (2.44), by $q$, we have $-QV_{1\mathrm{e}}q + A^T V_{2\mathrm{e}} q = V_{2\mathrm{e}}\Gamma_{22} q \Rightarrow A^T V_{2\mathrm{e}} q = \mu V_{2\mathrm{e}} q$. If $V_{2\mathrm{e}}q$ is nonzero, then it is a left-eigenvector of $A$. However, from equation (2.45) we can infer that $(V_{2\mathrm{e}}q)^T b = 0$. This means that the system $(A, b)$ is uncontrollable. This is a contradiction. Therefore, $V_{2\mathrm{e}}q$ must be a zero vector. Now from the fact that $V_{2\mathrm{e}}$ is full column-rank (Statement (1) of this lemma), it is evident that $q = 0$, which contradicts equation (2.56). Thus, our initial assumption that there exists a nonzero vector $y$ in $\mathtt{ker}\,(V_{2\mathrm{e}}^T V_{1\mathrm{e}})$ is not true. Hence, $\mathtt{ker}\,(V_{2\mathrm{e}}^T V_{1\mathrm{e}}) = \{0\}$.

Recall that we have assumed $V_{\mathrm{g}}\beta_1 + V_{1\mathrm{e}}\beta_2 = 0$. Pre-multiplying this equation with $V_{2\mathrm{e}}^T$, we have $V_{2\mathrm{e}}^T V_{\mathrm{g}}\beta_1 + V_{2\mathrm{e}}^T V_{1\mathrm{e}}\beta_2 = 0$. Using equation (2.46) and the fact that $\mathtt{ker}\,(V_{2\mathrm{e}}^T V_{1\mathrm{e}}) = \{0\}$, we have $V_{2\mathrm{e}}^T V_{1\mathrm{e}}\beta_2 = 0 \Rightarrow \beta_2 = 0$. Thus, we have $V_{\mathrm{g}}\beta_1 + V_{1\mathrm{e}}\beta_2 = 0 \Rightarrow V_{\mathrm{g}}\beta_1 = 0$. However, since $V_{\mathrm{g}}$ is full column-rank, we must have $\beta_1 = 0$. This is a contradiction to the fact that $\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} \neq 0$.

Hence, $\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \end{bmatrix}$ is full column-rank. ■

Now using Lemma 2.35, we proceed to prove Theorem 2.32.

*Proof of Theorem 2.32:* Since $\mathtt{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} = \mathtt{img}\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}} \end{bmatrix}$, where $V_{\mathtt{g}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$, and $V_{1\mathtt{e}}, V_{2\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$ is as defined in Lemma 2.35, we must have $\mathtt{img}\, V_{1\Lambda} = \mathtt{img}\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \end{bmatrix}$. Note that the number of columns of $V_{1\Lambda}$ and $\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \end{bmatrix}$ are the same. Hence, from Statement *(2)* of Lemma 2.35, it follows that $V_{1\Lambda}$ is full column-rank. ■

Since $V_{1\Lambda}$ is full column-rank, it follows from Definition 2.22 that $\mathtt{img}\, V_{\mathtt{e}\Lambda}$ is disconjugate. This property of disconjugacy is crucially used to prove Theorem 2.30. Apart from this property, there are a few more identities that are required to prove Theorem 2.30. We present these identities as two lemmas in the next section. However, before progressing to the next section, we present a figure next (Figure 2.2) that shows the decomposition of the state-space of $\Sigma_{\text{Ham}}$ in terms of the subspaces we introduced in this section.

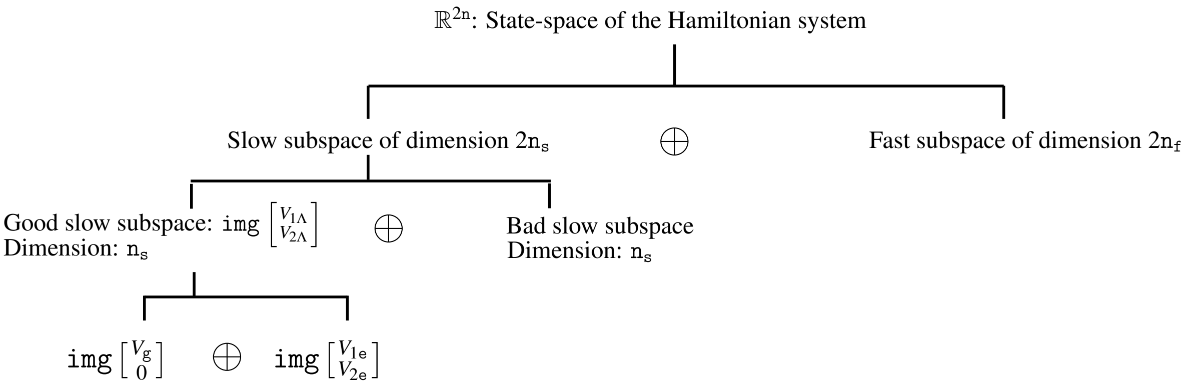



Figure 2.2: A direct-sum decomposition of the state-space of the Hamiltonian system $\Sigma_{\text{Ham}}$

### 2.4.2 Auxiliary results for the proof of Theorem 2.30

In this section we present two lemmas that establish a few identities involving the system matrices $(A, b)$, Markov parameters of $\Sigma_{\text{Ham}}$, cost matrix $Q$ and a solution $K$ of the LQR LMI (2.9). These identities are crucially used in the proof of Theorem 2.30.

Identities involving the Markov parameters of the Hamiltonian system and $Q$

**Lemma 2.36.** *Let* $(\widehat{A}, \widehat{b}, \widehat{c})$, $Q$ *and* $\mathtt{n_f}$ *be as defined in Theorem 2.30. Then, the following statements are true:*

*(1)* $\widehat{c}\widehat{A}^k\widehat{b} = 0$ *for* $k \in \{0, 1, \ldots, 2(\mathtt{n_f} - 1)\}$.

*(2)* $QA^\ell b = 0$ *for* $\ell \in \{0, 1, \ldots, \mathtt{n_f} - 2\}$.

*(3)* $\widehat{A}^\ell\widehat{b} = \mathtt{col}(A^\ell b, 0)$ *and* $\widehat{c}\widehat{A}^\ell = \begin{bmatrix} 0 & (-1)^\ell (A^\ell b)^T \end{bmatrix}$ *for* $\ell \in \{0, 1, \ldots, \mathtt{n_f} - 1\}$.

*Proof:* *(1):* We define $P := \widehat{A}$, $L := \widehat{b}$, $M := \widehat{c}$, $U_1 := E$, and $U_2 := H$ in Lemma 2.23. Further, we have $\mathtt{degdet}(sE - H) = 2\mathtt{n_s}$. Therefore, $\mathtt{N_s} = 2\mathtt{n_s}$ and $\mathtt{N_f} = \mathtt{N} - \mathtt{N_s} = 2\mathtt{n} - 2\mathtt{n_s} = 2\mathtt{n_f}$.

Therefore from Lemma 2.23 Statement *(1)* immediately follows.

*(2)* and *(3)*: Now, we use induction to prove these statements.

*Base case:* ($\ell = 0$) Using Statement *(1)* of this lemma we have

$$\widehat{c}\widehat{A}\widehat{b} = 0 \Rightarrow \begin{bmatrix} 0_{1,\mathtt{n}} & b^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} b \\ 0_{\mathtt{n},1} \end{bmatrix} = b^T Q b = 0. \tag{2.57}$$

Since $Q \geqslant 0$, using the property of positive-semidefinite matrices in equation (2.57) we get $Qb = 0$. Further, $\widehat{b} = \mathtt{col}(b, 0)$ and $\widehat{c} = \begin{bmatrix} 0 & b^T \end{bmatrix}$ by definition.

*Induction step:* Let $QA^\ell b = 0_{\mathtt{n},1}$, $\widehat{A}^\ell \widehat{b} = \mathtt{col}(A^\ell b, 0_{\mathtt{n},1})$, and $\widehat{c}\widehat{A}^\ell = \begin{bmatrix} 0_{1,\mathtt{n}} & (-1)^\ell b^T (A^T)^\ell \end{bmatrix}$, where $\ell < \mathtt{n_f} - 2$. We prove that

$$QA^{\ell+1} b = 0_{\mathtt{n},1}, \quad \widehat{A}^{\ell+1}\widehat{b} = \mathtt{col}(A^{\ell+1}b, 0_{\mathtt{n},1}), \quad \text{and} \quad \widehat{c}\widehat{A}^{\ell+1} = \begin{bmatrix} 0_{1,\mathtt{n}} & (-1)^{\ell+1} b^T (A^T)^{\ell+1} \end{bmatrix}.$$

Note that

$$\widehat{A}^{\ell+1}\widehat{b} = \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} A^\ell b \\ 0_{\mathtt{n},1} \end{bmatrix} = \begin{bmatrix} A^{\ell+1} b \\ -QA^\ell b \end{bmatrix} = \begin{bmatrix} A^{\ell+1} b \\ 0_{\mathtt{n},1} \end{bmatrix},$$

$$\widehat{c}\widehat{A}^{\ell+1} = \begin{bmatrix} 0_{1,\mathtt{n}} & (-1)^\ell (A^\ell b)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} (-1)^\ell (QA^\ell b)^T & (-1)^{\ell+1} (A^{\ell+1} b)^T \end{bmatrix} = \begin{bmatrix} 0_{1,\mathtt{n}} & (-1)^{\ell+1} (A^{\ell+1} b)^T \end{bmatrix}.$$

Since $\ell < \mathtt{n_f} - 2 \Rightarrow 2\ell + 3 < 2\mathtt{n_f} - 1$, using Statement *(1)* of this lemma and the induction hypothesis, we have

$$\widehat{c}\widehat{A}^{2\ell+3}\widehat{b} = 0 \Rightarrow (\widehat{c}\widehat{A}^{\ell+1})\widehat{A}(\widehat{A}^{\ell+1}\widehat{b}) = 0 \Rightarrow \begin{bmatrix} 0 & (-1)^{\ell+1} (A^{\ell+1} b)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} A^{\ell+1} b \\ 0 \end{bmatrix} = 0$$
$$\Rightarrow (A^{\ell+1} b)^T Q (A^{\ell+1} b) = 0 \Rightarrow QA^{\ell+1} b = 0 \text{ (Since } Q \geqslant 0).$$

This completes the proof of Statement *(2)*, and Statement *(3)* for $\ell \in \{0, 1, \ldots, \mathtt{n_f} - 2\}$.

In what follows we complete the proof of Statement *(3)* by proving the identity for the case $\ell = \mathtt{n_f} - 1$. Using the fact that $QA^{\mathtt{n_f}-2} b = 0$ from Statement *(2)* of this lemma, we have

$$\widehat{A}^{\mathtt{n_f}-1}\widehat{b} = \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} A^{\mathtt{n_f}-2} b \\ 0_{\mathtt{n},1} \end{bmatrix} = \begin{bmatrix} A^{\mathtt{n_f}-1} b \\ -QA^{\mathtt{n_f}-2} b \end{bmatrix} = \begin{bmatrix} A^{\mathtt{n_f}-1} b \\ 0_{\mathtt{n},1} \end{bmatrix}.$$

Similarly,

$$\widehat{c}\widehat{A}^{\mathtt{n_f}-1} = \begin{bmatrix} (-1)^{\mathtt{n_f}-1} (A^{\mathtt{n_f}-2} b)^T Q & (-1)^{\mathtt{n_f}-1} (A^{\mathtt{n_f}-1} b)^T \end{bmatrix} = \begin{bmatrix} 0_{1,\mathtt{n}} & (-1)^{\mathtt{n_f}-1} (A^{\mathtt{n_f}-1} b)^T \end{bmatrix}.$$

This completes the proof of Statement *(3)* of this lemma. ■

Algebraic relations satisfied by the solutions of an LQR LMI

**Lemma 2.37.** *Let $K$ be any solution of the singular LQR LMI* (2.9) *with* $\mathtt{degdet}(sE - H) = 2\mathtt{n_s}$ *and* $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$*, where* $(E, H)$ *are as defined in Theorem 2.30. Then, for any* $\alpha \in \{1, \ldots, \mathtt{n_f} - 1\}$*,* $KA^\alpha b = 0$*.*

*Proof:* We prove this using induction and Lemma 2.36.
*Base case:* ($\alpha = 0$) Since $K$ is a solution of the LQR LMI (2.9), $Kb = 0$ is trivially true.
*Inductive step:* Suppose $\alpha \leqslant \mathtt{n_f} - 1$. Assume $KA^{(\alpha-1)}b = 0$, we show that $KA^{\alpha}b = 0$ . Pre- and post-multiplying $\mathscr{L}(K) := A^T K + KA + Q$ by $(A^{(\alpha-1)}b)^T$ and $A^{(\alpha-1)}b$, respectively, we get $(A^{(\alpha-1)}b)^T \mathscr{L}(K)(A^{(\alpha-1)}b) \leqslant 0$. Opening the brackets and using the inductive hypothesis this inequality becomes $(A^{(\alpha-1)}b)^T Q(A^{(\alpha-1)}b) \leqslant 0$. Further, using Statement (1) of Lemma (2.36) in this inequality, we get $\left(A^{(\alpha-1)}b\right)^T Q \left(A^{(\alpha-1)}b\right) = 0 \Rightarrow \mathscr{L}(K)A^{(\alpha-1)}b = 0$ (Since $\mathscr{L}(K) \geqslant 0$). Expanding this equation and using inductive hypothesis with Statement (1) of Lemma 2.36 gives $KA^{\alpha}b = 0$. ■

Now that we have developed all the crucial results required to prove Theorem 2.30, in the ensuing section we prove Theorem 2.30.

### 2.4.3 Proof of Theorem 2.30

*Proof of Statement (1) of Theorem 2.30:* Partition $W =: \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$, where $W_1, W_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_f}}$. Using Statement *(3)* of Lemma 2.36, it is evident that

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \widehat{b} & \widehat{\widehat{Ab}} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix} = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \\ 0 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow \begin{cases} W_1 = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix}, \\ W_2 = 0_{\mathtt{n},\mathtt{n_f}}. \end{cases} \tag{2.58}$$

Therefore, $X_\Lambda = \begin{bmatrix} X_\Lambda & W \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} & W_1 \\ V_{2\Lambda} & 0_{\mathtt{n},\mathtt{n_f}} \end{bmatrix} = \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix}$. Then, we need to prove that $X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix}$ is invertible.

Note that since $V_{1\Lambda}$ is full column-rank (Theorem 2.32), there exists $F \in \mathbb{R}^{1\times\mathtt{n}}$ such that $V_{3\Lambda} = FV_{1\Lambda}$. Thus, from equation (2.39), $(A + bF)V_{1\Lambda} = V_{1\Lambda}\Gamma$. Define

$$W_{1\mathtt{F}} := \begin{bmatrix} b & (A+bF)b & \cdots & (A+bF)^{\mathtt{n_f}-1}b \end{bmatrix}.$$

Then, clearly $\mathtt{img}\, W_1 = \mathtt{img}\, W_{1\mathtt{F}}$. Since $\Sigma$ is controllable, $W_1$ is full column-rank $\Leftrightarrow$ $W_{1\mathtt{F}}$ is also full column-rank. Thus, proving $X_{1\Lambda}$ is invertible is equivalent to proving $\widetilde{X}_{1\Lambda} := \begin{bmatrix} V_{1\Lambda} & W_{1\mathtt{F}} \end{bmatrix}$ is invertible.

Now, we extend the columns of $V_{1\Lambda}$ to form a basis of $\mathbb{R}^{\mathtt{n}}$, say $\mathbb{B}$. Without loss of generality, we assume that the matrices $A, b$ are represented in the basis $\mathbb{B}$. Since $V_{1\Lambda}$ is $(A,b)$-invariant, in the new basis $(A+bF)$ must have the following structure $A + bF = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$, where $\bar{A}_{11} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ and $\bar{A}_{22} \in \mathbb{R}^{(\mathtt{n}-\mathtt{n_s})\times(\mathtt{n}-\mathtt{n_s})}$. Conforming to the partition in $A+bF$, we partition $b =: \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}$. Note $V_{1\Lambda}$ in the basis $\mathbb{B}$ is of the form $\begin{bmatrix} I_{\mathtt{n_s}} \\ 0 \end{bmatrix}$. Further, $W_{1\mathtt{F}}$ in this new basis $\mathbb{B}$ has the following structure

$$W_{1\mathtt{F}} = \begin{bmatrix} \bar{b}_1 & \star & \cdots & \star \\ \bar{b}_2 & \bar{A}_{22}\bar{b}_2 & \cdots & \bar{A}^{\mathtt{n_f}-1}\bar{b}_2 \end{bmatrix}, \text{ where } \star \text{ are suitable matrices with elements from } \mathbb{R}.$$

Since the system is controllable, we have $(A,b)$ controllable $\Leftrightarrow$ $(A+bF,b)$ controllable $\Rightarrow$ $(\bar{A}_{22},\bar{b}_2)$ is controllable. Therefore, $T := \begin{bmatrix}\bar{b}_2 & \bar{A}_{22}\bar{b}_2 & \cdots & \bar{A}^{\mathtt{n_f}-1}\bar{b}_2\end{bmatrix} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}$ is a nonsingular matrix. Now, note that the matrix $\widetilde{X}_{1\Lambda} = \begin{bmatrix}V_{1\Lambda} & W_{1\mathtt{F}}\end{bmatrix}$ in the basis $\mathbb{B}$ takes the form $\begin{bmatrix} I_{\mathtt{n_s}} & \star \\ 0 & T\end{bmatrix}$. Thus, $\widetilde{X}_{1\Lambda}$ is a block upper-triangular matrix with the diagonal blocks $I_{\mathtt{n_s}}$ and $T$ being nonsingular. Therefore, $\widetilde{X}_{1\Lambda}$ is invertible and hence $X_{1\Lambda}$ is invertible. ■

*Proof of Statement (2) of Theorem 2.30:* To prove $X_{2\Lambda}X_{1\Lambda}^{-1} = (X_{2\Lambda}X_{1\Lambda}^{-1})^T$ is equivalent to proving $X_{1\Lambda}^T X_{2\Lambda} = X_{2\Lambda}^T X_{1\Lambda}$. Hence instead of proving $X_{2\Lambda}X_{1\Lambda}^{-1} = (X_{2\Lambda}X_{1\Lambda}^{-1})^T$ we prove that $X_{1\Lambda}^T X_{2\Lambda} - X_{2\Lambda}^T X_{1\Lambda} = 0$. Now, using equation (2.58) to evaluate $X_{1\Lambda}^T X_{2\Lambda} - X_{2\Lambda}^T X_{1\Lambda}$, we get

$$X_{1\Lambda}^T X_{2\Lambda} - X_{2\Lambda}^T X_{1\Lambda} = \begin{bmatrix}V_{1\Lambda}^T \\ W_1^T\end{bmatrix}\begin{bmatrix}V_{2\Lambda} & 0_{\mathtt{n},\mathtt{n_f}}\end{bmatrix} - \begin{bmatrix}V_{2\Lambda}^T \\ 0_{\mathtt{n_f},\mathtt{n}}\end{bmatrix}\begin{bmatrix}V_{1\Lambda} & W_1\end{bmatrix} = \begin{bmatrix}V_{1\Lambda}^T V_{2\Lambda} - V_{2\Lambda}^T V_{1\Lambda} & -V_{2\Lambda}^T W_1 \\ W_1^T V_{2\Lambda} & 0_{\mathtt{n_f},\mathtt{n_f}}\end{bmatrix}. \quad (2.59)$$

From Proposition 2.31, we have $V_{1\Lambda}^T V_{2\Lambda} = V_{2\Lambda}^T V_{1\Lambda}$. Hence, to prove $X_{1\Lambda}^T X_{2\Lambda} - X_{2\Lambda}^T X_{1\Lambda} = 0$, we need to prove that $V_{2\Lambda}^T W_1 = 0$. From equation (2.39), we have

$$-QV_{1\Lambda} - A^T V_{2\Lambda} = V_{2\Lambda}\Gamma \Rightarrow V_{1\Lambda}^T Q + V_{2\Lambda}^T A = -\Gamma^T V_{2\Lambda}^T. \quad (2.60)$$

We first prove that $V_{2\Lambda}^T A^k b = 0$ for $k \in \{0,1,\dots,\mathtt{n_f}-1\}$ using mathematical induction.
*Base case:* $(k=0)$ $V_{2\Lambda}^T b = 0$ follows from equation (2.39).
*Induction step:* Let $V_{2\Lambda}^T A^k b = 0$ for $k < \mathtt{n_f}-1$. We prove that $V_{2\Lambda}^T A^{k+1} b = 0$. Post-multiplying equation (2.60) with $A^k b$ gives $V_{1\Lambda}^T Q A^k b + V_{2\Lambda}^T A^{k+1} b = -\Gamma^T V_{2\Lambda}^T A^k b$. Since $k < \mathtt{n_f}-1$, we know that $QA^k b = 0$ (Lemma 2.36). This equation along with the inductive hypothesis imply that $V_{2\Lambda}^T A^{k+1} b = 0$. Hence, by mathematical induction, we have proved that $V_{2\Lambda}^T A^k b = 0$ for $k \in \{0,1,2,\dots,\mathtt{n_f}-1\}$. In other words, we proved that

$$V_{2\Lambda}^T \begin{bmatrix}b & Ab & \cdots & A^{\mathtt{n_f}-1}b\end{bmatrix} = 0 \Rightarrow V_{2\Lambda}^T W_1 = 0. \quad (2.61)$$

Thus, from equation (2.59), we have $X_{1\Lambda}^T X_{2\Lambda} = X_{2\Lambda}^T X_{1\Lambda}$. Therefore, $X_{2\Lambda}X_{1\Lambda}^{-1}$ is symmetric. ■

*Proof of Statement (3) of Theorem 2.30:* Define $\mathscr{L}(K_{\mathtt{max}}) := A^T K_{\mathtt{max}} + K_{\mathtt{max}}A + Q$. Evaluating $X_{1\Lambda}^T \mathscr{L}(K_{\mathtt{max}}) X_{1\Lambda}$, we get

$$X_{1\Lambda}^T \mathscr{L}(K_{\mathtt{max}}) X_{1\Lambda} = \begin{bmatrix}V_{1\Lambda}^T \mathscr{L}(K_{\mathtt{max}})V_{1\Lambda} & V_{1\Lambda}^T \mathscr{L}(K_{\mathtt{max}})W_1 \\ W_1^T \mathscr{L}(K_{\mathtt{max}})V_{1\Lambda} & W_1^T \mathscr{L}(K_{\mathtt{max}})W_1\end{bmatrix}. \quad (2.62)$$

Note that

$$K_{\mathtt{max}}V_{1\Lambda} = X_{2\Lambda}X_{1\Lambda}^{-1}V_{1\Lambda} = \begin{bmatrix}V_{2\Lambda} & W_2\end{bmatrix}\begin{bmatrix}V_{1\Lambda} & W_1\end{bmatrix}^{-1} V_{1\Lambda} = V_{2\Lambda}$$

$$K_{\mathtt{max}}W_1 = \begin{bmatrix}V_{2\Lambda} & W_2\end{bmatrix}\begin{bmatrix}V_{1\Lambda} & W_1\end{bmatrix}^{-1} W_1 = W_2 = 0 \text{ (From equation (2.58))}.$$

Using the fact that $K_{\mathtt{max}}V_{1\Lambda} = V_{2\Lambda}$ and evaluating $V_{1\Lambda}^T \mathscr{L}(K_{\mathtt{max}})V_{1\Lambda}$ gives

$$V_{1\Lambda}^T \mathscr{L}(K_{\mathtt{max}})V_{1\Lambda} = V_{1\Lambda}(A^T K_{\mathtt{max}} + K_{\mathtt{max}}A + Q)V_{1\Lambda} = \begin{bmatrix}V_{2\Lambda}^T & -V_{1\Lambda}^T\end{bmatrix}\begin{bmatrix}A & 0 \\ -Q & -A^T\end{bmatrix}\begin{bmatrix}V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix} \quad (2.63)$$

Using equation (2.39) and Proposition 2.31 in equation (2.63), we have

$$V_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})V_{1\Lambda} = \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \end{bmatrix} \left( \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} \Gamma - \begin{bmatrix} b \\ 0 \end{bmatrix} V_{3\Lambda} \right) = -V_{2\Lambda}^T b V_{3\Lambda} = 0. \tag{2.64}$$

Using $W_2 = 0$ to evaluate $V_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})W_1$ gives

$$V_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})W_1 = V_{2\Lambda}^T AW_1 + V_{1\Lambda}^T QW_1. \tag{2.65}$$

Post-multiplying equation (2.60) by $W_1$ and using it in equation (2.65) gives

$$V_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})W_1 = V_{1\Lambda}^T QW_1 + V_{2\Lambda}^T AW_1 = -\Gamma^T V_{2\Lambda}^T W_1. \tag{2.66}$$

From equation (2.61), we have $V_{2\Lambda}^T W_1 = 0$. Thus, $V_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})W_1 = 0$.
Since $K_{\mathtt{max}}W_1 = 0$, we must have

$$W_1^T\mathscr{L}(K_{\mathtt{max}})W_1 = W_1^T A^T K_{\mathtt{max}}W_1 + W_1^T K_{\mathtt{max}}AW_1 + W_1^T QW_1 = W_1^T QW_1.$$

Now, using Statement *(1)* of Lemma 2.36, we have

$$W_1^T QW_1 = \begin{bmatrix} 0_{(\mathtt{n_f}-1),(\mathtt{n_f}-1)} & 0_{(\mathtt{n_f}-1),1} \\ 0_{1,(\mathtt{n_f}-1)} & (A^{\mathtt{n_f}-1}b)^T QA^{\mathtt{n_f}-1}b \end{bmatrix}. \tag{2.67}$$

Thus, using equation (2.67) in equation (2.62), we have

$$X_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})X_{1\Lambda} = \begin{bmatrix} 0_{(\mathtt{n}-1),(\mathtt{n}-1)} & 0_{(\mathtt{n}-1),1} \\ 0_{1,(\mathtt{n}-1)} & (A^{\mathtt{n_f}-1}b)^T QA^{\mathtt{n_f}-1}b \end{bmatrix}. \tag{2.68}$$

Since $Q \geqslant 0$, we have $(A^{\mathtt{n_f}-1}b)^T QA^{\mathtt{n_f}-1}b \geqslant 0$. Therefore, $X_{1\Lambda}^T\mathscr{L}(K_{\mathtt{max}})X_{1\Lambda} \geqslant 0$. Since $X_{1\Lambda}$ is invertible and $\mathscr{L}(K_{\mathtt{max}})$ is symmetric, by Sylvester's law of inertia[2] $\mathscr{L}(K_{\mathtt{max}}) \geqslant 0$. Next using Statement *(2)* of this theorem and the fact that $V_{2\Lambda}^T b = 0$ from equation (2.39) we have

$$K_{\mathtt{max}}b = X_{2\Lambda}X_{1\Lambda}^{-1}b = (X_{1\Lambda}^{-1})^T X_{2\Lambda}^T b = (X_{1\Lambda}^{-1})^T \begin{bmatrix} V_{2\Lambda}^T \\ 0 \end{bmatrix} b = 0. \tag{2.69}$$

Thus, $K_{\mathtt{max}}$ is a solution of the LQR LMI (2.9). From equation (2.68), we therefore infer that rank of $\mathscr{L}(K_{\mathtt{max}})$ is either 0 or 1.

Note that $\mathtt{rank}(\mathscr{L}(K_{\mathtt{max}})) = 0$ is equivalent to $\mathscr{L}(K_{\mathtt{max}}) = 0$, i.e., $A^T K_{\mathtt{max}} + K_{\mathtt{max}}A + Q = 0$ and $K_{\mathtt{max}}b = 0$. The equations $A^T K + KA + Q = 0$ and $Kb = 0$ are the constrained generalized continuous ARE (CGCARE) corresponding to the LQR Problem 2.21 (see [FN14], [FN18] for

[2] The inertia of a matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is as the set $\{\mathtt{n}_+, \mathtt{n}_0, \mathtt{n}_-\}$, where $\mathtt{n}_+$ and $\mathtt{n}_-$ are the number of eigenvalues of $A$ with positive and negative real parts, respectively (counted with multiplicity) and $\mathtt{n}_0$ is the number of eigenvalues of $A$ on the imaginary axis (counted with multiplicity).
*Sylvester's law of inertia:* Consider two symmetric matrices $A, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then, there exists a nonsingular matrix $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $A = P^T BP$ if and only if the inertia of $A$ and $B$ are the same. [Ber08, Corollary 5.4.7]

more on CGCARE). Interestingly, in Chapter 3 we show that a necessary condition for solvability of CGCARE is $\mathtt{det}(sE-H)=0$. Since in this theorem $\mathtt{det}(sE-H)\neq 0$ by assumption, CGCARE is not solvable here. This implies that $\mathscr{L}(K)=0$, i.e., $\mathtt{rank}(\mathscr{L}(K))=0$ is not possible in our case. Therefore, the minimum rank that can be attained by LQR LMI (2.9) is 1 and $\mathscr{L}(K_{\mathtt{max}})$ attains this rank. ■

*Proof of Statement (4) and (5) of Theorem 2.30:* Note that proving Statement *(4)* of this theorem is equivalent to proving that $K-K_{\mathtt{max}}\leqslant 0$ for all $K$ that satisfies the LQR LMI (2.9). We prove this in two steps. First, we prove that for $V_{1\Lambda}$ defined in the theorem, $\Delta := V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda}$ satisfies a suitable Lyapunov inequality (see equation (2.76) below). Then, using this Lyapunov inequality we finally show that $K-K_{\mathtt{max}}\leqslant 0$ for all $K$ that satisfies the LQR LMI (2.9).

*Step 1:* Note that for all $(x,u)$ that satisfies $\frac{d}{dt}x=Ax+bu$, evaluation of $\frac{d}{dt}(x^TKx)+x^TQx$ results in the following equation:

$$\begin{aligned}\frac{d}{dt}(x^TKx)+x^TQx &= \dot{x}^TKx+x^TK\dot{x}+x^TQx\\ &= (Ax+bu)^TKx+x^TK(Ax+bu)+x^TQx\\ &= \begin{bmatrix}x\\u\end{bmatrix}^T\begin{bmatrix}A^TK+KA+Q & Kb\\ b^TK & 0\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}, \text{ for all } t\in\mathbb{R}. \end{aligned}\tag{2.70}$$

Since $K$ is a solution of the LQR LMI (2.9), using the fact that $\begin{bmatrix}A^TK+KA+Q & Kb\\ b^TK & 0\end{bmatrix}\geqslant 0$ in equation (2.70), we have

$$\frac{d}{dt}(x^TKx)+x^TQx=\begin{bmatrix}x\\u\end{bmatrix}^T\begin{bmatrix}A^TK+KA+Q & Kb\\ b^TK & 0\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}\geqslant 0, \text{ for all } t\in\mathbb{R}. \tag{2.71}$$

From equation (2.39), we know that $AV_{1\Lambda}+bV_{3\Lambda}=V_{1\Lambda}\Gamma$. Further, since $V_{1\Lambda}$ is full column-rank (Theorem 2.32), we infer that there exists $F\in\mathbb{R}^{1\times \mathrm{n_s}}$ such that $FV_{1\Lambda}=V_{3\Lambda}$. Therefore, we have $(A+bF)V_{1\Lambda}=V_{1\Lambda}\Gamma$. Thus, corresponding to an initial condition $x_0=V_{1\Lambda}\beta$, where $\beta\in\mathbb{R}^{\mathrm{n_s}}$, $\bar{x}_{\mathrm{s}}:=V_{1\Lambda}e^{\Gamma t}\beta$, $\bar{u}_{\mathrm{s}}:=FV_{1\Lambda}e^{\Gamma t}\beta$ must satisfy $\frac{d}{dt}x=Ax+bu$. Using $\bar{x}_{\mathrm{s}}$ in equation (2.71), we have

$$\frac{d}{dt}(\bar{x}_{\mathrm{s}}^TK\bar{x}_{\mathrm{s}})+\bar{x}_{\mathrm{s}}^TQ\bar{x}_{\mathrm{s}}\geqslant 0 \Rightarrow \frac{d}{dt}(\bar{x}_{\mathrm{s}}^TK\bar{x}_{\mathrm{s}})\geqslant -\bar{x}_{\mathrm{s}}^TQ\bar{x}_{\mathrm{s}}, \text{ for all } t\in\mathbb{R}. \tag{2.72}$$

Note that $\dot{\bar{x}}_{\mathrm{s}}=V_{1\Lambda}\Gamma e^{\Gamma t}\beta=(A+bF)V_{1\Lambda}e^{\Gamma t}\beta$ (Since $(A+bF)V_{1\Lambda}=V_{1\Lambda}\Gamma$). Since $K_{\mathtt{max}}$ is a solution of the LQR LMI (2.9), using the fact that $K_{\mathtt{max}}b=0$ we have

$$\begin{aligned}&\frac{d}{dt}(\bar{x}_{\mathrm{s}}^TK_{\mathtt{max}}\bar{x}_{\mathrm{s}})+\bar{x}_{\mathrm{s}}^TQ\bar{x}_{\mathrm{s}}=\dot{\bar{x}}_{\mathrm{s}}^TK_{\mathtt{max}}\bar{x}_{\mathrm{s}}+\bar{x}_{\mathrm{s}}^TK_{\mathtt{max}}\dot{\bar{x}}_{\mathrm{s}}+\bar{x}_{\mathrm{s}}^TQ\bar{x}_{\mathrm{s}}\\ &=\beta^Te^{\Gamma^Tt}V_{1\Lambda}^T(A+bF)^TK_{\mathtt{max}}V_{1\Lambda}e^{\Gamma t}\beta+\beta^Te^{\Gamma^Tt}V_{1\Lambda}^TK_{\mathtt{max}}(A+bF)V_{1\Lambda}e^{\Gamma^Tt}\beta+\bar{x}_{\mathrm{s}}^TQ\bar{x}_{\mathrm{s}}\\ &=\beta^Te^{\Gamma^Tt}V_{1\Lambda}^T(A^TK_{\mathtt{max}}+K_{\mathtt{max}}A+Q)V_{1\Lambda}e^{\Gamma t}\beta, \text{ for all } t\in\mathbb{R}.\end{aligned}\tag{2.73}$$

From equation (2.64), it is evident that the right hand side of equation (2.73) is 0. Therefore,

$$\frac{d}{dt}\left(\bar{x}_{\mathrm{s}}^TK_{\mathtt{max}}\bar{x}_{\mathrm{s}}\right)=-\bar{x}_{\mathrm{s}}^TQ\bar{x}_{\mathrm{s}}, \text{ for all } t\in\mathbb{R}. \tag{2.74}$$

Subtracting equation (2.74) from inequality (2.72) gives $\frac{d}{dt}\left(\bar{x}_{\mathtt{s}}^T(K-K_{\mathtt{max}})\bar{x}_{\mathtt{s}}\right) = \dot{\bar{x}}_{\mathtt{s}}{}^T(K-K_{\mathtt{max}})\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{s}}^T(K-K_{\mathtt{max}})\dot{\bar{x}}_{\mathtt{s}} \geqslant 0,$ for all $t \in \mathbb{R}$. On using $\bar{x}_{\mathtt{s}} = V_{1\Lambda}e^{\Gamma t}\beta$ and $\dot{\bar{x}}_{\mathtt{s}} = V_{1\Lambda}\Gamma e^{\Gamma t}\beta$ in this inequality, we get

$$(V_{1\Lambda}e^{\Gamma t}\Gamma\beta)^T(K-K_{\mathtt{max}})(V_{1\Lambda}e^{\Gamma t}\beta) + (V_{1\Lambda}e^{\Gamma t}\beta)^T(K-K_{\mathtt{max}})(V_{1\Lambda}e^{\Gamma t}\Gamma\beta) \geqslant 0, \text{ for all } t \in \mathbb{R} \tag{2.75}$$

Since inequality (2.75) is true for all $t$, evaluating it at $t=0$, in particular, we get $\beta^T(\Gamma^T V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda} + V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda}\Gamma)\beta = \beta^T(\Gamma^T\Delta_\Lambda + \Delta_\Lambda\Gamma)\beta \geqslant 0$, where $\Delta_\Lambda := V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda}$. Since this inequality is true for all $\beta \in \mathbb{R}^{\mathtt{n_s}}$, we have

$$\Gamma^T\Delta_\Lambda + \Delta_\Lambda\Gamma \geqslant 0, \text{ where } \Delta_\Lambda = V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda}. \tag{2.76}$$

This ends the first step of the proof.

*Step 2:* Note that since $X_{1\Lambda}$ is nonsingular and $K-K_{\mathtt{max}}$ is symmetric, proving $K-K_{\mathtt{max}} \leqslant 0$ is equivalent to proving that $X_{1\Lambda}^T(K-K_{\mathtt{max}})X_{1\Lambda} \leqslant 0$ (by Sylvester's law of inertia). Hence, we prove $X_{1\Lambda}^T(K-K_{\mathtt{max}})X_{1\Lambda} \leqslant 0$ in the sequel.

Note that $X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix}$, where $W_1$ is as defined in equation (2.58). On evaluating $X_{1\Lambda}^T(K_{\mathtt{max}}-K)X_{1\Lambda}$, we therefore have

$$X_{1\Lambda}^T(K-K_{\mathtt{max}})X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda} & V_{1\Lambda}^T(K-K_{\mathtt{max}})W_1 \\ W_1^T(K-K_{\mathtt{max}})V_{1\Lambda} & W_1^T(K-K_{\mathtt{max}})W_1 \end{bmatrix}. \tag{2.77}$$

Since $W_1 = \begin{bmatrix} b & Ab & \cdots A^{\mathtt{n_f}-1}b \end{bmatrix}$ (equation (2.58)), we have from Lemma 2.37, $KW_1 = 0$ and $K_{\mathtt{max}}W_1 = 0$. Therefore, $(K-K_{\mathtt{max}})W_1 = 0$. Thus, from equation (2.77) it follows that

$$X_{1\Lambda}^T(K-K_{\mathtt{max}})X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Delta_\Lambda & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.78}$$

Since $\sigma(\Gamma) \subsetneq \mathbb{C}_-$, from equation (2.76), we have $\Delta_\Lambda \leqslant 0$. Using this negative-semidefiniteness property of $\Delta_\Lambda$ in equation (2.78), we infer $X_{1\Lambda}^T(K-K_{\mathtt{max}})X_{1\Lambda} \leqslant 0 \Leftrightarrow K-K_{\mathtt{max}} \leqslant 0$. This completes the proof of Statement *(4)* of the theorem.

Note that 0 is a solution of the LQR LMI (2.9). Thus, from Statement (4) of Theorem (2.30) we must have $0 \leqslant K_{\mathtt{max}}$. Thus, Statement *(5)* of the theorem is proved. ∎

Interestingly, from Step 1 of the Proof of Statement *(4)* of Theorem 2.30 above, we can infer that the difference between the maximal rank-minimizing solution $K_{\mathtt{max}}$ of the LQR LMI (2.9) with any other solution $K$ of the LQR LMI when restricted to the space $\mathtt{img}\, V_{1\Lambda}$ satisfies a Lyapunov inequality of the form given in equation (2.76). We present this as a lemma next. For the ease of exposition we call the difference $K-K_{\mathtt{max}}$ the *maximal gap* of $K$ (see [Wil71] for more on the use of the term gap).

Maximal gap of $K$ restricted to $\mathtt{img}\, V_{1\Lambda}$ satisfy a Lyapunov inequality

**Lemma 2.38.** *Consider the singular LQR Problem 2.21. Let $V_{1\Lambda}, K_{\mathtt{max}}$, and $\Gamma$ be as defined in Theorem 2.30. Assume $K$ to be any solution of the LQR LMI* (2.9). *Define $\Delta_\Lambda := V_{1\Lambda}^T(K - K_{\mathtt{max}})V_{1\Lambda} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$. Then, $\Delta_\Lambda$ satisfies the following Lyapunov inequality:*

$$\Gamma^T\Delta_\Lambda + \Delta_\Lambda\Gamma \geqslant 0.$$

*Proof:* The proof follows from Step 2 of the proof of Statement *(4)* of Theorem 2.30. ∎

In order to demonstrate that Theorem 2.30 finds the maximal rank-minimizing solution of the LQR LMI (2.3), we revisit Example 2.20 that we have previously failed to solve using Proposition 2.19.

**Example 2.39.** *Note that in Example 2.20, we have* $\mathtt{n} = 3$ *and* $\mathtt{n_s} = 1$. *Thus,* $\mathtt{n_f} = \mathtt{n} - \mathtt{n_s} = 2$. *Therefore, using Theorem 2.30, we have*

$$\begin{bmatrix} V_\Lambda & W \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} & b & Ab \\ V_{2\Lambda} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} X_{1\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \\ X_{2\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{cases}$$

*Therefore,* $K_{\mathtt{max}} = X_{2\Lambda}X_{1\Lambda}^{-1} = \begin{bmatrix} 2 & & \\ & 0 & \\ & & 0 \end{bmatrix}$. *It can be verified that LQR LMI* (2.9) *evaluated at* $K_{\mathtt{max}}$ *gives* $\mathscr{L}(K_{\mathtt{max}}) := \begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geqslant 0$. *Further,* $\mathtt{rank}(K_{\mathtt{max}}) = 1$. *This is the minimum rank achievable by the LQR LMI* (2.9) *(see proof of Statement (3) of Theorem 2.30 for the justification of the LQR LMI's minimum rank being* 1 *in this case). Further,* $K_{\mathtt{max}}$ *is also the maximal solution of the LQR LMI* (2.9) *(see proof of Statement (4) of Theorem 2.30 in Section 2.4.3 for a justification of this claim). Thus, from the example it is clear that Theorem 2.30 indeed provides a method to compute the maximal rank-minimizing solution of an LQR LMI corresponding to a singular LQR problem.*

Recall from Statement *(1)* of Theorem 2.30 and equation (2.58) that $X_{1\Lambda}$ can be written as

$$X_\Lambda = \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} & W_1 \\ V_{2\Lambda} & 0_{\mathtt{n},\mathtt{n_f}} \end{bmatrix}.$$

Thus, we have $X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix}$ and $X_{2\Lambda} = \begin{bmatrix} V_{2\Lambda} & 0_{\mathtt{n},\mathtt{n_f}} \end{bmatrix}$. Further, from Lemma 2.35, we know that $\mathtt{img}\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}} \end{bmatrix} = \mathtt{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}$ and $\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \end{bmatrix}$ is full column-rank. Hence, the matrix $X_{1\Lambda}$, without loss of generality, is given by $X_{1\Lambda} = \begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W_1 \end{bmatrix}$ and the corresponding $X_{2\Lambda}$ matrix is then $X_{2\Lambda} = \begin{bmatrix} 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}} & 0_{\mathtt{n},\mathtt{n_f}} \end{bmatrix}$. Since $X_{1\Lambda}$ is invertible (Statement *(1)* of Theorem 2.30), it is evident that the columns of $X_{1\Lambda}$ can be assumed to be a basis for $\mathbb{R}^{\mathtt{n}}$. Hence, the state-space of the system $\Sigma$ can be decomposed in the following subspaces:

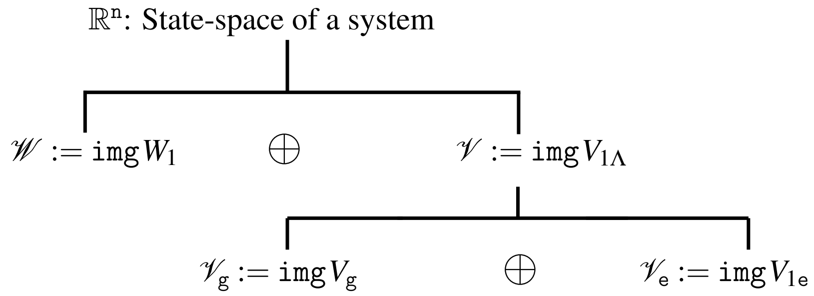



Figure 2.3: A direct-sum decomposition of the state-space $\mathbb{R}^{\mathtt{n}}$

Note that the subspace $\mathscr{V}_{\mathtt{g}}$ in Figure 2.3 is the g-dimensional good slow subspace of the system $\Sigma$. Further, the dimension of $\mathscr{W}$ is $\mathtt{n_f}$ and that of $\mathscr{V}_{\mathtt{e}}$ is $\mathtt{n}-(\mathtt{g}+\mathtt{n_f})$. From Figure 2.3, it is evident that any initial condition $x_0 \in \mathbb{R}^{\mathtt{n}}$ of the system $\Sigma$ can be decomposed as

$$x_0 =: x_{\mathtt{gs}} + x_{0\mathtt{f}} + x_{\mathtt{es}}, \text{ where } x_{\mathtt{gs}} \in \mathscr{V}_{\mathtt{g}}, x_{0\mathtt{f}} \in =: \mathscr{W}, \text{ and } x_{\mathtt{es}} \in \mathscr{V}_{\mathtt{e}}. \tag{2.79}$$

This leads to some interesting facts about the optimal cost of a singular LQR problem. The first among them is as follows:

Optimal cost of an LQR problem

**Corollary 2.40.** *Consider the LQR Problem 2.21 and let $K_{\mathtt{max}}$ be the maximal rank-minimizing solution of the corresponding LQR LMI* (2.9). *Assume $x_0 =: x_{\mathtt{gs}} + x_{0\mathtt{f}} + x_{\mathtt{es}}$ to be an initial condition of the system $\Sigma$ as defined in equation* (2.79). *Then the following statements hold:*

*(1)* $x_{\mathtt{gs}}^T K_{\mathtt{max}} x_{\mathtt{gs}} = 0$.

*(2)* $x_{0\mathtt{f}}^T K_{\mathtt{max}} x_{0\mathtt{f}} = 0$.

*(3) The optimal cost of the LQR problem is $x_{\mathtt{es}}^T K_{\mathtt{max}} x_{\mathtt{es}}$.*

*Proof: (1):* Let $x_{\mathtt{gs}} := V_{\mathtt{g}}\alpha$, where $\alpha \in \mathbb{R}^{\mathtt{g}}$. Note that

$$K_{\mathtt{max}} x_{\mathtt{gs}} = K_{\mathtt{max}} V_{\mathtt{g}}\alpha = X_{2\Lambda} X_{1\Lambda}^{-1} V_{\mathtt{g}}\alpha = \begin{bmatrix} 0_{\mathtt{n,g}} & V_{2\mathtt{e}} & 0_{\mathtt{n,f}} \end{bmatrix} \begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W_1 \end{bmatrix}^{-1} V_{\mathtt{g}}\alpha = 0. \tag{2.80}$$

Therefore, $x_{\mathtt{gs}}^T K_{\mathtt{max}} x_{\mathtt{gs}} = \alpha^T V_{\mathtt{g}}^T K_{\mathtt{max}} V_{\mathtt{g}}\alpha = 0$.
*(2):* Let $x_{0\mathtt{f}} := W_1\beta$, where $\beta \in \mathbb{R}^{\mathtt{n_f}}$. Note that

$$K_{\mathtt{max}} x_{0\mathtt{f}} = K_{\mathtt{max}} W_1\beta = \begin{bmatrix} 0_{\mathtt{n,g}} & V_{2\mathtt{e}} & 0_{\mathtt{n,f}} \end{bmatrix} \begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W_1 \end{bmatrix}^{-1} W_1\beta = 0. \tag{2.81}$$

Therefore, $x_{0\mathtt{f}}^T K_{\mathtt{max}} x_{0\mathtt{f}} = \beta^T W_1^T K_{\mathtt{max}} W_1\beta = 0$.
*(3):* From [Sch83], it is known that the optimal cost corresponding to the LQR Problem 2.21 is given by $x_0^T K_{\mathtt{max}} x_0$, where $K_{\mathtt{max}}$ is the maximal rank-minimizing solution of the LQR LMI (2.9). Hence, using equations (2.80) and (2.81) and evaluating the optimal cost for the LQR Problem 2.21, we have

$$x_0^T K_{\mathtt{max}} x_0 = (x_{\mathtt{gs}} + x_{\mathtt{es}} + x_{0\mathtt{f}})^T K_{\mathtt{max}} (x_{\mathtt{gs}} + x_{\mathtt{es}} + x_{0\mathtt{f}}) = x_{\mathtt{es}}^T K_{\mathtt{max}} x_{\mathtt{es}}. \tag{2.82}$$

This completes the proof of the corollary. ∎

From Corollary 2.40 it is evident that if the initial condition of the system is from $\mathscr{W}$ or $\mathscr{V}_{\mathtt{g}}$ then the cost incurred by the system is zero. This corroborates the findings in [WKS86]. Thus, the optimal cost of an LQR problem depends only on the maximal rank-minimizing solution of the corresponding LQR LMI and the projection of the initial condition of the system onto the subspace $\mathscr{V}_{\mathtt{e}}$.

Next we look at a special case of LQR problems when the system admits the zero matrix as the only solution to the corresponding LQR LMI.

A sufficiency condition for $K_{\mathtt{max}} = 0$

**Corollary 2.41.** *Consider the singular LQR Problem 2.21 with assumptions as given in Theorem 2.30. Consider* $\mathtt{dim}(\mathtt{sup}\mathscr{B}_\Sigma) = \mathtt{n_s}$, *where* $\mathscr{B}_\Sigma$ *is as defined in Lemma 2.34. Then,* $K_{\mathtt{max}} = 0_{\mathtt{n},\mathtt{n}}$.

*Proof:* Since $\mathtt{dim}(\mathtt{sup}\mathscr{B}_\Sigma) = \mathtt{n_s}$ and $\mathtt{dim}(\mathscr{O}_{wg}) = \mathtt{n_s}$, from Lemma 2.35 it is evident that $\mathtt{img}\begin{bmatrix} V_{\mathtt{g}} \\ 0_{\mathtt{n},\mathtt{n_s}} \end{bmatrix} = \mathtt{img}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}$. Therefore, $V_{2\Lambda} = 0_{\mathtt{n},\mathtt{n_s}}$. Further, from equation (2.58) we have $W_2 = 0$. Therefore, $X_{2\Lambda} = 0$ and hence using Theorem 2.30, we must have $K_{\mathtt{max}} = 0_{\mathtt{n},\mathtt{n}}$. ∎

The next corollary states that if the transfer function induced by the cost-matrix $Q$ and the system $\Sigma$ is minimum-phase, then the optimal cost of the corresponding LQR problem is zero.

Optimal cost of LQR problems: minimum-phase case

**Corollary 2.42.** *Consider the singular LQR Problem 2.21 with* $\mathtt{rank}\, Q = 1$ *and* $(Q,A)$ *observable. Let* $c \in \mathbb{R}^{1\times \mathtt{n}}$ *be such that* $Q = c^T c$. *Define* $G(s) := c(sI_{\mathtt{n}} - A)^{-1}b$. *If the system* $G(s)$ *is minimum-phase, then the optimal cost of the LQR problem is zero.*

*Proof:* Recall $\widehat{A}, \widehat{b}, \widehat{c}$ are as defined in equation (2.12). Define $\mathtt{det}(sI_{\mathtt{n}} - A) =: d(s)$. Therefore, $\mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) = d(s)d(-s)$. Since the system is $(A,b)$ controllable and $(Q,A)$ observable, there exists a real-polynomial $n(s)$ such that $G(s) = \frac{n(s)}{d(s)}$ with $n(s)$ and $d(s)$ are coprime.

Note that $\mathtt{det}(sE - H) = \mathtt{det}\begin{bmatrix} sI_{2\mathtt{n}} - \widehat{A} & -\widehat{b} \\ -\widehat{c} & 0 \end{bmatrix} = \widehat{c}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{b} \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) =: p(s)$. Further, by simple multiplication it can be seen that

$$G(-s)G(s) = \widehat{c}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{b} \Rightarrow \frac{n(-s)n(s)}{d(-s)d(s)} = \frac{p(s)}{d(-s)d(s)},$$

Therefore, $p(s) = n(-s)n(s)$. Since $|\sigma(E,H)| = 2\mathtt{n_s} \Rightarrow |\mathtt{roots}\, p(s)| = 2\mathtt{n_s} \Rightarrow |\mathtt{roots}\,(n(s))| = \mathtt{n_s}$. Since $G(s)$ is minimum-phase, $\mathtt{roots}(n(s)) \subsetneq \mathbb{C}_-$.

Consider the system $\frac{d}{dt}x = Ax + bu$ and $y := cx$. Note that this is a SISO system which is $(A,b)$ controllable and $(Q,A)$ observable $\Rightarrow$ $(c,A)$ observable. Therefore, as discussed in Section 2.2.4, $\sigma((A+bF)|_{\mathtt{sup}\mathscr{B}_\Sigma}) = \mathtt{rootnum}(G(s))$. Therefore, $\mathtt{dim}(\mathtt{sup}\mathscr{B}_\Sigma) = \mathtt{n_s}$. Hence, by Corollary 2.41 we have $K_{\mathtt{max}} = 0_{\mathtt{n},\mathtt{n}} \Rightarrow$ the optimal cost is zero. ∎

Note that Corollary 2.42, albeit for single-input systems, corroborates the findings on minimum-phase systems in [Fra79, Theorem 2] and [KS72].

## 2.5 Summary

In this chapter, we presented a method to compute the maximal rank-minimizing solution of an LQR LMI corresponding to a single-input system (Theorem 2.30). We developed this method using the notion of fast subspaces (strongly reachable subspace) and slow subspaces (weakly unobservable subspace) of Hamiltonian systems. We showed that augmenting the basis of the good slow subspace of the Hamiltonian system $\Sigma_{\mathrm{Ham}}$ with the basis of a subspace of the fast subspace of $\Sigma_{\mathrm{Ham}}$ is the crucial idea that leads to the method. While developing this method, we also showed that the fast subspace and the slow subspace of a SISO system can be characterized in terms of its Rosenbrock system matrix (Theorem 2.24 and Theorem 2.25). Further, we also showed that the good slow subspace of the Hamiltonian system is disconjugate (Theorem 2.32).

The relation between slow and fast subspaces with singular optimal control was already known in the literature. In this chapter, in order to get the maximal rank-minimizing solution of the LQR LMI we linked these well-known notions to the corresponding Hamiltonian system. Application of the notion of slow and fast subspaces to the Hamiltonian system not only leads to a method to compute the maximal rank-minimizing solution of the LQR LMI, but also leads to results that corroborate some of the findings in the literature (Corollary 2.42). Hence, the primary contribution of this chapter is the idea that, unlike the approach in [HS83], [Wil81], [WKS86] where the notion of slow and fast subspaces were applied to the system, the application of these notions of slow and fast subspaces to the Hamiltonian system brings out further insight into the singular optimal problem. These ideas also lead to design of state-feedback controllers to solve a singular LQR problem. We develop the theory behind the design of such controllers in the next chapter.

# Chapter 3

# Almost every single-input LQR problem admits a PD-feedback solution

## 3.1 Introduction

A regular LQR problem, as motivated in Chapter 2, can be solved using controllers designed using maximal solution of the corresponding ARE. From a system-trajectory viewpoint, such a feedback law $u(t) = Fx(t)$ confines the set of trajectories of the system to the optimal ones. However, it is known that for the singular LQR case such a confinement, using the feedback law $u(t) = Fx(t)$, might not be always possible [HS83]. As seen in Chapter 2, one of the reasons is that $R$ is non-invertible for the singular LQR case. Moreover, the ARE itself does not exist either. However, apart from these arguments, there is a system-theoretic explanation for the non-existence of such a static state-feedback in the singular LQR case. Note that for regular LQR problems, it is known that, for any arbitrary initial condition $x_0$, the optimal trajectories of the system are of the form $x(t) = e^{(A+BF)t}x_0$ and $u(t) = Fx(t)$ (see [Kir04, Chapter 5]). These optimal state-trajectories are clearly *restrictions* to $\mathbb{R}_+ := [0,\infty)$ of functions from the space of infinitely differentiable functions $\mathfrak{C}^\infty$ (from $\mathbb{R}$ to $\mathbb{R}$). However, for singular LQR problems it is known that the optimal inputs, and hence the optimal state-trajectories, of the system are from the space of impulsive-smooth distributions (see Definition 2.12, [HS83], [WKS86]). This can easily be verified with the help of a simple example [HS83, Example 2.11].

**Example 3.1.** *Consider the system $\frac{d}{dt}x = u$ and $x_0 = 1$. Let the performance index be $J = \int_0^\infty x^2(t)dt$. Note that $J$ can be made arbitrarily small by a suitable choice of $u$, e.g., let*

$$u(t) = \begin{cases} -\dfrac{1}{\varepsilon} & \textit{for} \quad 0 \leqslant t \leqslant \varepsilon \\ 0 & \textit{for} \quad t > \varepsilon. \end{cases} \tag{3.1}$$

*On using this input, the state can be computed to be*

$$x(t) = \begin{cases} 1 - \dfrac{t}{\varepsilon} & \text{for} \quad 0 \leqslant t \leqslant \varepsilon \\ 0 & \text{for} \quad t > \varepsilon. \end{cases}$$

*The perfomance index therefore becomes* $J = \int_0^{\varepsilon} \left(1 - \frac{t}{\varepsilon}\right)^2 dt = \frac{\varepsilon}{3}$. *Thus, using the input defined in equation* (3.1)*,* $J$ *can be made arbitrarily small. However, no piecewise continuous or measurable* $u(t)$ *can make the performance index zero.*
*However, if we use* $u(t) = -\delta(t)$*, then* $x(t) = 0$ *for* $t > 0$*. Thus, we have* $J = 0$*.*

Example 3.1 shows that for singular LQR problems a solution may not exist if inputs are from the space of infinitely differentiable functions. The inputs need to be from from the space of impulsive-smooth distributions. A static state-feedback is incapable of producing such impulsive-smooth distributions, and hence, incapable of solving the LQR problem for the singular case. This can also be verified with the help of Example 3.1.

**Example 3.2.** *Let* $u(t) = -kx(t)$ *for* $k \in \mathbb{R}$*. On application of this static state-feedback law on the system in Example 3.1, the state becomes* $x(t) = e^{-kt}$*. Evaluation of the performance index gives* $J = \int_0^{\infty} e^{-2kt} dt = \frac{1}{2k} \neq 0$*. Thus, no static feedback can make the performance index zero.*

Interestingly, there are certain singular LQR problems that can be solved using static state-feedback control law. The authors in [FN14, FN16, NF19] established that a singular LQR problem is solvable using a static state-feedback control law if and only if such a problem admits solutions to the *constrained generalized continuous ARE (CGCARE)*. This is because in such a case the optimal trajectories continue to be trajectories in $\mathfrak{C}^{\infty}$. However, in Chapter 4 we show that for almost all singular LQR problems, CGCARE is not solvable and hence such a static state-feedback solution generically cannot solve a singular LQR problem.

In this chapter, we show that for (almost) every singular LQR problem, with the underlying state-space system having a single-input, the impulsive-smooth optimal state-trajectories can be obtained via a state-feedback that is a static linear function of not just the state but also its first derivative. For obvious reasons we call this feedback a *proportional plus derivative (PD) state-feedback*. Evidently, presence of the derivative feedback forces the closed-loop system to be a singular descriptor system. We show that a suitable PD feedback always exists such that the impulsive-smooth state-trajectories of the closed-loop singular descriptor system are precisely the impulsive-smooth optimal state-trajectories of the singular LQR problem. We present this as Theorem 3.12, in Section 3.4. This is the main result of this chapter.

Two important results, Theorem 2.30 and Theorem 3.9, play crucial roles in the derivation of Theorem 3.12. Both these results are based on properties of the Hamiltonian system corresponding to the LQR problem. It is well-known that the Hamiltonian system, given by the equation (2.11), arises on application of Pontryagin's maximum principle (PMP) to the LQR problem [IOW99]. It follows from PMP that, for the regular case, the optimal solutions of the

LQR problem are nothing but suitably chosen trajectories of the Hamiltonian system [Kir04, Section 5.2]. For the singular case, however, this Hamiltonian system becomes a singular descriptor system, and PMP becomes applicable to only the smooth trajectories of this system. We show in Theorem 3.9 that, not only the smooth trajectories of the Hamiltonian system, even the impulsive-smooth ones, when suitably chosen, are optimal. We prove this using two results, first a result in [Sch83] that shows that the optimal value of the cost functional for any LQR problem is induced by the maximal solution, among all the rank-minimizing solutions, of the LQR LMI (2.3). Recall from Chapter 2 that we call such a solution the maximal rank-minimizing solution of the LQR LMI. The second result is Theorem 2.30 that provides a method to compute the maximal rank-minimizing solution of an LQR LMI in terms of the Hamiltonian pencil. The only assumption that we make here is that the Hamiltonian matrix pair does not have any finite eigenvalue on the imaginary axis. We show that this assumption can be guaranteed by ensuring that the matrix of rational functions $C(sI_{\mathtt{n}}-A)^{-1}B$, where $Q =: C^TC$ has no finite zero on the imaginary axis (Lemma 3.17). This is true for almost all $A,B,C$ matrices; the word "almost" in the title is added hence.

## 3.2 Preliminaries

### 3.2.1 Half-line solution of a state-space equation

Recall that in line with the definition in [HS83], [WKS86], we defined the space of impulsive-smooth distributions in Definition 2.12. Since we deal with distributions from $\mathfrak{C}^{\bullet}_{\mathtt{imp}}$, it is essential to define what is meant by the solution of a system $\Sigma$ with state-space equation $\frac{d}{dt}x = Ax+Bu$ and initial condition $x_0$. In this chapter, we use the term solution in the distributional sense as introduced in [HS83] [HSW00].

**Definition 3.3.** [HSW00, Equation 3.7] *Consider a system $\Sigma$ with a state-space dynamics $\frac{d}{dt}x = Ax+Bu$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, $\mathtt{col}(x,u) \in \mathfrak{C}^{\mathtt{n+m}}_{\mathtt{imp}}$ is called a trajectory in $\Sigma$, corresponding to an initial condition $x_0$, if $\mathtt{col}(x,u)$ satisfies the following equation: $\frac{d}{dt}x = Ax+Bu+x_0\delta$.*

For a detailed justification of Definition 3.3 refer to the discussion in [HS83, Section 3].

### 3.2.2 Admissible inputs

In the LQR Problem 2.1 it is of paramount importance that the inputs $u$ of the system $\Sigma$ be such that the integral in equation (2.8) is well-defined. In this chapter we follow the same line of reasoning as in [HS83] to ensure that equation (2.8) is well-defined. Note that since $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$, there exists a full row-rank matrix $\begin{bmatrix} C & D \end{bmatrix} \in \mathbb{R}^{\mathtt{r}\times(\mathtt{n+m})}$ such that $\begin{bmatrix} C^T \\ D^T \end{bmatrix}[C\ D] = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, where $\mathtt{rank}\left(\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\right) = \mathtt{r}$. Define $y(t) := Cx(t)+Du(t)$. In the sequel, the trajectories $x(t)$ and $y(t)$

that result from an initial condition $x_0$ and an input $u(t)$ are denoted by $x(t;x_0,u)$ and $y(t;x_0,u)$, respectively.

As motivated in Section 3.1, for singular LQR problems the optimal inputs $u(t)$ are from the space $\mathfrak{C}^{\mathtt{m}}_{\mathtt{imp}}$. Further, the class of impulsive-smooth distributions are known to be closed under convolution, in particular under differentiation and integration [HS83, Proposition 3.2]. Therefore, corresponding to $u(t) \in \mathfrak{C}^{\mathtt{m}}_{\mathtt{imp}}$, we must have $x(t) \in \mathfrak{C}^{\mathtt{n}}_{\mathtt{imp}} \Rightarrow y(t) \in \mathfrak{C}^{\mathtt{r}}_{\mathtt{imp}}$. Thus, in terms of Definition 2.12, we have $y(t) = y_{\mathtt{reg}} + y_{\mathtt{imp}}$. Note that the functional (2.8) in terms of $y(t)$ takes the following form $J(x_0,u) = \int_0^\infty ||y(t)||^2 dt$. For this integral to be well-defined, we need $y(t)$ to be from the space $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{r}})|_{\mathbb{R}_+}$ and hence, we define *admissible inputs* as follows:

**Definition 3.4.** [HS83, Section 3] *We call an input $u(t) \in \mathfrak{C}^{\mathtt{m}}_{\mathtt{imp}}$* admissible *if $y(t;x_0,u) \in \mathfrak{C}^{\mathtt{r}}_{\mathtt{imp}}$ is such that $y_{\mathtt{imp}} = 0$. The space of admissible inputs is represented by $\mathscr{U}_\Sigma$.*

Using this notion of admissible inputs, we are now in a position to restate Problem 2.21 specifying explicitly the space from which the inputs $u$ need to belong.

**Problem 3.5. (Single-input singular LQR problem)** *Consider a controllable system $\Sigma$ with state-space dynamics $\frac{d}{dt}x = Ax + bu$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $b \in \mathbb{R}^{\mathtt{n}}$. Then, for every initial condition $x_0$, find an input $u \in \mathscr{U}_\Sigma$ that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \left(x^T Qx\right) dt, \text{ where } Q \geqslant 0. \tag{3.2}$$

## 3.3 Characterization of optimal trajectories

As stated in Section 3.1, our primary objective in this chapter is to design a state-feedback controller that solves a singular LQR problem. In this chapter we take the first step to attain this objective by characterizing the optimal trajectories of a system corresponding to a singular LQR problem.

Recall from equation (2.58) that on partitioning $W$ (defined in Theorem 2.30) as $W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$, where $W_1, W_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_f}}$, we must have

$$W_1 = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix}. \tag{3.3}$$

Now, define $\mathscr{V} := \mathtt{img}\, V_{1\Lambda}$ and $\mathscr{W} = \mathtt{img}\, W_1$. Recall from Figure 2.3 in Chapter 2 that the state-space of the system $\Sigma$ can be decomposed as $\mathbb{R}^{\mathtt{n}} = \mathscr{V} \oplus \mathscr{W}$. Thus, any initial condition $x_0$ of the system $\Sigma$ can be uniquely decomposed as $x_0 = x_{0\mathtt{s}} + x_{0\mathtt{f}}$, where $x_{0\mathtt{s}} \in \mathscr{V}$, and $x_{0\mathtt{f}} \in \mathscr{W}$. In the next two lemmas, we characterize those trajectories of $\Sigma$ that are *candidate* optimal trajectories corresponding to an initial condition first for $x_0 \in \mathscr{V}$ and then for $x_0 \in \mathscr{W}$. Later in Section 3.3.3 we show that these candidate optimal trajectories are indeed the optimal trajectories that we are looking for.

### 3.3.1 Characterization of the candidate optimal fast trajectories

In the next lemma, we characterize those trajectories of $\Sigma$ that are candidate optimal trajectories corresponding to initial condition $x_0 \in \mathscr{W}$.

Candidate fast optimal trajectories of the system $\Sigma$

**Lemma 3.6.** *Let $x_{0k}$ be as given in the table below and define $z_{0k} := 0$, where $k \in \{0,1,\dots,\mathtt{n_f}-1\}$. Define $\bar{x}_{\mathtt{f}k}, \bar{z}_{\mathtt{f}k} \in \mathfrak{C}^{\mathtt{n}}_{\mathtt{imp}}$ and $\bar{u}_{\mathtt{f}k} \in \mathfrak{C}_{\mathtt{imp}}$ as given in the table.*

| $k$ | $x_{0k}$ | $z_{0k}$ | $\bar{x}_{\mathtt{f}k}$ | $\bar{z}_{\mathtt{f}k}$ | $\bar{u}_{\mathtt{f}k}$ |
|---|---|---|---|---|---|
| 0 | $\alpha_0 b$ | 0 | 0 | 0 | $-\alpha_0\delta$ |
| 1 | $\alpha_1 Ab$ | 0 | $-\alpha_1 b\delta$ | 0 | $-\alpha_1\delta^{(1)}$ |
| 2 | $\alpha_2 A^2 b$ | 0 | $-\alpha_2\left(b\delta^{(1)} + Ab\delta\right)$ | 0 | $-\alpha_2\delta^{(2)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathtt{n_f}-1$ | $\alpha_{\mathtt{n_f}-1}A^{\mathtt{n_f}-1}b$ | 0 | $-\alpha_{\mathtt{n_f}-1}\sum_{i=0}^{\mathtt{n_f}-2}A^{\mathtt{n_f}-2-i}b\delta^{(i)}$ | 0 | $-\alpha_{\mathtt{n_f}-1}\delta^{(\mathtt{n_f}-1)}$ |

*Let $x_{0\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} x_{0k}$, $\bar{x}_{\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} \bar{x}_{\mathtt{f}k}$, $\bar{z}_{\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} \bar{z}_{\mathtt{f}k}$ and $\bar{u}_{\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} \bar{u}_{\mathtt{f}k}$. Then,*

*(1) $\mathtt{col}\,(\bar{x}_{\mathtt{f}}, \bar{z}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}) \in \Sigma_{\mathrm{Ham}}$ corresponding to initial condition $\mathtt{col}\,(x_{0\mathtt{f}}, 0_{\mathtt{n},1})$.*

*(2) $\mathtt{col}\,(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}) \in \Sigma$ corresponding to initial condition $x_{0\mathtt{f}}$.*

*Proof: (1):* Using Lemma 2.36 it is easy to verify that $\widehat{A}^k\widehat{b} = \mathtt{col}(A^k b, 0_{\mathtt{n},1})$ for $k \in \{0,1,\dots,\mathtt{n_f}-1\}$. Hence, $\mathtt{col}\,(x_{0k}, z_{0k}) \in \mathbb{R}^{2\mathtt{n}}$ from the table above can be rewritten as

$$\mathtt{col}\,(x_{0k}, z_{0k}) = \mathtt{col}\left(\alpha_k A^k b, 0_{\mathtt{n},1}\right) = \alpha_k \widehat{A}^k \widehat{b}, \text{ for } k \in \{1,2,\dots,\mathtt{n_f}-1\}.$$

Now corresponding to the initial condition $\alpha_k \widehat{A}^k\widehat{b}$ of the $\Sigma_{\mathrm{Ham}}$ we compute the trajectories of $\Sigma_{\mathrm{Ham}}$. Define $s(t)$ to be the unit step function, i.e.,

$$s(t) = \begin{cases} 1, & \text{for } t \geqslant 0 \\ 0, & \text{for } t < 0. \end{cases}$$

Corresponding to the initial condition $\mathtt{col}\,(x_{0k}, z_{0k})$ and input $u(t) = -\alpha_k\delta^{(k)}$, we therefore have

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = e^{\widehat{A}t}\widehat{A}^k\widehat{b}\alpha_k s(t) + \int_0^t e^{\widehat{A}(t-\tau)}\widehat{b}\left(-\alpha_k\delta^{(k)}(\tau)\right)d\tau = e^{\widehat{A}t}\widehat{A}^k\widehat{b}\alpha_k s(t) - \alpha_k \frac{d^k}{dt^k}\left(e^{\widehat{A}t}\widehat{b}s(t)\right) \tag{3.4}$$

We first prove that

$$\frac{d^k}{dt^k}\left(e^{\widehat{A}t}\widehat{b}s(t)\right) = e^{\widehat{A}t}\widehat{A}^k\widehat{b}s(t) + \sum_{i=0}^{k-1}\widehat{A}^{k-1-i}\widehat{b}\delta^{(i)}. \tag{3.5}$$

We use the principle of mathematical induction to prove it.
*Base case:* $(k=1)$: Expanding using the chain rule of differentiation, we have

$$\frac{d}{dt}\left(e^{\widehat{A}t}\widehat{b}s(t)\right) = e^{\widehat{A}t}\widehat{A}\widehat{b}s(t) + \widehat{b}\delta.$$

*Induction step:* We assume that

$$\frac{d^k}{dt^k}\left(e^{\widehat{A}t}\widehat{b}s(t)\right) = e^{\widehat{A}t}\widehat{A}^k\widehat{b}s(t) + \sum_{i=0}^{k-1}\widehat{A}^{k-1-i}\widehat{b}\delta^{(i)}.$$

We show that

$$\frac{d^{k+1}}{dt^{k+1}}\left(e^{\widehat{A}t}\widehat{b}s(t)\right) = e^{\widehat{A}t}\widehat{A}^{k+1}\widehat{b}s(t) + \sum_{i=0}^{(k+1)-1}\widehat{A}^{(k+1)-1-i}\widehat{b}\delta^{(i)}.$$

Now, using the chain rule of differentiation and applying the induction hypothesis, we have

$$\begin{aligned}\frac{d^{k+1}}{dt^{k+1}}\left(e^{\widehat{A}t}\widehat{b}s(t)\right) &= \frac{d^k}{dt^k}\left(e^{\widehat{A}t}\widehat{A}\widehat{b}s(t) + \widehat{b}\delta\right) = \frac{d^k}{dt^k}\left(e^{\widehat{A}t}\widehat{A}\widehat{b}s(t)\right) + \widehat{b}\delta^{(k)}\\ &= e^{\widehat{A}t}\widehat{A}^{k+1}\widehat{b}s(t) + \sum_{i=0}^{k-1}\widehat{A}^{k-i}\widehat{b}\delta^{(i)} + \widehat{b}\delta^{(k)} = e^{\widehat{A}t}\widehat{A}^{k+1}\widehat{b}s(t) + \sum_{i=0}^{(k+1)-1}\widehat{A}^{(k+1)-1-i}\widehat{b}\delta^{(i)}.\end{aligned}$$

This proves equation (3.5). Using equation (3.5) in equation (3.4), we have

$$\begin{aligned}\begin{bmatrix} x(t)\\ z(t)\end{bmatrix} &= e^{\widehat{A}t}\widehat{A}^k\widehat{b}\alpha_k s(t) - \alpha_k\frac{d^k}{dt^k}\left(e^{\widehat{A}t}\widehat{b}s(t)\right)\\ &= e^{\widehat{A}t}\widehat{A}^k\widehat{b}s(t)\alpha_k - e^{\widehat{A}t}\widehat{A}^k\widehat{b}s(t)\alpha_k - \sum_{i=0}^{k-1}\widehat{A}^{k-1-i}\widehat{b}\delta^{(i)} = -\sum_{i=0}^{k-1}\widehat{A}^{k-1-i}\widehat{b}\delta^{(i)}.\end{aligned} \tag{3.6}$$

Using Statement *(3)* of Lemma 2.36, we can rewrite equation (3.6) as:

$$\begin{bmatrix} x(t)\\ z(t)\end{bmatrix} = -\sum_{i=0}^{k-1}\widehat{A}^{k-1-i}\widehat{b}\delta^{(i)} = \sum_{i=0}^{k-1}\begin{bmatrix} A^{k-1-i}b\\ 0_{\mathtt{n},1}\end{bmatrix}\delta^{(i)} = \begin{bmatrix}\bar{x}_{\mathtt{f}k}\\ \bar{z}_{\mathtt{f}k}\end{bmatrix},\ \text{for } k\in\{1,2,\ldots,\mathtt{n_f}-1\}. \tag{3.7}$$

Thus, the trajectory $\mathtt{col}(\bar{x}_{\mathtt{f}k},\bar{z}_{\mathtt{f}k},\bar{u}_{\mathtt{f}k})$ satisfies the equation $\frac{d}{dt}\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right] = \widehat{A}\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right] + \widehat{b}u$. Next, we show that the trajectory $\mathtt{col}(\bar{x}_{\mathtt{f}k},\bar{z}_{\mathtt{f}k},\bar{u}_{\mathtt{f}k})$ satisfies $\widehat{c}\left[\begin{smallmatrix}\bar{x}_{\mathtt{f}k}\\ \bar{z}_{\mathtt{f}k}\end{smallmatrix}\right] = 0$.

$$\widehat{c}\begin{bmatrix}\bar{x}_{\mathtt{f}k}\\ \bar{z}_{\mathtt{f}k}\end{bmatrix} = -\widehat{c}\Big(\alpha_k\sum_{i=0}^{k-1}\left(\widehat{A}^{(k-1-i)}\widehat{b}\delta^{(i)}\right)\Big) = -\alpha_k\sum_{i=0}^{k-1}\left(\widehat{c}\widehat{A}^{(k-1-i)}\widehat{b}\delta^{(i)}\right). \tag{3.8}$$

From Lemma 2.36, $\widehat{c}\widehat{A}^{\ell}\widehat{b} = 0$ for $\ell\in\{0,1,2,\ldots,\mathtt{n_f}-1\}$. Therefore, the right-hand side of equation (3.8) is equal to 0, i.e., $\widehat{c}\left[\begin{smallmatrix}\bar{x}_{\mathtt{f}k}\\ \bar{z}_{\mathtt{f}k}\end{smallmatrix}\right] = 0$. Thus, the trajectories $\left[\begin{smallmatrix}\bar{x}_{\mathtt{f}k}\\ \bar{z}_{\mathtt{f}k}\end{smallmatrix}\right]$ satisfies the output-nulling equation (2.12) of $\Sigma_{\text{Ham}}$ for $k\in\{1,2,\ldots,\mathtt{n_f}-1\}$ in the distributional sense. To complete the proof we need to show that $\begin{bmatrix} \bar{x}_{\mathtt{f}0} \\ \bar{z}_{\mathtt{f}0} \end{bmatrix}$ satisfies equation (2.12), as well. We prove this next. When $k = 0$ i.e. $\begin{bmatrix} x_{00} \\ z_{00} \end{bmatrix} = \alpha_0 \widehat{b}$ and $u(t) = -\alpha_0 \delta$, from equation (3.4) we have

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \alpha_0 e^{\widehat{A}t}\widehat{b} - \int_0^t e^{\widehat{A}(t-\tau)}\widehat{b}\left(-\alpha_0 \delta^{(0)}(\tau)\right) d\tau = \alpha_0 e^{\widehat{A}t}\widehat{b} - \alpha_0 e^{\widehat{A}t}\widehat{b} = 0_{2\mathrm{n},1} = \begin{bmatrix} \bar{x}_{\mathtt{f}0} \\ \bar{z}_{\mathtt{f}0} \end{bmatrix}.$$

Clearly, $\widehat{c}\begin{bmatrix} \bar{x}_{\mathtt{f}0} \\ \bar{z}_{\mathtt{f}0} \end{bmatrix} = 0$. Therefore, $\mathtt{col}(\bar{x}_{\mathtt{f}0}, \bar{z}_{\mathtt{f}0}, \bar{u}_{\mathtt{f}0}) \in \Sigma_{\mathrm{Ham}}$. Thus, $\mathtt{col}(\bar{x}_{\mathtt{f}k}, \bar{z}_{\mathtt{f}k}, \bar{u}_{\mathtt{f}k}) \in \Sigma_{\mathrm{Ham}}$ corresponding to initial condition $\mathtt{col}(x_{0k}, z_{0k})$, where $k \in \{0, 1, \ldots, \mathtt{n_f} - 1\}$. Since $\Sigma_{\mathrm{Ham}}$ is a linear system, by the principle of superposition, Statement *(1)* of the lemma directly follows.
*(2):* We present a table next which explicitly validates that the trajectories $\mathtt{col}(\bar{x}_{\mathtt{f}k}, \bar{u}_{\mathtt{f}k})$ characterized in Lemma 3.6 satisfies the state-space dynamics of $\Sigma$ in a distributional sense, i.e., in the sense of Definition 3.3.

| $k$ | $x_0 = x_{0k}$ | $x(t) = \bar{x}_{\mathtt{f}k}$ | $u(t) = \bar{u}_{\mathtt{f}k}$ | $\frac{d}{dt}x$ | $Ax + bu$ |
|---|---|---|---|---|---|
| 0 | $\alpha_0 b$ | 0 | $-\alpha_0\delta$ | 0 | $-\alpha_0 b\delta$ |
| 1 | $\alpha_1 Ab$ | $-\alpha_1 b\delta$ | $-\alpha_1\delta^{(1)}$ | $-\alpha_1 b\delta^{(1)}$ | $-(\alpha_1 b\delta^{(1)} + \alpha_1 Ab\delta)$ |
| 2 | $\alpha_2 A^2 b$ | $-\alpha_2\left(b\delta^{(1)} + Ab\delta\right)$ | $-\alpha_2\delta^{(2)}$ | $-\alpha_2\left(b\delta^{(2)} + Ab\delta^{(1)}\right)$ | $-\alpha_2\left(b\delta^{(2)} + Ab\delta^{(1)} + A^2 b\delta\right)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathtt{n_f} - 1$ | $\alpha_{\mathtt{n_f}-1} A^{\mathtt{n_f}-1} b$ | $-\alpha_{\mathtt{n_f}-1}\sum_{i=0}^{\mathtt{n_f}-2} A^{\mathtt{n_f}-2-i} b\delta^{(i)}$ | $-\alpha_{\mathtt{n_f}-1}\delta^{(\mathtt{n_f}-1)}$ | $-\alpha_{\mathtt{n_f}-1}\sum_{i=0}^{\mathtt{n_f}-2} A^{\mathtt{n_f}-1-i} b\delta^{(i)}$ | $-\alpha_{\mathtt{n_f}-1}\sum_{i=0}^{\mathtt{n_f}-1} A^{\mathtt{n_f}-1-i} b\delta^{(i)}$ |

Table 3.1: Table to show the validity of $\frac{d}{dt}x = Ax + bu + x_0\delta$ for different initial conditions.

From the table it is evident that the trajectories $\mathtt{col}(\bar{x}_{\mathtt{f}k}, \bar{u}_{\mathtt{f}k})$ satisfy $\frac{d}{dt}x = Ax + bu + x_0\delta$. Thus, $\mathtt{col}(\bar{x}_{\mathtt{f}k}, \bar{u}_{\mathtt{f}k}) \in \Sigma$ corresponding to initial condition $x_{0k}$. Since $\Sigma$ is a linear system, by principle of superposition Statement *(2)* directly follows. ■

Note that using $W_1$ defined in equation (3.3), the candidate optimal state-trajectory $\bar{x}_{\mathtt{f}}$ can also be written as:

$$\bar{x}_{\mathtt{f}} = -W_1 \begin{bmatrix} 0 & \delta & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(\mathtt{n_f}-2)} \\ 0 & 0 & \delta & \delta^{(1)} & \cdots & \delta^{(\mathtt{n_f}-3)} \\ 0 & 0 & 0 & \delta & \cdots & \delta^{(\mathtt{n_f}-4)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \delta^{(1)} \\ 0 & 0 & 0 & 0 & \cdots & \delta \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{\mathtt{n_f}-1} \end{bmatrix}. \tag{3.9}$$

This form of $\bar{x}_{\mathtt{f}}$ will be of importance to us in the sequel.

### 3.3.2 Characterization of the candidate optimal slow trajectories

Next we characterize the candidate optimal trajectories of $\Sigma$ when the initial condition $x_0$ is from $\mathscr{V}$. For Lemma 3.7, it is important to note the following: since $V_{1\Lambda}$ is full column-rank by Theorem 2.32, there must exists $F \in \mathbb{R}^{1\times \mathtt{n}}$ such that $V_{3\Lambda} = FV_{1\Lambda}$, where $V_{3\Lambda}$ is as defined in equation (2.39). We use such a matrix $F$ in the next lemma.

Candidate slow optimal trajectories of the system $\Sigma$

**Lemma 3.7.** *Let* $\mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ *be such that equation* (2.39) *is satisfied. Assume the initial condition of the system to be* $x_0 = V_{1\Lambda}\beta$*, where* $\beta \in \mathbb{R}^{\mathtt{n_s}}$*. Define* $z_0 := V_{2\Lambda}\beta$*,* $\bar{x}_{\mathtt{s}} := V_{1\Lambda}e^{\Gamma t}\beta$*,* $\bar{u}_{\mathtt{s}} := FV_{1\Lambda}e^{\Gamma t}\beta$*, and* $\bar{z}_{\mathtt{s}} := V_{2\Lambda}e^{\Gamma t}\beta$*, where* $F \in \mathbb{R}^{1\times\mathtt{n}}$ *satisfies* $V_{3\Lambda} = FV_{1\Lambda}$*. Then,*

*(1)* $\mathtt{col}\,(\bar{x}_{\mathtt{s}}, \bar{z}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}) \in \Sigma_{\mathrm{Ham}}$ *corresponding to initial condition* $\mathtt{col}\,(x_0, z_0)$*.*

*(2)* $\mathtt{col}\,(\bar{x}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}) \in \Sigma$ *corresponding to initial condition* $x_0$*.*

*Proof:* $(1)$ : Define $\widehat{F} := \begin{bmatrix} F & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2\mathtt{n}}$. Using $\widehat{F}$ as the state-feedback in $\Sigma_{\mathrm{Ham}}$, i.e., $u = \widehat{F}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$, the output-nulling representation (2.12) of $\Sigma_{\mathrm{Ham}}$ takes the following form

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \widehat{A}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{b}u = (\widehat{A} + \widehat{b}\widehat{F})\begin{bmatrix} x \\ z \end{bmatrix}, \text{ and } \widehat{c}\begin{bmatrix} x \\ z \end{bmatrix} = 0. \tag{3.10}$$

Therefore, the trajectories in $\Sigma_{\mathrm{Ham}}$ corresponding to the initial condition $(x_0, z_0)$ takes the following form:

$$\begin{bmatrix} x \\ z \end{bmatrix} = e^{(\widehat{A}+\widehat{b}\widehat{F})t}\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = e^{(\widehat{A}+\widehat{b}\widehat{F})t}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}\beta = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} e^{\Gamma t}\beta = \begin{bmatrix} \bar{x}_{\mathtt{s}} \\ \bar{z}_{\mathtt{s}} \end{bmatrix}. \tag{3.11}$$

From equation (2.39) we know that $\widehat{c}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} = 0$. Therefore,

$$\widehat{c}\begin{bmatrix} \bar{x}_{\mathtt{s}} \\ \bar{z}_{\mathtt{s}} \end{bmatrix} = \widehat{c}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} e^{\Gamma t}\beta = 0. \tag{3.12}$$

Thus, from equation (3.11) and equation (3.12) it is evident that $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{z}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}) \in \Sigma_{\mathrm{Ham}}$.
$(2)$ : From equation (2.11) it is evident that $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{u}_{\mathtt{s}})$ satisfy the state-space dynamics $\frac{d}{dt}x = Ax + bu$. Thus, $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}) \in \Sigma$. ■

Now we claim that the trajectories defined in Lemmas 3.7 and 3.6 are indeed the optimal trajectories of $\Sigma$.

### 3.3.3 Optimal trajectories of the system

Recall that the DAEs in equation (2.11) are obtained on applying PMP in regular LQR problems. Therefore, for regular LQR problems, the trajectories that satisfy the DAEs in equation (2.11) are the optimal trajectories (see [Kir04]). However, we cannot invoke PMP here to claim that the trajectories characterized in Lemma 3.6 and Lemma 3.7 that satisfy the DAEs in equation (2.11) are optimal. This is because the trajectories characterized in Lemma 3.6 are not bounded: see [PBGM62, Chapter II] for more on inputs admissible for PMP. Hence, instead of invoking PMP we use a result in [Sch83] to prove that the trajectories in Lemma 3.6 and Lemma 3.7 minimizes the functional (2.8). We first review the result in [Sch83] for the ease of exposition.

**Proposition 3.8.** *[Sch83, Theorem 2] Consider the singular LQR Problem 3.5. Let the corresponding LMI be as given in inequality* (2.9). *Let $K_{\mathtt{max}}$ be the maximal rank-minimizing solution of the LQR LMI. Then,*

$$\mathtt{min} \int_0^\infty \left(x^T Q x\right) dt = x_0^T K_{\mathtt{max}} x_0.$$

The next theorem shows that the candidate optimal trajectories characterized in Lemma 3.6 and Lemma 3.7 are the optimal ones for the LQR Problem 3.5.

Optimal trajectories of the system $\Sigma$

**Theorem 3.9.** *Consider an initial condition of the system $\Sigma$ to be $x_0 = V_{1\Lambda}\beta + W_1\alpha$, where $V_{1\Lambda}$ and $W_1$ are as defined in Theorem 2.30 and equation* (3.3)*, respectively with $\beta \in \mathbb{R}^{\mathtt{n_s}}$ and $\alpha \in \mathbb{R}^{\mathtt{n_f}}$. Let $\bar{x} := \bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}$ and $\bar{u} := \bar{u}_{\mathtt{s}} + \bar{u}_{\mathtt{f}}$, where $\mathtt{col}(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}})$ and $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{u}_{\mathtt{s}})$ are as defined in Lemma 3.6 and Lemma 3.7, respectively. Then,*

*$\bar{u}$ is an admissible input, i.e., $\bar{u} \in \mathscr{U}_\Sigma$.*

*Further, the following statements hold:*

*(1)* $\mathtt{col}(\bar{x}, \bar{u}) \in \Sigma$.

*(2)* $\int_0^\infty \left(\bar{x}_{\mathtt{s}}^T Q \bar{x}_{\mathtt{s}}\right) dt = x_0^T K_{\mathtt{max}} x_0$.

*(3)* $\int_0^\infty \left(\bar{x}_{\mathtt{f}}^T Q \bar{x}_{\mathtt{f}}\right) dt = 0$.

*(4)* $\int_0^\infty \left(\bar{x}^T Q \bar{x}\right) dt = x_0^T K_{\mathtt{max}} x_0$.

*(5)* $\mathtt{col}(\bar{x}, \bar{u})$ *is the optimal trajectory of the LQR Problem 3.5.*

*Proof:* From Lemma 3.6 we rewrite the trajectories $\bar{x}_{\mathtt{f}k}$, $\bar{u}_{\mathtt{f}k}$ and initial condition $x_{0\mathtt{f}k}$ as $\bar{x}_{\mathtt{f}k} = -\alpha_k \sum_{i=0}^{k-1} A^{k-1-i} b \delta^{(i)}$, $\bar{u}_{\mathtt{f}k} = -\alpha_k \delta^{(k)}$ , and $x_{0\mathtt{f}k} = A^k b \alpha_k$, respectively with $k \in \{1, 2, \ldots, \mathtt{n_f} - 1\}$. Hence, we have

$$Q\bar{x}_{\mathtt{f}k} = -Q \sum_{i=0}^{k-1} A^{k-1-i} b \delta^{(i)} \alpha_k = \sum_{i=0}^{k-1} Q A^{k-1-i} b \delta^{(i)} \alpha_k, \text{ for } k \in \{1, 2, \cdots, \mathtt{n_f} - 1\}. \tag{3.13}$$

From Lemma 2.36 we know that $QA^\ell b = 0$ for $\ell \in \{0, 1, \cdots, \mathtt{n_f} - 2\}$. Using this identity in equation (3.13), we infer that $Q\bar{x}_{\mathtt{f}k} = 0$ for $k \in \{1, \ldots, \mathtt{n_f} - 1\}$. Further, for initial condition

$x_0 = b\alpha_0$, we have $\bar{x}_{\mathtt{f}0} = 0$ from Lemma 3.6. Hence, $Q\bar{x}_{\mathtt{f}0} = 0$. Thus, we can infer that

$$Q\bar{x}_{\mathtt{f}} = Q\sum_{k=0}^{\mathtt{n_f}-1}\bar{x}_{\mathtt{f}k} = \sum_{k=0}^{\mathtt{n_f}-1}Q\bar{x}_{\mathtt{f}k} = 0 \text{ for } k \in \{0,1,2,\cdots,\mathtt{n_f}-1\}. \tag{3.14}$$

Now, define $C \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$ such that $Q =: C^TC$, where $\mathtt{rank}(Q) = \mathtt{r}$. Further, define $y(t) := Cx(t)$. Using the fact that $Q\bar{x}_{\mathtt{f}} = 0 \Rightarrow C\bar{x}_{\mathtt{f}} = 0$, we must have $y(t;x_0,\bar{u}) = C(\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}) = C(V_{1\Lambda}e^{\Gamma t}\beta + \bar{x}_{\mathtt{f}}) = CV_{1\Lambda}e^{\Gamma t}\beta \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{r}})|_{\mathbb{R}_+}$. Thus, by Definition 3.4, we have $\bar{u} \in \mathscr{U}_{\Sigma}$.
*(1):* Using Lemmas 3.7, 3.6, and linearity of $\Sigma$, it is evident that $\mathtt{col}(\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{s}} + \bar{u}_{\mathtt{f}}) \in \Sigma$.
*(2):* Since $V_{3\Lambda} \in \mathbb{R}^{1\times\mathtt{n_s}}$ and $V_{1\Lambda}$ is full column-rank, there exists an $F \in \mathbb{R}^{1\times\mathtt{n}}$ such that $V_{3\Lambda} = FV_{1\Lambda}$. Thus, from equation (2.39), we have $AV_{1\Lambda} + bV_{3\Lambda} = (A+bF)V_{1\Lambda} = V_{1\Lambda}\Gamma$. Therefore, $\dot{\bar{x}}_{\mathtt{s}} = V_{1\Lambda}\Gamma e^{\Gamma t}\beta = (A+bF)V_{1\Lambda}e^{\Gamma t}\beta$. Hence, on using $K_{\mathtt{max}}b = 0$, we get

$$\begin{aligned}
&\frac{d}{dt}(\bar{x}_{\mathtt{s}}^TK_{\mathtt{max}}\bar{x}_{\mathtt{s}}) = \dot{\bar{x}}_{\mathtt{s}}{}^TK_{\mathtt{max}}\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{s}}^TK_{\mathtt{max}}\dot{\bar{x}}_{\mathtt{s}}\\
&= (V_{1\Lambda}\Gamma e^{\Gamma t}\beta)^TK_{\mathtt{max}}(V_{1\Lambda}e^{\Gamma t}\beta) + (V_{1\Lambda}e^{\Gamma t}\beta)^TK_{\mathtt{max}}(V_{1\Lambda}\Gamma e^{\Gamma t}\beta)\\
&= \beta^Te^{\Gamma^Tt}V_{1\Lambda}^T(A+bF)^TK_{\mathtt{max}}V_{1\Lambda}e^{\Gamma t}\beta + \beta^Te^{\Gamma^Tt}V_{1\Lambda}^TK_{\mathtt{max}}(A+bF)V_{1\Lambda}e^{\Gamma^Tt}\beta\\
&= \beta^Te^{\Gamma^Tt}(V_{1\Lambda}^TA^TK_{\mathtt{max}}V_{1\Lambda} + V_{1\Lambda}^TK_{\mathtt{max}}AV_{1\Lambda})e^{\Gamma t}\beta.
\end{aligned} \tag{3.15}$$

Note that $K_{\mathtt{max}}V_{1\Lambda} = \begin{bmatrix}V_{2\Lambda} & 0\end{bmatrix}\begin{bmatrix}V_{1\Lambda} & W_1\end{bmatrix}^{-1}V_{1\Lambda} = V_{2\Lambda}$. Using this in equation (3.15), we have

$$\begin{aligned}
V_{1\Lambda}^T(A^TK_{\mathtt{max}} + K_{\mathtt{max}}A + Q)V_{1\Lambda} &= V_{1\Lambda}^TA^TV_{2\Lambda} + V_{2\Lambda}^TAV_{1\Lambda} + V_{1\Lambda}^TQV_{1\Lambda}\\
&= \begin{bmatrix}V_{2\Lambda}^T & -V_{1\Lambda}^T\end{bmatrix}\widehat{A}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}.
\end{aligned} \tag{3.16}$$

From equation (2.39), we have

$$\widehat{A}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix} + \widehat{b}V_{3\Lambda} = \begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}\Gamma \Rightarrow \widehat{A}\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix} = \begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}\Gamma - \widehat{b}V_{3\Lambda} \tag{3.17}$$

Using equation (3.17) in equation (3.16) combined with the facts that $V_{1\Lambda}^TV_{2\Lambda} = V_{2\Lambda}^TV_{1\Lambda}$ (see Proposition 2.31) and $b^TV_{2\Lambda} = 0$ , we have

$$\begin{aligned}
V_{1\Lambda}^T(A^TK_{\mathtt{max}} + K_{\mathtt{max}}A + Q)V_{1\Lambda} &= \begin{bmatrix}V_{2\Lambda}^T & -V_{1\Lambda}^T\end{bmatrix}\left(\begin{bmatrix}V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}\Gamma - \begin{bmatrix}b\\ 0\end{bmatrix}\right) = 0\\
&\Rightarrow V_{1\Lambda}^T(A^TK_{\mathtt{max}} + K_{\mathtt{max}}A)V_{1\Lambda} = -V_{1\Lambda}^TQV_{1\Lambda}.
\end{aligned} \tag{3.18}$$

Using equation (3.18) in equation (3.15), we get

$$\frac{d}{dt}\left(\bar{x}_{\mathtt{s}}^TK_{\mathtt{max}}\bar{x}_{\mathtt{s}}\right) = -\beta^Te^{\Gamma^Tt}V_{1\Lambda}^TQV_{1\Lambda}e^{\Gamma t}\beta = -\bar{x}_{\mathtt{s}}^TQ\bar{x}_{\mathtt{s}}. \tag{3.19}$$

Hence, cost due to input $\bar{u}_{\mathtt{s}}$ and initial condition $x_{0\mathtt{s}}$ is

$$\int_0^{\infty}\left(\bar{x}_{\mathtt{s}}^TQ\bar{x}_{\mathtt{s}}\right)dt = -\int_0^{\infty}\frac{d}{dt}\left(\bar{x}_{\mathtt{s}}^TK_{\mathtt{max}}\bar{x}_{\mathtt{s}}\right)dt = x_0^TK_{\mathtt{max}}x_0 - x_{\infty}^TK_{\mathtt{max}}x_{\infty}, \text{ where } \lim_{t\to\infty}x(t) =: x_{\infty}.$$

Note that the integral above is well-defined since $\bar{x}_{\mathtt{s}} = \mathtt{img}\left(V_{1\Lambda}e^{\Gamma t}\right)$, where $\sigma(\Gamma) \subsetneq \mathbb{C}_-$. Further, since $\sigma(\Gamma) \subsetneq \mathbb{C}_-$, $\lim_{t\to\infty} x = x_\infty = 0$. Thus, Statement 2 follows.

*(3):* Since $Q\bar{x}_{\mathtt{f}} = 0$ from equation (3.14), the cost due to input $\bar{u}_{\mathtt{f}}$ and initial condition $x_{0\mathtt{f}}$ is

$$\int_0^\infty \left(\bar{x}_{\mathtt{f}}^T Q \bar{x}_{\mathtt{f}}\right) dt = 0.$$

*(4):* From Statement *(1)* of this theorem, we know that corresponding to initial condition $x_0$ we have $\mathtt{col}(\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{s}} + \bar{u}_{\mathtt{f}}) \in \Sigma$. Therefore, using Statement *(2)*, *(3)* of the theorem and the fact that $Q\bar{x}_{\mathtt{f}} = 0$, we have

$$\begin{aligned}\int_0^\infty \left((\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}})^T Q (\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}})\right) dt &= \int_0^\infty (\bar{x}_{\mathtt{s}}^T Q \bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{s}}^T Q \bar{x}_{\mathtt{f}} + \bar{x}_{\mathtt{f}}^T Q \bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}^T Q \bar{x}_{\mathtt{f}}) dt \\ &= \int_0^\infty (\bar{x}_{\mathtt{s}}^T Q \bar{x}_{\mathtt{s}}) dt = x_0^T K_{\mathtt{max}} x_0.\end{aligned}$$

*(5):* From Statement *(3)* and *(4)* of Theorem 2.30, we know that $K_{\mathtt{max}}$ is the maximal rank-minimizing solution. Further, from Proposition 3.8 we know that the minimum value that can be attained by $\int_0^\infty (x^T Q x) dt$ is $x_0^T K_{\mathtt{max}} x_0$. Hence, from Statement *(4)* of this theorem, we infer that $\mathtt{col}(\bar{x}, \bar{u})$ are the minimizers of $\int_0^\infty (x^T Q x) dt$, i.e., $\mathtt{col}(\bar{x}, \bar{u})$ is the optimal trajectory of the LQR Problem 3.5. ■

Thus, Theorem 3.9 establishes that $\mathtt{col}(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}})$ are the optimal trajectories of $\Sigma$. Note that, on using Table 3.1, the optimal input for the LQR problem in Example 3.1 can be computed to be $-\delta(t)$. This corroborates with our analysis in Example 3.1.

In the next section, we show that the system $\Sigma$ can indeed be confined to the optimal trajectories $\mathtt{col}(\bar{x}, \bar{u})$ using a PD state-feedback control law of the form $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \frac{d}{dt} x$.

## 3.4 PD state-feedback controller for singular LQR problems: single-input case

In this section we present a method to design a PD state-feedback control law $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \frac{d}{dt} x$ that solves the singular LQR Problem 2.21. To this end we first define the feedback matrices $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$. Then we show that the application of the control law $u = F_{\mathtt{p}} x + F_{\mathtt{d}} \frac{d}{dt} x$ to the system $\Sigma$ confines its state-trajectories to the optimal ones $\bar{x}$ (characterized in Theorem 3.9).

Using the fact that $X_{1\Lambda}$ is nonsingular (Statement *(1)* of Theorem 2.30), we define the matrices $F_{\mathtt{p}}, F_{\mathtt{d}} \in \mathbb{R}^{1\times\mathtt{n}}$ as follows:

$$F_{\mathtt{p}} := \begin{bmatrix} V_{3\Lambda} & f_0 & f_1 & \cdots & f_{\mathtt{n_f}-1} \end{bmatrix} X_{1\Lambda}^{-1}, \tag{3.20}$$

$$F_{\mathtt{d}} := \begin{bmatrix} 0_{1,\mathtt{n_s}} & 1 & -f_0 & \cdots & -f_{\mathtt{n_f}-2} \end{bmatrix} X_{1\Lambda}^{-1}, \tag{3.21}$$

where $V_{3\Lambda}$ is as defined in Theorem 2.30 and $f_i \in \mathbb{R}$ for $i \in \{0,1,\dots,\mathtt{n_f}-1\}$. The closed loop system obtained on application of $u = F_{\mathtt{p}}x + F_{\mathtt{d}}\frac{d}{dt}x$ to $\Sigma$ is as follows:

$$E_{\mathtt{c}}\frac{d}{dt}x = A_{\mathtt{c}}x, \text{ where } (I_{\mathtt{n}} - bF_{\mathtt{d}}) =: E_{\mathtt{c}}, (A + bF_{\mathtt{p}}) =: A_{\mathtt{c}}. \tag{3.22}$$

We use the symbol $\Sigma_{\mathtt{closed}}$ to represent the closed loop system in equation (3.22). Note that

$$\begin{aligned} E_{\mathtt{c}}X_{1\Lambda} &= (I_{\mathtt{n}} - bF_{\mathtt{d}})X_{1\Lambda} = X_{1\Lambda} - bF_{\mathtt{d}}X_{1\Lambda} \\ &= \begin{bmatrix} V_{1\Lambda} & b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix} - b\begin{bmatrix} 0_{1,\mathtt{n_s}} & 1 & -f_0 & \cdots & -f_{\mathtt{n_f}-2} \end{bmatrix} \\ E_{\mathtt{c}} &= \underbrace{\begin{bmatrix} V_{1\Lambda} & 0_{\mathtt{n},1} & Ab + bf_0 & \cdots & A^{\mathtt{n_f}-1}b + bf_{\mathtt{n_f}-2} \end{bmatrix}}_{\widehat{E}_{\mathtt{c}}} X_{1\Lambda}^{-1} \end{aligned}$$

Clearly, $\widehat{E}_{\mathtt{c}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a singular matrix and therefore, $E_{\mathtt{c}}$ is the product of a singular matrix $\widehat{E}_{\mathtt{c}}$ and a nonsingular matrix $X_{1\Lambda}^{-1}$. Thus, $E_{\mathtt{c}}$ is a singular matrix. This implies that the system $\Sigma_{\mathtt{closed}}$ in equation (3.22) is a singular descriptor system. Recall from Proposition 2.8 that for the singular descriptor system $\Sigma_{\mathtt{closed}}$ if $\det(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$, then we get unique state-trajectories in $\Sigma_{\mathtt{closed}}$ (given by equation (2.5)) corresponding to an initial condition. Uniqueness in the state-trajectories of $\Sigma_{\mathtt{closed}}$ corresponding to an initial condition is crucial in the sequel to prove that the PD state-feedback control law, that we propose, confines the system to its optimal state-trajectories *only*. Hence, in the next lemma we show the existence of $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ such that $\det(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$.

Existence of $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ such that the matrix pencil $(sE_{\mathtt{c}} - A_{\mathtt{c}})$ is regular

**Lemma 3.10.** *Let $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ be as defined in equation* (3.20) *and equation* (3.21)*, respectively. Then, there exist $f_0,\dots,f_{\mathtt{n_f}-1} \in \mathbb{R}$ such that $\det(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$, where $E_{\mathtt{c}}$, $A_{\mathtt{c}}$ are as defined in equation*(3.22)*.*

*Proof:* In order to prove this, we construct two matrices $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ using equation (3.20) and equation (3.21), respectively such that $\det(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$. We define $\widehat{f} := \begin{bmatrix} f_0 & f_1 & \cdots & f_{\mathtt{n_f}-2} \end{bmatrix} \in \mathbb{R}^{1\times(\mathtt{n_f}-1)}$. Then using equation (3.21), we can write

$$E_{\mathtt{c}}X_{1\Lambda} = (I_{\mathtt{n}} - bF_{\mathtt{d}})X_{1\Lambda} = X_{1\Lambda}\begin{bmatrix} I_{\mathtt{n_s}} & 0 & 0 \\ 0 & 0 & \widehat{f} \\ 0 & 0 & I_{\mathtt{n_f}-1} \end{bmatrix}. \tag{3.23}$$

From equation (3.20), we have $F_{\mathtt{p}}A^k b = f_k$ for $k \in \{0,\dots,\mathtt{n_f}-1\}$. Therefore, we get

$$(A + bF_{\mathtt{p}})A^k b = A^{k+1}b + bF_{\mathtt{p}}A^k b = A^{k+1}b + bf_k, \quad \text{for } k \in \{0,1,\dots,\mathtt{n_f}-1\}. \tag{3.24}$$

Note that for $k = \mathtt{n_f}-1$, we have $(A + bF_{\mathtt{p}})A^{\mathtt{n_f}-1}b = A^{\mathtt{n_f}}b + bf_{\mathtt{n_f}-1}$. Since $X_{1\Lambda}$ is invertible, using equation (3.3) we can infer that the columns of $X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} & b & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix}$

are independent. Therefore, there exists $\kappa_v \in \mathbb{R}^{\mathtt{n_s}}$ and $\kappa_0, \kappa_1, \ldots, \kappa_{\mathtt{n_f}-1} \in \mathbb{R}$ such that the vector $A^{\mathtt{n_f}} b$ can be uniquely written as

$$A^{\mathtt{n_f}} b := V_{1\Lambda}\kappa_v + b\kappa_0 + Ab\kappa_1 + \cdots + A^{\mathtt{n_f}-1} b\kappa_{\mathtt{n_f}-1}. \tag{3.25}$$

Defining $\kappa := \mathtt{col}(\kappa_1, \kappa_2, \ldots, \kappa_{\mathtt{n_f}-1})$ and using equation (3.24), equation (3.25) with the fact that $(A + bF_{\mathtt{p}})V_{1\Lambda} = V_{1\Lambda}\Gamma$ (From equation (2.39)), we have

$$A_{\mathtt{c}} X_{1\Lambda} = (A + bF_{\mathtt{p}}) X_{1\Lambda} = X_{1\Lambda} \begin{bmatrix} \Gamma & 0 & \kappa_v \\ 0 & \widehat{f} & f_{\mathtt{n_f}-1} + \kappa_0 \\ 0 & I_{\mathtt{n_f}-1} & \kappa \end{bmatrix}. \tag{3.26}$$

Define $Z_1 := \begin{bmatrix} I_{\mathtt{n_s}} & 0 & 0 \\ 0 & 1 & -\widehat{f} \\ 0 & 0 & I_{\mathtt{n_f}-1} \end{bmatrix}$. Note that $\mathtt{det}(Z_1) = 1$. Using equation (3.23) and equation (3.26), it can be verified by simple multiplication that

$$\begin{aligned} Z_1 X_{1\Lambda}^{-1}(sE_{\mathtt{c}} - A_{\mathtt{c}}) X_{1\Lambda} &= sZ_1 X_{1\Lambda}^{-1} E_{\mathtt{c}} X_{1\Lambda} - Z_1 X_{1\Lambda}^{-1} A_{\mathtt{c}} X_{1\Lambda} \\ &= \left[\begin{array}{c|ccccc} sI_{\mathtt{n_s}} - \Gamma & 0 & 0 & \cdots & 0 & -\kappa_v \\ \hline 0 & 0 & 0 & \cdots & 0 & -\kappa_0 - f_{\mathtt{n_f}-1} + \widehat{f}\kappa \\ 0 & -1 & s & \cdots & 0 & -\kappa_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s & -\kappa_{\mathtt{n_f}-2} \\ 0 & 0 & 0 & \cdots & -1 & s - \kappa_{\mathtt{n_f}-1} \end{array}\right]. \end{aligned} \tag{3.27}$$

Since $\mathtt{det}\left(Z_1 X_{1\Lambda}^{-1}(sE_{\mathtt{c}} - A_{\mathtt{c}}) X_{1\Lambda}\right) = \mathtt{det}\,(sE_{\mathtt{c}} - A_{\mathtt{c}})$, we have from equation (3.27),

$$\mathtt{det}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0 \Leftrightarrow f_{\mathtt{n_f}-1} - \widehat{f}\kappa \neq -\kappa_0.$$

In particular, if we choose $\widehat{f} = 0$ and $f_{\mathtt{n_f}-1} \neq -\kappa_0$, then the matrices $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ are such that $\mathtt{det}(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$. Thus, there exist at least two matrices $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ such that $\mathtt{det}(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$. This completes the proof of the lemma. ■

Note that there are infinitely many choices of $\widehat{f}$ and $f_{\mathtt{n_f}-1}$ such that $f_{\mathtt{n_f}-1} - \widehat{f}\kappa \neq -\kappa_0$. Thus, from Lemma 3.10 we can infer that there are uncountably many choices of $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ such that $\mathtt{det}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$ for $\Sigma_{\mathtt{closed}}$. Each of these choices leads to a closed loop system $\Sigma_{\mathtt{closed}}$ that admits unique state-trajectories. Next we prove that these unique state trajectories are nothing but the optimal state-trajectories $\bar{x}$ characterized in Theorem 3.9.

Trajectories of the closed loop system $\Sigma_{\mathtt{closed}}$

**Theorem 3.11.** *Let $\Sigma_{\mathtt{closed}}$ be the system defined in equation* (3.22)*, where $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ are as defined in equation* (3.20) *and equation* (3.21)*, respectively, with $\mathtt{det}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$. Consider an arbitrary initial condition of the system $\Sigma_{\mathtt{closed}}$ given as $x_0 =: V_{1\Lambda}\beta + W_1\alpha$, where $\beta \in \mathbb{R}^{\mathtt{n_s}}$, $\alpha \in \mathbb{R}^{\mathtt{n_f}}$, and $V_{1\Lambda}$, $W_1$ are as defined in Theorem 2.30, equation* (3.3)*, respectively. Let $\bar{x}$ be as defined in Theorem 3.9. Then, the unique trajectory in $\Sigma_{\mathtt{closed}}$ corresponding to $x_0$ is $\bar{x}$.*

*Proof:* First we transform the system $\Sigma_{\mathtt{closed}}$ to its canonical form. Recall $Z_1$ from the proof of Lemma 3.10. Using the co-ordinate transform $p := X_{1\Lambda}^{-1}x$ on $\Sigma_{\mathtt{closed}}$ and then pre-multiplying with $Z_1X_{1\Lambda}^{-1}$ gives

$$\underbrace{Z_1X_{1\Lambda}^{-1}E_{\mathtt{c}}X_{1\Lambda}}_{E_{\mathtt{new}}}\frac{d}{dt}p = \underbrace{Z_1X_{1\Lambda}^{-1}A_{\mathtt{c}}X_{1\Lambda}}_{A_{\mathtt{new}}}p. \tag{3.28}$$

From equation (3.23) and equation (3.26) we have that

$$E_{\mathtt{new}} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\mathtt{n_f}-1} \end{bmatrix}, \quad A_{\mathtt{new}} = \begin{bmatrix} \Gamma & 0_{\mathtt{n_s},\mathtt{n_f}-1} & \kappa_v \\ 0 & 0_{1,\mathtt{n_f}-1} & \gamma \\ 0_{\mathtt{n_f}-1,1} & I_{\mathtt{n_f}-1} & \kappa \end{bmatrix}, \tag{3.29}$$

where $\kappa$, $\kappa_v, \widehat{f}$ are as defined in the proof of Lemma 3.10 and $\gamma := \kappa_0 + f_{\mathtt{n_f}-1} - \widehat{f}\kappa$. Note that $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ are chosen such that $\gamma \neq 0$ to ensure $\mathtt{det}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$ (see Lemma 3.10). We use this fact to define the matrix:

$$Z_2 := \begin{bmatrix} I_{\mathtt{n_s}} & -\frac{\kappa_v}{\gamma} & 0_{\mathtt{n_s},\mathtt{n_f}-1} \\ 0 & -\frac{\kappa}{\gamma} & I_{\mathtt{n_f}-1} \\ 0 & \frac{1}{\gamma} & 0_{1,\mathtt{n_f}-1} \end{bmatrix}.$$

Note that $Z_2$ is nonsingular. On pre-multiplication of $E_{\mathtt{new}}$ and $A_{\mathtt{new}}$ in equation (3.28) with $Z_2$ it can be verified that

$$E_{\mathtt{closed}} := Z_2E_{\mathtt{new}} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & Y \end{bmatrix}, \text{ where } Y := \begin{bmatrix} 0 & I_{\mathtt{n_f}-1} \\ 0 & 0 \end{bmatrix} \tag{3.30}$$

$$A_{\mathtt{closed}} := Z_2A_{\mathtt{new}} = \begin{bmatrix} \Gamma & 0 \\ 0 & I_{\mathtt{n_f}} \end{bmatrix}. \tag{3.31}$$

Therefore, pre-multiplying equation (3.28) with $Z_2$ gives

$$\begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & Y \end{bmatrix}\frac{d}{dt}p = \begin{bmatrix} \Gamma & 0 \\ 0 & I_{\mathtt{n_f}} \end{bmatrix}p \tag{3.32}$$

Since $Z_2Z_1X_{1\Lambda}^{-1}$ and $X_{1\Lambda}$ are nonsingular, equation (3.32) is a canonical form of the system $\Sigma_{\texttt{closed}}$. Now we prove the theorem in two steps: first we assume the initial condition to be $x_0 = V_{1\Lambda}\beta$ and then we consider $x_0 = W_1\alpha$.

*Step 1:* Let $x_0 = V_{1\Lambda}\beta$, where $\beta \in \mathbb{R}^{\mathtt{n_s}}$. Then, in the transformed co-ordinates the initial condition is $X_{1\Lambda}^{-1}x_0 = X_{1\Lambda}^{-1}V_{1\Lambda}\beta = X_{1\Lambda}^{-1}\left[V_{1\Lambda}\ W_1\right]\begin{bmatrix}\beta\\0\end{bmatrix} = \texttt{col}(\beta, 0_{\mathtt{n_f},1})$. Using the fact that the nilpotency index of $Y$ is $\mathtt{n_f}$, from equation (2.5) the state-trajectories of $\Sigma$ corresponding to initial condition $V_{1\Lambda}\beta$ is:

$$x(t) = X_{1\Lambda}\begin{bmatrix}I_{\mathtt{n_s}}\\0\end{bmatrix}e^{\Gamma t}\begin{bmatrix}I_{\mathtt{n_s}} & 0\end{bmatrix}X_{1\Lambda}^{-1}x_0 - X_{1\Lambda}\begin{bmatrix}0\\I_{\mathtt{n_f}}\end{bmatrix}\sum_{i=1}^{\mathtt{n_f}-1}\delta^{(i-1)}Y^i\begin{bmatrix}0 & I_{\mathtt{n_f}}\end{bmatrix}X_{1\Lambda}^{-1}x_0$$

$$= V_{1\Lambda}e^{\Gamma t}\begin{bmatrix}I_{\mathtt{n_s}} & 0\end{bmatrix}\begin{bmatrix}\beta\\0_{\mathtt{n_f},1}\end{bmatrix} - W_1\sum_{i=1}^{\mathtt{n_f}-1}\delta^{(i-1)}Y^i\begin{bmatrix}0 & I_{\mathtt{n_f}}\end{bmatrix}\begin{bmatrix}\beta\\0_{\mathtt{n_f},1}\end{bmatrix} = V_{1\Lambda}e^{\Gamma t}\beta = \bar{x}_{\mathtt{s}}.$$

*Step 2:* Let $x_0 = W_1\alpha = \sum_{k=0}^{\mathtt{n_f}-1}A^kb\alpha_k$, where $\alpha \in \mathbb{R}^{\mathtt{n_f}}$ and $\alpha =: \texttt{col}(\alpha_0, \alpha_1, \ldots, \alpha_{\mathtt{n_f}-1})$. Then, in the transformed co-ordinates the initial condition of the system $\Sigma_{\texttt{closed}}$ is $X_{1\Lambda}^{-1}x_0 = X_{1\Lambda}^{-1}W_1\alpha = \texttt{col}(0_{\mathtt{n_s},1}, \alpha)$. Hence, from equation (2.5) the state-trajectories of $\Sigma$ due to initial condition $W_1\alpha$ is $x(t) = -W_1\sum_{i=1}^{\mathtt{n_f}-1}\delta^{(i-1)}Y^i\alpha$, which in matrix form is:

$$x(t) = -W_1\begin{bmatrix}0 & \delta & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(\mathtt{n_f}-2)}\\ 0 & 0 & \delta & \delta^{(1)} & \cdots & \delta^{(\mathtt{n_f}-3)}\\ 0 & 0 & 0 & \delta & \cdots & \delta^{(\mathtt{n_f}-4)}\\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \delta^{(1)}\\ 0 & 0 & 0 & 0 & \cdots & \delta\\ 0 & 0 & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{\mathtt{n_f}-1}\end{bmatrix}.$$

Using equation (3.9), we therefore have $x(t) = \bar{x}_{\mathtt{f}}$.

It then follows from Step 1 and Step 2, and linearity of $\Sigma_{\texttt{closed}}$ that corresponding to an initial condition $x_0 = V_{1\Lambda}\beta + W_1\alpha$, the trajectory of the system $\Sigma_{\texttt{closed}}$ is $\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}$. From Theorem 3.9 we know that $\bar{x} = \bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}}$. Since $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ are chosen such that $\texttt{det}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$, corresponding to initial condition $x_0$, $\bar{x}$ must be unique. ■

Using Theorems 3.9 and 3.11 we present the main result next.

Trajectories of the closed loop system $\Sigma_{\texttt{closed}}$ are the optimal ones

**Theorem 3.12.** *Consider the singular LQR Problem 2.21. Assume* $F_{\mathtt{p}} \in \mathbb{R}^{1\times\mathtt{n}}$ *and* $F_{\mathtt{d}} \in \mathbb{R}^{1\times\mathtt{n}}$ *to be as defined in equation* (3.20) *and equation* (3.21)*, respectively with* $\texttt{det}\left(s(I_{\mathtt{n}} - bF_{\mathtt{d}}) - (A + bF_{\mathtt{p}})\right) \neq 0$*. Let the closed loop system obtained on application of the PD state-feedback law* $u = F_{\mathtt{p}}x + F_{\mathtt{d}}\frac{d}{dt}x$ *to* $\Sigma$ *be as defined in equation* (3.22)*. Then, for an arbitrary initial condition* $x_0$*, the corresponding trajectory of the closed loop system* $\Sigma_{\texttt{closed}}$ *minimizes the functional* (2.8).

*Proof:* From Theorem 3.11 it is clear that corresponding to an initial condition $x_0$ the unique state-trajectory of the system $\Sigma_{\texttt{closed}}$ is $\bar{x}$. Further, from Theorem 3.9, we know that the integral $\int_0^\infty(\bar{x}^T Q\bar{x})dt$ is well-defined and $\bar{x}$ is the optimal state-trajectory of the LQR Problem 2.21. Thus, the state-trajectories of $\Sigma_{\texttt{closed}}$ corresponding to any initial condition $x_0$ minimizes the functional (2.8). ■

Recall that Theorem 3.12 is not applicable to singular LQR problems that admit Hamiltonian matrix pairs with imaginary axis eigenvalues (condition $\sigma(E,H)\cap j\mathbb{R}=\emptyset$ in Theorem 2.30). This assumption is true for almost all singular LQR problems. Thus, using the fact that regular LQR problems are solvable using static (P) state-feedback control laws, we can infer from Theorem 3.12 that almost every single-input LQR problem can be solved using a PD state-feedback control law.

Note that since $\Sigma_{\texttt{closed}}$ is a singular descriptor system, from Section 2.2.3 we know that $\Sigma_{\texttt{closed}}$ admits a slow and a fast subspace. In what follows, we show that the slow and fast subspaces of $\Sigma_{\texttt{closed}}$ are nothing but the subspaces $\texttt{img}\, V_{1\Lambda}$ and $\texttt{img}\, W_1$, respectively with $V_{1\Lambda}$ and $W$ as defined in Theorem 2.30 and equation (3.3), respectively.

Pre- and post-multiplying $E_{\texttt{c}}$ and $A_{\texttt{c}}$ in equation (3.22) with the nonsingular matrices $Z_2 Z_1 X_{1\Lambda}^{-1}$ and $X_{1\Lambda}$ takes $\Sigma_{\texttt{closed}}$ to its canonical form as in equation (3.32). Therefore, recall from Section 2.2.3 that the subspace spanned by the first $\texttt{n}_{\texttt{s}}$ columns of $X_{1\Lambda}$, i.e., the subspace $\texttt{img}\, V_{1\Lambda}$ is the slow subspace of $\Sigma_{\texttt{closed}}$. On the other hand, the subspace spanned by the last $\texttt{n}_{\texttt{f}}$ columns of $X_{1\Lambda}$, i.e., $\texttt{img}\, W_1$ is the fast subspace of $\Sigma_{\texttt{closed}}$. We formally present this in the form of a corollary next.

Slow and fast subspaces of the closed-loop system that solves LQR Problem 3.5

**Corollary 3.13.** *Consider the system $\Sigma_{\texttt{closed}}$ with the state-space equation of the form given in equation* (3.22)*, where $F_{\texttt{p}}$ and $F_{\texttt{d}}$ are as defined in Theorem 3.12. Define $\mathscr{V} := \texttt{img}\, V_{1\Lambda}$ and $\mathscr{W} := \texttt{img}\, W_1$ with $V_{1\Lambda}$ and $W_1$ as defined in Theorem 2.30. Then, $\mathscr{V}$ and $\mathscr{W}$ are the slow subspace and fast subspace of the system $\Sigma_{\texttt{closed}}$, respectively.*

*Proof:* This directly follows from the discussion before this corollary. ■

Now let us apply Theorem 3.12 for the single-input regular LQR Problem 2.1 with the system given by $\frac{d}{dt}x = Ax+bu$ and the objective function as defined in equation (2.1). For the regular LQR problem, we can directly use Proposition 2.19. Hence, there exists $(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ such that

$$\begin{bmatrix} A & 0 & b \\ -Q & -A^T & -S \\ S^T & b^T & R \end{bmatrix} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma, \text{ where } \sigma(\Gamma) \subsetneq \mathbb{C}_-. \tag{3.33}$$

Thus, we have from equation (3.33)

$$S^T V_{1\Lambda} + b^T V_{2\Lambda} + R V_{3\Lambda} = 0 \Rightarrow V_{3\Lambda} = -R^{-1}(b^T V_{2\Lambda} + S^T V_{1\Lambda}).$$

Using the fact that $X_{1\Lambda} = V_{1\Lambda}$ here, from Theorem 3.12 we get

$$F_{\mathtt{p}} = V_{3\Lambda}X_{1\Lambda}^{-1} = V_{3\Lambda}V_{1\Lambda}^{-1} = -R^{-1}(b^T V_{2\Lambda} + S^T V_{1\Lambda})V_{1\Lambda}^{-1}$$
$$= -R^{-1}(b^T V_{2\Lambda}V_{1\Lambda}^{-1} + S^T) = -R^{-1}(b^T K_{\mathtt{max}} + S^T).$$

Further, from equation (3.21), we have $F_{\mathtt{d}} = 0$. Thus, the state-feedback law $u = -R^{-1}(b^T K_{\mathtt{max}} + S^T)x$ solves the regular LQR problem. This corroborates the well-known results on regular LQR problem in the literature. Hence, Theorem 3.12 is indeed a generalization of the solution to the regular LQR problem.

**Example 3.14.** *Recall Example 2.20 from Section 2.2.6. For this example* $X_{1\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, *and* $V_{3\Lambda} = 0$. *Assigning* $f_0 = 0$ *and defining* $f_1 =: f$ *in equation* (3.20), *we have*

$$F_{\mathtt{p}} = \begin{bmatrix} 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2f & 0 & f \end{bmatrix}$$

*Similarly, from equation* (3.21), *we have*

$$F_{\mathtt{d}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

*Thus,*

$$I_3 - BF_{\mathtt{d}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A + BF_{\mathtt{p}} = \begin{bmatrix} 1 & 0 & 1 \\ 1+2f & 0 & 1+f \\ 1 & 1 & 0 \end{bmatrix}.$$

*Note that* $\mathtt{det}\left(s(I_3 - BF_{\mathtt{d}}) - (A + BF_{\mathtt{p}})\right) = -f(s+1)$. *Thus, if we chose any* $f \in \mathbb{R} \setminus 0$ *then* $\mathtt{det}\left(s(I_3 - BF_{\mathtt{d}}) - (A + BF_{\mathtt{p}})\right) \neq 0$. *Hence, for any value of* $f \in \mathbb{R} \setminus 0$, *we have a PD-controller that solves the singular LQR problem. Note that there are uncountable numbers of PD-controllers that solve this singular optimal control problem.*

*For initial condition* $x_0 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\beta + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\alpha_0 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\alpha_1$, *the optimal input for this problem is* $u^* = -2e^{-t}\beta - \alpha_0\delta - \alpha_1\dot{\delta}$.

We revisit Example 1.2 introduced in Chapter 1 to design a controller for the damped spring-mass system such that its trajectories are confined to the ones that minimize the total energy of the system.

**Example 3.15.** *Consider the damped spring-mass system in Example 1.2 with normalized spring constant k, damping constant c and mass m. Then, the dynamics of the system is*

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

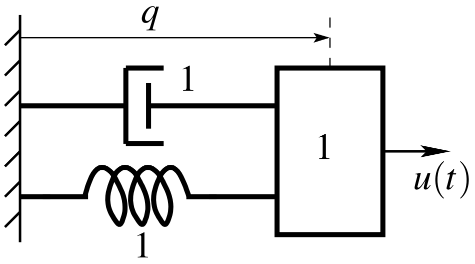



Figure 3.1: A damped spring-mass system with $m = 1$ kg, $c = 1$ Ns/m, and $k = 1$ N/m.

*The objective is to find an input u, for all initial conditions $x_0 \in \mathbb{R}^2$, that minimizes the functional*

$$J(x_0, u) = \frac{1}{2}\int_0^{\infty} \left( \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \right) dt$$

*On computing* $\mathtt{det}(sE - H)$*, we have* $\mathtt{n_s} = 1$ *and* $\mathtt{n_f} = 1$*. The set of eigenvalues of the corresponding Hamiltonian system is* $\Lambda = \{1, -1\}$ *and hence the lambda-set is* $\{-1\}$*. Corresponding to this lambda-set an eigenvector is* $V_{\mathtt{e}\Lambda} = \mathtt{col}(2, -2, 1, 0, 2)$*. Therefore,* $V_{1\Lambda} = \mathtt{col}(2, -2)$*,* $V_{2\Lambda} = \mathtt{col}(1, 0)$*, and* $V_{3\Lambda} = 2$*. Using Theorem 2.30, we get*

$$X_\Lambda = \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix} = \left[\begin{array}{cc} 2 & 0 \\ -2 & 1 \\ \hline 1 & 0 \\ 0 & 0 \end{array}\right] \Rightarrow K_{\mathtt{max}} = X_{2\Lambda}X_{1\Lambda}^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$

*Therefore, minimum energy the damped spring mass system can attain is* $x_0^T K_{\mathtt{max}} x_0 = \dfrac{q(0)^2}{2}$*, where* $q(0)$ *is the initial position of the damped spring-mass system. We design a controller using Theorem 3.12 to confine the trajectories of the system in Figure 3.1 to its optimal trajectories.*

$$F_{\mathtt{p}} = \begin{bmatrix} V_{3\Lambda} & f \end{bmatrix} X_{1\Lambda}^{-1} = \begin{bmatrix} 2 & f \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1+f & f \end{bmatrix}$$

$$F_{\mathtt{d}} = \begin{bmatrix} 0 & 1 \end{bmatrix} X_{1\Lambda}^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

*The closed-loop system obtained on application of the feedback* $u(t) = F_{\mathtt{p}}x(t) + F_{\mathtt{d}}\frac{d}{dt}x(t)$ *is*

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f & f \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

*On chosing any* $f \in \mathbb{R} \setminus \{0\}$*, we have an autonomous closed loop system that confines the trajectories of the damped spring-mass system to the ones that minimize the total energy of the system.*

From Theorem 3.12 it is evident that LQR problems can be solved using PD state-feedback controllers. However, Theorem 3.12 is applicable only for those LQR problems that admit Hamiltonian systems $\Sigma_{\mathrm{Ham}}$ with $\sigma(E,H)\cap j\mathbb{R}=\emptyset$ and $\mathtt{det}(sE-H)\neq 0$ (see Theorem 2.30 for these assumptions). Hence, a relevant question here is: how do we solve the singular LQR Problem 3.5 when the aforementioned assumptions are not met? We discuss about the implications of these assumptions for single-input LQR problems next.

(i) $\underline{\mathtt{det}(sE-H)=0}$ : A singular Hamiltonian matrix pencil implies that the corresponding Hamiltonian system $\Sigma_{\mathrm{Ham}}$ is non-autonomous. However, we show next that for a single-input system with $Q\neq 0$, the Hamiltonian system is always autonomous.

A single-input singular LQR problem admits an autonomous Hamiltonian system

**Lemma 3.16.** *Consider the singular LQR Problem 3.5 with the corresponding Hamiltonian system $\Sigma_{\mathrm{Ham}}$ as defined in equation* (2.11) *with $Q\neq 0$. Then, the Hamiltonian system $\Sigma_{\mathrm{Ham}}$ is autonomous.*

*Proof:* To the contrary, assume the system $\Sigma_{\mathrm{Ham}}$ to be non-autonomous. Note that the transfer function of the Hamiltonian system $\Sigma_{\mathrm{Ham}}$ is given by the rational function $H(s):=\widehat{c}(sI_{2\mathrm{n}}-\widehat{A})^{-1}\widehat{b}$, where $\widehat{A},\widehat{b},\widehat{c}$ are as defined in equation (2.11). From Proposition 2.17 it is evident that $\Sigma_{\mathrm{Ham}}$ is non-autonomous if and only if $H(s)=0$. For $H(s)$ to be identically zero, all the Markov parameters of $H(s)$ must be zero, i.e., $\widehat{c}\widehat{A}^{\ell}\widehat{b}=0$ for all $\ell\in\mathbb{N}\cup\{0\}$.

We first claim that if $\widehat{c}\widehat{A}^{\ell}\widehat{b}=0$ for all $\ell\in\mathbb{N}\cup\{0\}$, then $QA^kb=0$ for all $k\in\mathbb{N}\cup\{0\}$. We prove this using induction and the fact that $Q\geqslant 0$.
*Base case:* ($k=0$) For $\ell=1$, we know that

$$\widehat{c}\widehat{A}\widehat{b}=0\Rightarrow\begin{bmatrix}0 & b^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}b\\ 0\end{bmatrix}=b^TQb=0\Rightarrow Qb=0.$$

Assume $QA^ib=0$ for $0\leqslant i\leqslant k-1$. We prove that $QA^kb=0$.

$$\begin{aligned}\widehat{c}\widehat{A}^{(2k+1)}\widehat{b}&=\begin{bmatrix}0 & b^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}^{2k-1}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}b\\ 0\end{bmatrix}\\ &=\begin{bmatrix}-(Qb)^T & -(Ab)^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}^{2k-1}\begin{bmatrix}Ab\\ -Qb\end{bmatrix}\\ &=\begin{bmatrix}0 & -(Ab)^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}^{2k-3}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}Ab\\ 0\end{bmatrix}\\ &=\begin{bmatrix}0 & (A^2b)^T\end{bmatrix}\begin{bmatrix}A & 0\\ -Q & -A^T\end{bmatrix}^{2k-3}\begin{bmatrix}A^2b\\ 0\end{bmatrix}.\quad(\text{Using } QAb=0)\end{aligned}\tag{3.34}$$

Proceeding in a similar way and using the assumption that $QA^ib = 0$ for all $0 \leqslant i \leqslant k-1$, we infer from equation (3.34) that

$$CA^{(2k+1)}b = \begin{bmatrix} 0 & (-1)^k(A^kb)^T \end{bmatrix} \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} (-1)^k(A^kb) \\ 0 \end{bmatrix} = -(A^kb)^T Q(A^kb) = 0$$

$$\Rightarrow QA^kb = 0.$$

Thus, we can write $Q\begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n}-1}b \end{bmatrix} = 0$. However, for a single-input controllable system $\begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n}-1}b \end{bmatrix}$ is a nonsingular matrix. Therefore, we must have $Q = 0$. This is a contradiction and hence the transfer function of $\Sigma_{\text{Ham}}$ is a non-zero rational function. Thus, $\Sigma_{\text{Ham}}$ is an autonomous system. ■

Note that for a regular LQR problem, $\Sigma_{\text{Ham}}$ is known to be autonomous. Thus, all LQR problems with the underlying system being single-input admit autonomous Hamiltonian system. Interestingly, this is not the case for multi-input systems: see Example 4.12 and Example 4.13.

(ii) $\underline{\sigma(E,H) \cap j\mathbb{R} \neq \emptyset}$ : This is the class of LQR problems that admit Hamiltonian systems with eigenvalues on the imaginary axis. Since the unmixing condition in Definition 2.18 will be violated in such a case, we cannot partition the eigenvalues of $(E,H)$ into Lambda-set. However, taking a cue from [FMX02], we can relax the condition of unmixing for the eigenvalues of $(E,H)$ on the imaginary axis. Such a relaxation would mean that all the Lambda-sets of $\texttt{det}(sE-H)$ would contain the eigenvalues of $(E,H)$ on $j\mathbb{R}$. We might still be able to compute the maximal solution of the corresponding LQR LMI using Theorem 2.30. However, the closed-loop system $\Sigma_{\texttt{closed}}$ obtained would always admit eigenvalues on the imaginary axis. This implies that the states of the closed loop system would be periodic in nature and hence, would not converge to zero. In such a case the functional in equation (3.2) won't converge. Therefore, for infinite-horizon LQR problems it is a common practice to implicitly assume $\sigma(E,H) \cap j\mathbb{R} = \emptyset$ in order to guarantee solution to the problem at hand.

Interestingly, for the LQR Problem 3.5 with $(Q,A)$ observable, the assumption $\sigma(E,H) \cap j\mathbb{R} = \emptyset$ is equivalent to the fact the system $\Sigma$ does not admit any transmission zeros on the $j\mathbb{R}$. We establish this in the next lemma.

$\sigma(E,H) \cap j\mathbb{R} = \emptyset$ if and only if the system $\Sigma$ has no transmission zeros on $j\mathbb{R}$

**Lemma 3.17.** *Consider the singular LQR Problem 3.5 with Hamiltonian matrix pair $(E,H)$ as defined in equation* (2.11). *Define $Q =: C^TC$ and let $y(t) := Cx(t)$, where $C \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$ with $\texttt{rank}(Q) = \mathtt{r}$. Let the system $\Sigma$ be $(A,B)$ controllable and $(C,A)$ observable. Define $G(s) := C(sI_{\mathtt{n}} - A)^{-1}B$. Then,*

$$j\omega \in \sigma(E,H) \text{ if and only if } j\omega \text{ is a transmission zero of } G(s).$$

*Proof:* Since the system is $(A,B)$ is controllable and $(C,A)$ is observable, without loss of generality, we can assume a right co-prime factorization of $G(s)$ to be given by $G(s) =: N(s)D(s)^{-1}$.

Since $G(s)$ is a column-vector of rational functions, we must have $D(s)$ to be a polynomial (denoted by d(s)), and $N(s)$ to be an $\mathtt{m} \times 1$ vector of polynomials. Note that, with this description of $G(s) = \dfrac{N(s)}{d(s)}$, the set of transmission zeros of $G(s)$ is given by the zeros of the polynomial vector $N(s)$: see [SF77] for the link between zeros and transmission zeros of a system.

From Statement *(1)* of Lemma 2.29, we know that

$$\mathtt{det}(sE - H) = N(-s)^T N(s). \tag{3.35}$$

Note that $j\omega$ is a zero of $N(s)$ if and only if $j\omega$ is a zero of $N(-s)^T N(s)$. Using this along with the fact that zeros of $N(s)$ are the transmission zeros of $G(s)$, we infer from equation (3.35): $j\omega \in \sigma(E,H)$ if and only $j\omega$ is a transmission zero of $G(s)$. ■

Thus the assumption that $\sigma(E,H) \cap j\mathbb{R} = \emptyset$ can be guaranteed by ensuring that the matrix of rational functions $C(sI_{\mathtt{n}} - A)^{-1}B$ has not finite zero on the imaginary axis. This is true for almost all A,B,C matrices; the word "almost" in the title is added hence.

## 3.5 Summary

In this chapter, using the subspaces involved in the computation of the maximal rank-minimizing solution $K_{\mathtt{max}}$ of an LQR LMI, we characterized the optimal trajectories of a system corresponding to a singular LQR problem. We showed that if the initial condition of the system are written in a suitable basis (columns of $V_{1\Lambda}$ and $W_1$ from Theorem 2.30) then the optimal trajectory shows a nice structure (Lemmas 3.7 and 3.6). Taking a clue from this structure we design controllers for the system that confines the trajectories of the system to its optimal trajectories. We show that such controllers need to be PD-controllers (Theorem 3.12). Further, we explicitly characterize the slow and fast subspaces of the singular descriptor system obtained on application of the proposed PD-control law (Theorem 3.13).

Interestingly, it has been shown in [FN14] and [FN16] that, contrary to the notion that singular LQR problems cannot be solved using static state-feedback, singular LQR problems can indeed be solved using static state-feedback law provided such problems admit solutions to a special form of the ARE called the *constrained generalized continuous ARE* (CGCARE). Hence, a natural question is: What is the link between CGCARE and the theory that we have developed in this chapter? We explore this link in the next chapter.

# Chapter 4

# Constrained generalized continuous ARE (CGCARE)

## 4.1 Introduction

As motivated in Section 3.4 of Chapter 3 a singular LQR problem is known to admit optimal trajectories from the space of impulsive-smooth distributions. Hence, singular LQR problems might not be solvable using static state-feedback. However, in [FN14] and [FN16] it has been established that the singular LQR Problem 2.1 can be solved using static state-feedback if and only if the problem admits solution to an equation of the following form:

$$\begin{cases} A^T K + KA + Q - (KB + S)R^{\dagger}(B^T K + S^T) = 0 \\ \texttt{ker}\, R \subseteq \texttt{ker}\,(S + KB). \end{cases} \tag{4.1}$$

The condition $\texttt{ker}\, R \subseteq \texttt{ker}\,(S + KB)$ in equation (4.1) pertains to the algebraic relations that the solutions of an LQR LMI, corresponding to a singular LQR problem, has to satisfy. Due to the presence of an ARE and a set of constrained (algebraic) equations in equation (4.1), such an equation is known in the literature as the *constrained generalized continuous ARE (CGCARE)*. Since solvability of CGCARE guarantees solution of a singular LQR problem using static state-feedback, for the case when a singular LQR problem admits a CGCARE solution the optimal trajectories are from the space of infinitely differentiable functions. A natural question, therefore, is: when does a CGCARE admit a solution? In this chapter, we formulate necessary and sufficient conditions for existence of a solution to the CGCARE. This is the first main result of this chapter (Theorem 4.8). A direct corollary of this result reveals that a CGCARE admits a solution only if the determinant of the corresponding Hamiltonian pencil is identically zero. Another consequence of the first main result of this chapter is that, for a singular LQR problem, in order for the corresponding CGCARE to have solutions, it is necessary and sufficient that the Hamiltonian system is non-autonomous with input cardinality precisely equal to the dimension of nullspace of input cost matrix $R$.

Having formulated the necessary and sufficient conditions for CGCARE solvability, the next relevant question is: how often are these conditions satisfied? To this end, in Section 4.4, we first show that the determinant of a Hamiltonian pencil is *generically* nonzero. This is the second main result of this chapter (Theorem 4.16). Note that, it is tempting here to argue that the determinant of a matrix pencil being nonzero is a generic property because it evaluates to zero over a proper *algebraic variety*. However, since we are dealing with singular LQR problems, and since an infinite-horizon LQR problem is generically *not* singular, loosely speaking, we need to prove genericity of the determinant of a matrix pencil being nonzero over a curved hypersurface in the Euclidean space. This makes the problem challenging and non-trivial. In order to overcome this challenge, we use the notion of genericity based on perturbation (see Definition 4.15). We elaborate on this in Section 4.4. Finally, using the second main result (Theorem 4.16), we infer that CGCAREs corresponding to singular infinite-horizon LQR problems are generically unsolvable. This is the third main result of this chapter (Theorem 4.22). This result implies that almost all singular LQR problems cannot be solved using static state-feedback. Hence, singular LQR problems need to be solved using PD state-feedback controllers as described in Chapter 3. Note that all the results in this chapter are for multi-input systems and hence we revisit the preliminaries for Hamiltonian systems in the next section. Although most of the results in the next section (Section 4.2) are known in the literature, we present these results as lemmas and reprove them for the sake of completeness and ease of exposition.

## 4.2 Hamiltonian system for multi-input systems

Recall from Section 2.2.6 that the matrix pair $(E,H)$ is called the Hamiltonian matrix pair (see equation (2.7)). Similar to equation (2.11), the Hamiltonian system corresponding to the Hamiltonian matrix pair $(E,H)$ in equation (2.7) is given by the following equation:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{m},\mathtt{m}} \end{bmatrix}}_{E} \frac{d}{dt}\begin{bmatrix} x \\ z \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & R \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u \end{bmatrix} \tag{4.2}$$

Then, the output-nulling representation corresponding to the system in equation (4.2) is:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \widehat{A}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{B}u, \quad 0 = \widehat{C}\begin{bmatrix} x \\ z \end{bmatrix} + Ru, \tag{4.3}$$

where $\widehat{A} := \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}$, $\widehat{B} := \begin{bmatrix} B \\ -S \end{bmatrix}$ and $\widehat{C} := \begin{bmatrix} S^T & B^T \end{bmatrix}$. We represent the system in equation (4.3) by $\Sigma_{\text{Ham}}$. In order to present the main results of this chapter we first need to show that the singular LQR Problem 2.1 can be transformed to a convenient form without loss of generality. Hence, in the next lemma, we show how Problem 2.1 can be transformed to such a convenient form using a change of basis on the input space of the system. Note that for a given singular LQR Problem 2.1, since $R$ is symmetric, there exists a orthogonal matrix $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ such that $U^T RU = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \widehat{R} \end{smallmatrix}\right]$, where $\widehat{R} = \widehat{R}^T \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$ and $\widehat{R}$ is nonsingular. We use such orthogonal matrices to define a transformation on the input-space of the system $\Sigma$ defined in Problem 2.1. This leads to a transformed form of the LQR Problem 2.1 which we use in the sequel.

A method to transform the singular LQR Problem 2.1 to a convenient form

**Lemma 4.1.** *Consider the singular LQR Problem 2.1 where* $\mathtt{rank}(R) =: \mathtt{r} < \mathtt{m}$. *Let* $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ *be an orthogonal matrix such that* $U^T RU = \mathtt{diag}(0, \widehat{R})$, *where* $\widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$ *and* $\widehat{R} > 0$. *Define* $BU =: \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ *and* $SU =: \begin{bmatrix} S_1 & S_2 \end{bmatrix}$, *where* $B_1, S_1 \in \mathbb{R}^{\mathtt{n}\times(\mathtt{m}-\mathtt{r})}$ *and* $B_2, S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. *Then, the following statements hold:*

*(1)* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0 \Leftrightarrow \begin{cases} S_1 = 0, \\ Q - S_2\widehat{R}^{-1}S_2^T \geqslant 0. \end{cases}$

*(2)* $u^*$ *is a solution to the singular LQR Problem 2.1 if and only if* $U^T u^* := \mathtt{col}(u_1^*, u_2^*)$ *minimizes*

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix} dt. \tag{4.4}$$

*(3)* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *is a solution of the CGCARE* (4.1) *if and only if* $K$ *is a solution of the following equations*

$$A^T K + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^T K + S_2^T) = 0, \text{ and } KB_1 = 0. \tag{4.5}$$

*Proof: (1):* Define $L := \left[\begin{smallmatrix} I_{\mathtt{n}} & 0 \\ 0 & U \end{smallmatrix}\right] \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}$. It is evident that $L^T \left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right] L = \left[\begin{smallmatrix} Q & S_1 & S_2 \\ S_1^T & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{smallmatrix}\right]$. Since $L$ is invertible and symmetric, by Sylvester's law of inertia, it follows that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0 \quad \text{if and only if} \quad \begin{bmatrix} Q & S_1 & S_2 \\ S_1^T & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0. \tag{4.6}$$

Since $\widehat{R} > 0$, taking Schur complement with respect to $\widehat{R}$ it follows that

$$\begin{bmatrix} Q & S_1 & S_2 \\ S_1^T & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0 \text{ if and only if } \begin{bmatrix} Q - S_2\widehat{R}^{-1}S_2^T & S_1 \\ S_1^T & 0 \end{bmatrix} \geqslant 0 \Leftrightarrow \begin{cases} S_1 = 0, \text{ and} \\ Q - S_2\widehat{R}^{-1}S_2^T \geqslant 0. \end{cases} \tag{4.7}$$

Thus, from equation (4.6) and equation (4.7), Statement *(1)* follows.
*(2):* Clearly, for all $\mathtt{col}(x(t),u(t))$ that satisfies the state-space system dynamics $\frac{d}{dt}x = Ax+Bu$,

$$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt = \int_0^\infty \begin{bmatrix} x \\ U^T u \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x \\ U^T u \end{bmatrix} dt.$$

Thus, $u^*$ is a solution to the singular LQR Problem 2.1 if and only if $U^T u^*$ is a solution to the singular LQR Problem 4.4.
*(3):* Since $K$ is a solution to the CGCARE (4.1), we have

$$\begin{aligned} & A^T K + KA + Q - (KB+S)R^\dagger(B^T K + S^T) = 0 \\ \Leftrightarrow\ & A^T K + KA + Q - (KB+S)U \begin{bmatrix} 0 & 0 \\ 0 & \widehat{R}^{-1} \end{bmatrix} U^T (B^T K + S^T) = 0 \\ \Leftrightarrow\ & A^T K + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^T K + S_2^T) = 0. \end{aligned}$$

Further, we also have

$$\begin{aligned} \mathtt{ker}\, R \subseteq \mathtt{ker}\,(S+KB) & \Leftrightarrow \mathtt{ker}\,(U^T R U) \subseteq \mathtt{ker}\,(U^T(S+KB)U) \\ & \Leftrightarrow \mathtt{ker} \begin{bmatrix} 0 & 0 \\ 0 & \widehat{R} \end{bmatrix} \subseteq \mathtt{ker} \begin{bmatrix} KB_1 & KB_2 + S_2 \end{bmatrix} \Leftrightarrow KB_1 = 0. \end{aligned}$$

This completes the proof of the lemma. ■

Hence the singular infinite-horizon LQR Problem 2.1 can be rewritten in terms of the new input $\tilde{u}$ as follows:

**Problem 4.2.** *Consider a system with state-space dynamics* $\frac{d}{dt}x = Ax + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, *where* $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B_1 \in \mathbb{R}^{\mathtt{n}\times(\mathtt{m}-\mathtt{r})}$, $B_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. *Then, for every initial condition* $x_0$, *find an input* $\tilde{u} :=$ $\mathtt{col}(u_1,u_2)$ *that minimizes the functional*

$$J(x_0,\tilde{u}) := \int_0^\infty \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix} dt, \text{ where } \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0, \widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}, \text{ and } \widehat{R} > 0. \quad (4.8)$$

Clearly, the CGCARE corresponding to the transformed LQR Problem 4.2 is given by equation (4.5). Further, the Hamiltonian system (4.2) corresponding to the transformed LQR

Problem 4.2 is:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & 0_{\mathtt{m-r,m-r}} & 0 \\ 0 & 0 & 0 & 0_{\mathtt{r,r}} \end{bmatrix}}_{\widehat{E}} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B_1 & B_2 \\ -Q & -A^T & 0 & -S_2^T \\ 0 & B_1^T & 0 & 0 \\ S_2^T & B_2^T & 0 & \widehat{R} \end{bmatrix}}_{\widehat{H}} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \end{bmatrix} \tag{4.9}$$

Since $\widehat{R}$ is nonsingular in equation (4.9), we can eliminate $u_2$ from the equation: this gives the following system of differential-algebraic equations (DAEs) equivalent to equation (4.9):

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{m-r,m-r}} \end{bmatrix}}_{E_{\mathtt{r}}} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \widetilde{A} & -\widetilde{M} & B_1 \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & B_1^T & 0 \end{bmatrix}}_{H_{\mathtt{r}}} \begin{bmatrix} x \\ z \\ u_1 \end{bmatrix}, \tag{4.10}$$

where $\widetilde{A} := A - B_2\widehat{R}^{-1}S_2^T$, $\widetilde{M} := B_2\widehat{R}^{-1}B_2^T$ and $\widetilde{Q} := Q - S_2\widehat{R}^{-1}S_2^T$. We call the matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$, the *reduced Hamiltonian pencil* and the system governed by the DAEs in equation (4.10) the *reduced Hamiltonian system*. The output-nulling representation of the reduced Hamiltonian system (4.10) is therefore given by the following equations:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = A_{\mathtt{r}} \begin{bmatrix} x \\ z \end{bmatrix} + B_{\mathtt{r}} u_1 \text{ and } 0 = C_{\mathtt{r}} \begin{bmatrix} x \\ z \end{bmatrix}, \tag{4.11}$$

where $A_{\mathtt{r}} := \begin{bmatrix} \widetilde{A} & -\widetilde{M} \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times 2\mathtt{n}}$, $B_{\mathtt{r}} := \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times(\mathtt{m-r})}$ and $C_{\mathtt{r}} := \begin{bmatrix} 0 & B_1^T \end{bmatrix} \in \mathbb{R}^{(\mathtt{m-r})\times 2\mathtt{n}}$.

In what follows, we establish a relation among the Hamiltonian matrix pair $(E,H)$ corresponding to LQR Problem 2.1 (equation (4.2)), the Hamiltonian matrix pair $(\widehat{E},\widehat{H})$ corresponding to the transformed LQR Problem 4.2 (equation (4.9)) and the matrix pair $(E_{\mathtt{r}},H_{\mathtt{r}})$ corresponding to the reduced Hamiltonian system in equation 4.10.

Relation between the characteristic polynomials of $(E,H)$, $(\widehat{E},\widehat{H})$, and $(E_{\mathtt{r}},H_{\mathtt{r}})$

**Lemma 4.3.** *Let the matrix pairs* $(E,H)$, $(\widehat{E},\widehat{H})$, *and* $(E_{\mathtt{r}},H_{\mathtt{r}})$ *be as defined in equations* (4.2), (4.9), *and* (4.10), *respectively, with the transformation from* $(E,H)$ *to* $(\widehat{E},\widehat{H})$ *being done through an orthogonal matrix* $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ *as in Lemma 4.1. Then,*

$$\mathtt{det}(sE - H) = \mathtt{det}(s\widehat{E} - \widehat{H}) = (-1)^{\mathtt{r}}\mathtt{det}(\widehat{R}) \times \mathtt{det}(sE_{\mathtt{r}} - H_{\mathtt{r}}). \tag{4.12}$$

*Proof:* Recall from Lemma 4.1 that the orthogonal matrix $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ is such that $U^TRU = \mathtt{diag}\left(0_{\mathtt{m-r,m-r}}, \widehat{R}\right)$, $BU = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, and $SU = \begin{bmatrix} 0_{\mathtt{n,m-r}} & S_2 \end{bmatrix}$. Define $V := \mathtt{diag}(I_{\mathtt{n}}, I_{\mathtt{n}}, U) \in$

$\mathbb{R}^{(2\mathtt{n}+\mathtt{m})\times(2\mathtt{n}+\mathtt{m})}$. It is easy to verify that

$$V^T H V = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & U^T \end{bmatrix} \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & R \end{bmatrix} \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & U \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2 \\ -Q & -A^T & 0 & -S_2 \\ 0 & B_1^T & 0 & 0 \\ S_2^T & B_2^T & 0 & \widehat{R} \end{bmatrix} = \widehat{H}, \tag{4.13}$$

$$V^T E V = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & U^T \end{bmatrix} \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{m},\mathtt{m}} \end{bmatrix} \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & U \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{m},\mathtt{m}} \end{bmatrix} = \widehat{E}. \tag{4.14}$$

Since $V$ is orthogonal, $\mathtt{det}(V) = \pm 1$. Therefore, using equation (4.13) and equation (4.14) to compute $\mathtt{det}(sE - H)$, we have

$$\mathtt{det}(sE-H) = \mathtt{det}(sV\widehat{E}V^T - V\widehat{H}V^T) = \mathtt{det}(V)\mathtt{det}(s\widehat{E}-\widehat{H})\mathtt{det}(V^T) = \mathtt{det}(s\widehat{E}-\widehat{H}). \tag{4.15}$$

Next, define

$$Z_1 := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & I_{\mathtt{m}-\mathtt{r}} & 0 \\ -\widehat{R}^{-1}S_2^T & -\widehat{R}^{-1}B_2^T & 0 & I_{\mathtt{r}} \end{bmatrix}, \text{ and } Z_2 := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & -B_2\widehat{R}^{-1} \\ 0 & I_{\mathtt{n}} & 0 & S_2\widehat{R}^{-1} \\ 0 & 0 & I_{\mathtt{m}-\mathtt{r}} & 0 \\ 0 & 0 & 0 & I_{\mathtt{r}} \end{bmatrix}.$$

Clearly, $\mathtt{det}(Z_2) = \mathtt{det}(Z_1) = 1$. Further, $Z_2\widehat{E}Z_1 = \mathtt{diag}(E_{\mathtt{r}}, 0_{\mathtt{r},\mathtt{r}})$ and $Z_2\widehat{H}Z_1 = \mathtt{diag}(H_{\mathtt{r}}, \widehat{R})$. Upon defining $\widetilde{E} := N\widehat{E}M$, and $\widetilde{H} := N\widehat{H}M$, we get

$$\mathtt{det}(s\widehat{E}-\widehat{H}) = \mathtt{det}(s\widetilde{E}-\widetilde{H}) = (-1)^{\mathtt{r}}\mathtt{det}(\widehat{R}) \times \mathtt{det}(sE_{\mathtt{r}} - H_{\mathtt{r}}). \tag{4.16}$$

Equation (4.15) and equation (4.16) together gives equation (4.12). ■

Another result that would be required for the main theorem of this section is the relation between transfer function $G(s)$ of a system and the corresponding Hamiltonian pencil $(E, H)$. This is a generalization of Lemma 2.29 to the multi-input case.

Relation between Popov's function $G(-s)^T G(s)$ and $\sigma(E,H)$

**Lemma 4.4.** *Consider the singular LQR Problem 2.1. Assume* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} =: \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C\ D]$, *where* $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, *and* $\begin{bmatrix} C & D \end{bmatrix}$ *is full row-rank. Define* $G(s) := C(sI_{\mathtt{n}} - A)^{-1}B + D$. *Let* $G(s) =: \dfrac{N(s)}{d(s)}$, *where* $d(s) := \mathtt{det}(sI_{\mathtt{n}} - A)$ *and* $N(s) \in \mathbb{R}[s]^{\mathtt{p}\times\mathtt{m}}$. *Define* $\mathtt{rootnum}\left(G(-s)^T G(s)\right) := \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right)$. *Let the corresponding Hamiltonian pencil pair* $(E,H)$ *be as given in equation* (4.2). *Then, the following statements hold:*

*(1)* $G(-s)^T G(s) = \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} + R$.

*(2)* $\mathtt{rootnum}\left(G(-s)^T G(s)\right) = \sigma(E,H)$.

*Further, if* $\sigma(E,H) \cap j\mathbb{R} = \emptyset$, *then* $\mathtt{det}(sE - H)$ *admits a Lambda-set.*

*Proof: (1):* Note that $G(-s)^T = -B^T(sI_{\mathtt{n}} + A^T)^{-1}C^T$. Using this fact we have

$$\begin{aligned}
\widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} &= \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} sI_{\mathtt{n}} - A & 0 \\ Q & sI_{\mathtt{n}} + A^T \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} (sI_{\mathtt{n}} - A)^{-1} & 0 \\ -(sI_{\mathtt{n}} + A^T)^{-1}C^T C(sI_{\mathtt{n}} - A)^{-1} & (sI_{\mathtt{n}} + A^T)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \\
&= -B^T(sI_{\mathtt{n}} + A^T)^{-1}C^T C(sI_{\mathtt{n}} - A)^{-1}B = G(-s)^T G(s).
\end{aligned}$$

*(2):* Note that we can write $\mathtt{det}(sE - H)$ as follows:

$$\begin{aligned}
\mathtt{det}(sE - H) &= \mathtt{det}\begin{bmatrix} sI_{2\mathtt{n}} - \widehat{A} & -\widehat{B} \\ -\widehat{C} & -R \end{bmatrix} = \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) \times \mathtt{det}\left(-R - \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B}\right) \\
&= \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) \times (-1)^{\mathtt{m}} \mathtt{det}\left(G(-s)^T G(s)\right) \\
&= (-1)^{\mathtt{m}} \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) \times \frac{\mathtt{det}\left(R \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) + \widehat{C}\,\mathtt{adj}(sI_{2\mathtt{n}} - \widehat{A})\widehat{B}\right)}{\mathtt{det}(sI_{2\mathtt{n}} - \widehat{A})} \\
&= (-1)^{\mathtt{m}} \times \mathtt{det}\left(R \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) + \widehat{C}\,\mathtt{adj}(sI_{2\mathtt{n}} - \widehat{A})\widehat{B}\right). \qquad (4.17)
\end{aligned}$$

From Statement *(1)* of this lemma, we have

$$\begin{aligned}
G(-s)^T G(s) = \frac{\left(\widehat{C}\,\mathtt{adj}(sI_{2\mathtt{n}} - \widehat{A})\widehat{B}\right)\widehat{B}}{\mathtt{det}(sI_{2\mathtt{n}} - \widehat{A})} + R &= \frac{R \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) + \widehat{C}\,\mathtt{adj}(sI_{2\mathtt{n}} - \widehat{A})\widehat{B}}{\mathtt{det}(sI_{2\mathtt{n}} - \widehat{A})} \\
&= \frac{R \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) + \widehat{C}\,\mathtt{adj}(sI_{2\mathtt{n}} - \widehat{A})\widehat{B}}{d(-s)d(s)}. \qquad (4.18)
\end{aligned}$$

Since $G(s) = \frac{N(s)}{d(s)}$, we must have $G(-s)^T G(s) = \frac{N(-s)^T N(s)}{d(-s)d(s)}$. Therefore, from equation (4.18) we have

$$N(-s)^T N(s) = R \times \mathtt{det}(sI_{2\mathtt{n}} - \widehat{A}) + \widehat{C}\,\mathtt{adj}(sI_{2\mathtt{n}} - \widehat{A})\widehat{B}. \tag{4.19}$$

Using equation (4.19) in equation (4.17), we have

$$\mathtt{det}(sE - H) = (-1)^{\mathtt{m}} \times \mathtt{det}\left[N(-s)^T N(s)\right] \Rightarrow \sigma(E,H) = \mathtt{rootnum}\left(G(-s)^T G(s)\right). \tag{4.20}$$

This completes the proof of Statement *(2)* of the lemma.

Note that if $\lambda \in \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right)$, then $-\lambda \in \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right)$. Further, since $\mathtt{det}\left[N(-s)^T N(s)\right] \in \mathbb{R}[s]$, if $\lambda \in \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right)$, then we must have $\bar{\lambda} \in \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right)$. Thus, the roots of $\mathtt{det}\left[N(-s)^T N(s)\right]$ are symmetric about the real and imaginary-axis of the $\mathbb{C}$-plane. Therefore, $\mathtt{det}\left[N(-s)^T N(s)\right]$ is a even-degree polynomial. From Statement *(2)* we know that $\sigma(E,H) = \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right)$. Therefore, $\mathtt{det}(sE - H)$ is a even-degree polynomial as well. Let $\mathtt{degdet}(sE - H) =: 2\mathtt{n_s}$. Since $\sigma(E,H) \cap j\mathbb{R} = \emptyset \Rightarrow \mathtt{roots}\left(\mathtt{det}\left[N(-s)^T N(s)\right]\right) \cap j\mathbb{R} = \emptyset$, we must have $\mathtt{n_s}$ roots of $\mathtt{det}(sE - H)$ in $\mathbb{C}_-$ and the rest $\mathtt{n_s}$ in $\mathbb{C}_+$. By the Definition of Lambda-sets in Definition 2.18, the collection of the roots of $\mathtt{det}(sE - H)$ in $\mathbb{C}_-$ is a Lambda-set of $\mathtt{det}(sE - H)$. ∎

Now that we have proved some of the auxiliary results required in this chapter, in the next section we answer the question: When is the CGCARE solvable? Recall that the LQR LMI corresponding to LQR Problem 2.1 is given by:

$$\mathscr{L}(K) := \begin{bmatrix} A^T K + KA + Q & KB + S \\ B^T K + S^T & R \end{bmatrix} \geqslant 0. \tag{4.21}$$

An important result related to the rank of the LQR LMI (4.21), proposed in [Sch83], is crucially required to formulate conditions on solvability of CGCARE in the next section. We present this result as a proposition next for ease of reference.

**Proposition 4.5.** [Sch83, Theorem 1] *Consider the LQR LMI* (4.21) *and the transfer function $G(s)$ as defined in Lemma 4.4. The minimal rank of $\mathscr{L}(K)$, where $K$ varies over the set of symmetric matrices satisfying $\mathscr{L}(K) \geqslant 0$, is equal to* $\mathtt{nrank}\,(G(s))$.

Another result in [Sch83] related to the normal rank of a matrix with rational functions as element is required in this chapter and we present it next (see [Sch83, Lemma 2] for a proof). Before that we define normal rank of a matrix with rational functions.

**Definition 4.6.** [Kai80, Section 6.3] *The normal rank of a rational polynomial matrix $G(s) \in \mathbb{R}(s)^{\mathtt{n}\times\mathtt{p}}$, represented by* $\mathtt{nrank}(G(s))$, *is defined as*

$$\begin{aligned} \mathtt{nrank}(G(s)) :&= \mathtt{max}\left\{\mathtt{rank}(G(\lambda)) \,|\, \lambda \in \mathbb{C} \textit{ and } G(s) \textit{ is analytic at } \lambda\right\} \\ &= \mathtt{max}\left\{\mathtt{rank}(G(\lambda)) \,|\, \lambda \in j\mathbb{R} \textit{ and } G(s) \textit{ is analytic at } \lambda\right\}. \end{aligned}$$

**Proposition 4.7.** [Sch83, Lemma 2] *Consider $W(s) \in \mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$. Then,*

$$\mathtt{nrank}\,(W(s)) = \mathtt{nrank}\left(W(-s)^T W(s)\right).$$

## 4.3 Conditions for solvability of CGCARE

In this section we answer the first question we raised in Section 4.1: when is a CGCARE solvable? The next theorem, the first main result of this chapter, provides a set of necessary and sufficient conditions for the solvability of the CGCARE (4.5).

Necessary and sufficient conditions for the solvability of CGCARE

**Theorem 4.8.** *Consider the singular LQR Problem 2.1 and the transformed LQR Problem 4.2. Let the Hamiltonian pencil pair* $(E,H)$ *be as defined in equation* (4.2)*, and the corresponding reduced Hamiltonian system be as given in equation* (4.10)*. Assume* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} =: \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$*, where* $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$*,* $D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$*, and* $\begin{bmatrix} C & D \end{bmatrix}$ *is full row-rank. Define* $G(s) := C(sI_{\mathtt{n}} - A)^{-1}B + D$ *and* $\tilde{Q} := Q - S_2\widehat{R}^{-1}S_2^T$*. Then, the following statements are equivalent:*

*(1) CGCARE* (4.1) *admits a solution.*

*(2) CGCARE* (4.5) *admits a solution.*

*(3)* $\mathtt{nrank}(G(s)) = \mathtt{rank}(D)$.

*(4)* $\mathtt{nrank}(sE - H) = 2\mathtt{n} + \mathtt{rank}(R)$.

*(5)* $C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1}B_{\mathtt{r}} = 0$.

*(6)* $C_{\mathtt{r}}A_{\mathtt{r}}^{\ell}B_{\mathtt{r}} = 0$, *for all* $\ell \in \mathbb{N}$.

*(7)* $\widetilde{Q}\widetilde{A}^k B_1 = 0$, *for all* $k \in \mathbb{N}$.

*Proof: (1)* $\Leftrightarrow$ *(2):* This follows from Statement *(3)* of Lemma 4.1.
*(2)* $\Rightarrow$ *(3):* The LQR LMI corresponding to the transformed LQR Problem 4.2 takes the following form:

$$\mathscr{L}_{\mathtt{tran}}(K) := \begin{bmatrix} A^TK + KA + Q & KB_1 & KB_2 + S_2 \\ B_1^TK & 0 & 0 \\ B_2^TK + S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0. \tag{4.22}$$

Hence the underlying LMIs corresponding to the CGCAREs (4.1) and (4.5) are given by the inequalities (4.21) and (4.22), respectively. Define $Z_1 := \mathtt{diag}(I_{\mathtt{n}}, U) \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}$, where $U^TRU = \mathtt{diag}(0_{\mathtt{m-r},\mathtt{m-r}}, \widehat{R})$. Then, pre- and post-multiplying $\mathscr{L}(K)$ with $Z_1^T$ and $Z_1$, respectively gives

$$Z_1^T\mathscr{L}(K)Z_1 = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & U^T \end{bmatrix} \begin{bmatrix} A^TK + KA + Q & KB + S \\ B^TK + S^T & R \end{bmatrix} \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & U \end{bmatrix} = \mathscr{L}_{\mathtt{tran}}(K) \tag{4.23}$$

Since $Z_1$ is nonsingular and $Z_1^T \mathscr{L}(K) Z_1$ is symmetric, by Sylvester's law of inertia we have from equation (4.23)

$$\mathtt{rank}(\mathscr{L}(K)) = \mathtt{rank}(\mathscr{L}_{\mathtt{tran}}(K)). \tag{4.24}$$

Further, define $Z_2 := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{m-r}} & 0 \\ -\widehat{R}^{-1}(B_2^T K + S_2^T) & 0 & I_{\mathtt{r}} \end{bmatrix}$. Then, we have

$$Z_2^T \mathscr{L}_{\mathtt{tran}}(K) Z_2 = \begin{bmatrix} \mathscr{R}(K) & KB_1 & 0 \\ B_1^T K & 0 & 0 \\ 0 & 0 & \widehat{R} \end{bmatrix} =: \mathscr{L}_{\mathtt{red}}(K), \tag{4.25}$$

where $\mathscr{R}(K) := A^T K + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^T K + S_2^T)$. Since $Z_2$ is nonsingular and $Z_2^T \mathscr{L}_{\mathtt{tran}}(K) Z_2$ is symmetric, from equation (4.24) and equation (4.25), using Sylvester's law of inertia, we have

$$\mathtt{rank}(\mathscr{L}(K)) = \mathtt{rank}(\mathscr{L}_{\mathtt{tran}}(K)) = \mathtt{rank}(\mathscr{L}_{\mathtt{red}}(K)). \tag{4.26}$$

Further, note that for any $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, from equation (4.25) we infer that

$$\mathtt{rank}(\mathscr{L}_{\mathtt{red}}(K)) \geqslant \mathtt{rank}(\widehat{R}) \tag{4.27}$$

Let $K_o$ be a solution to the CGCARE (4.1), i.e. $\mathscr{R}(K_o) = 0$ and $K_o B_1 = 0$. Then, by equation (4.25), $\mathscr{L}_{\mathtt{red}}(K)$ evaluated at $K = K_o$ gives $\mathtt{rank}\,(\mathscr{L}_{\mathtt{red}}(K_o)) = \mathtt{rank}(\widehat{R})$. Using this fact along with equation (4.25), we have

$$\mathtt{rank}\,(\mathscr{L}_{\mathtt{red}}(K_o)) = \mathtt{rank}(\widehat{R}) = \mathtt{rank}\,(D) = \mathtt{rank}(\mathscr{L}(K_o)). \tag{4.28}$$

Thus, from equation (4.27) and equation (4.28) it is evident that the minimum rank of $\mathscr{L}(K)$ among all symmetric matrices $K$ that satisfies $\mathscr{L}(K) \geqslant 0$ is achieved at the solutions of its corresponding CGCARE (4.1). Hence, using Proposition 4.5 we have $\mathtt{rank}\,(\mathscr{L}(K_o)) = \mathtt{nrank}\,(G(s))$. Using this fact in equation (4.28) we have $\mathtt{rank}\,(\mathscr{L}(K_o)) = \mathtt{rank}(D) = \mathtt{nrank}\,(G(s))$.

*(3) $\Rightarrow$ (2):* Since $\mathtt{nrank}\,(G(s)) = \mathtt{rank}(D) = \mathtt{rank}(R)$, from Proposition 4.5 it is clear that the minimum rank that can be attained by $\mathscr{L}(K)$ is $\mathtt{rank}(R)$. From equation (4.25) and equation (4.26) it is clear that $\mathtt{rank}(\mathscr{L}(K)) = \mathtt{rank}(\mathscr{L}_{\mathtt{red}}(K)) = \mathtt{rank}(R)$ only if there exists a $K$ such that $\mathscr{R}(K) = A^T K + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^T K + S_2^T) = 0$ and $KB_1 = 0$. In other words, $\mathtt{nrank}(G(s)) = \mathtt{rank}(D)$ implies that there exists a $K$ that solves the CGCARE (4.1).

*(3) $\Leftrightarrow$ (4):* From Statement *(1)* of Lemma 4.4, we have $G(-s)^T G(s) = \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} + R$, where $\widehat{A}, \widehat{B}$, and $\widehat{C}$ are as defined in equation (4.3). Further, from Proposition 4.7 it is evident that $\mathtt{nrank}(G(s)) = \mathtt{nrank}(G(-s)^T G(s))$. Therefore, we have

$$\mathtt{nrank}(G(s)) = \mathtt{nrank}\left(G(-s)^T G(s)\right) = \mathtt{nrank}\left(\widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} + R\right) \tag{4.29}$$

Define the nonsingular matrices

$$U_1 := \begin{bmatrix} I_{2\mathtt{n}} & 0 \\ \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1} & I_{\mathtt{m}} \end{bmatrix}, \text{ and } U_2 := \begin{bmatrix} I_{2\mathtt{n}} & (sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} \\ 0 & I_{\mathtt{m}} \end{bmatrix}.$$

Recall that $sE - H = \left[\begin{smallmatrix} sI_{2\mathtt{n}} - \widehat{A} & -\widehat{B} \\ -\widehat{C} & -R \end{smallmatrix}\right]$. Therefore, pre-multiplying $(sE - H)$ with $U_1$ and post-multiplying it with $U_2$, we have

$$U_1(sE - H)U_2 = \begin{bmatrix} sI_{2\mathtt{n}} - \widehat{A} & 0 \\ 0 & -\widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} - R \end{bmatrix} \tag{4.30}$$

Since $U_1$ and $U_2$ are nonsingular and $\mathtt{nrank}(sI_{2\mathtt{n}} - \widehat{A}) = 2\mathtt{n}$, from equation (4.30) we have

$$\begin{aligned} \mathtt{nrank}(sE - H) = \mathtt{nrank}\left(U_1(sE - H)U_2\right) &= \mathtt{nrank}(sI_{2\mathtt{n}} - \widehat{A}) + \mathtt{nrank}\left(\widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} + R\right) \\ &= 2\mathtt{n} + \mathtt{nrank}\left(G(-s)^T G(s)\right) \end{aligned} \tag{4.31}$$

Using equation (4.29) and equation (4.31), we therefore infer that

$$\begin{aligned} \mathtt{nrank}(G(s)) = \mathtt{rank}(D) &\Leftrightarrow \mathtt{nrank}\left(G(-s)^T G(s)\right) = \mathtt{rank}(R) \\ &\Leftrightarrow \mathtt{nrank}(sE - H) = 2\mathtt{n} + \mathtt{rank}(R). \end{aligned}$$

*(4) $\Leftrightarrow$ (5):* Define the matrices

$$Z_1 := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & -B_2\widehat{R}^{-1} \\ 0 & I_{\mathtt{n}} & 0 & S_2\widehat{R}^{-1} \\ 0 & 0 & I_{\mathtt{m-r}} & 0 \\ 0 & 0 & 0 & I_{\mathtt{r}} \end{bmatrix}, \quad Z_2 := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & I_{\mathtt{m-r}} & 0 \\ -\widehat{R}^{-1}S_2^T & -\widehat{R}^{-1}B_2^T & 0 & I_{\mathtt{r}} \end{bmatrix}.$$

It is easy to verify that

$$Z_1HZ_2 = \begin{bmatrix} A - B_2\widehat{R}^{-1}S_2^T & -B_2\widehat{R}^{-1}B_2^T & B_1 & 0 \\ -Q + S_2\widehat{R}^{-1}S_2^T & -(A - B_2\widehat{R}^{-1}S_2^T)^T & 0 & 0 \\ 0 & B_1^T & 0 & 0 \\ 0 & 0 & 0 & \widehat{R} \end{bmatrix} = \begin{bmatrix} H_{\mathtt{r}} & 0 \\ 0 & \widehat{R} \end{bmatrix}, Z_1EZ_2 = \begin{bmatrix} E_{\mathtt{r}} & 0 \\ 0 & 0_{\mathtt{r,r}} \end{bmatrix}.$$

Therefore, $Z_1(sE - H)Z_2 = \mathtt{diag}\left(sE_{\mathtt{r}} - H_{\mathtt{r}}, -\widehat{R}\right)$. Since $Z_1$ and $Z_2$ are nonsingular matrices, we have $\mathtt{nrank}(sE - H) = \mathtt{nrank}\left(Z_1(sE - H)Z_2\right)$. Thus,

$$\mathtt{nrank}(sE - H) = \mathtt{nrank}\begin{bmatrix} sE_{\mathtt{r}} - H_{\mathtt{r}} & 0 \\ 0 & -\widehat{R} \end{bmatrix} = 2\mathtt{n} + \mathtt{rank}(R) \Leftrightarrow \mathtt{nrank}\left(sE_{\mathtt{r}} - H_{\mathtt{r}}\right) = 2\mathtt{n}. \tag{4.32}$$

Define the nonsingular matrices

$$U_3 := \begin{bmatrix} I_{2\mathtt{n}} & 0 \\ C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} & I_{\mathtt{m}-\mathtt{r}} \end{bmatrix}, \text{ and } U_4 := \begin{bmatrix} I_{2\mathtt{n}} & (sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}} \\ 0 & I_{\mathtt{m}-\mathtt{r}} \end{bmatrix}.$$

Therefore, pre- and post-multiplying $(sE_{\mathtt{r}} - H_{\mathtt{r}})$ with $U_3$ and $U_4$, respectively we have

$$U_3\,(sE_{\mathtt{r}} - H_{\mathtt{r}})\,U_4 = \begin{bmatrix} sI_{2\mathtt{n}} - A_{\mathtt{r}} & 0 \\ 0 & -C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}} \end{bmatrix} \tag{4.33}$$

Using equation (4.33) to compute $\mathtt{nrank}(sE_{\mathtt{r}} - H_{\mathtt{r}})$ we have

$$\begin{aligned} \mathtt{nrank}\,(sE_{\mathtt{r}} - H_{\mathtt{r}}) &= \mathtt{nrank}\,(U_3\,(sE_{\mathtt{r}} - H_{\mathtt{r}})\,U_4) \\ &= \mathtt{nrank}\left( \begin{bmatrix} sI_{2\mathtt{n}} - A_{\mathtt{r}} & 0 \\ 0 & -C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}} \end{bmatrix} \right) \\ &= \mathtt{nrank}(sI_{2\mathtt{n}} - A_{\mathtt{r}}) + \mathtt{nrank}\left(C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}}\right) \end{aligned} \tag{4.34}$$

Since $\mathtt{nrank}\,(sI_{2\mathtt{n}} - A_{\mathtt{r}}) = 2\mathtt{n}$, using equation (4.32) and equation (4.34) we infer that

$$\mathtt{nrank}\,(sE_{\mathtt{r}} - H_{\mathtt{r}}) = 2\mathtt{n} \Leftrightarrow \mathtt{nrank}\left(C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}}\right) = 0 \Leftrightarrow C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}} = 0. \tag{4.35}$$

Thus, from equation (4.32) and equation (4.35), we have Statement *(4)* $\Leftrightarrow$ Statement *(5)*.
*(5)* $\Leftrightarrow$ *(6):* The impulse response corresponding to $C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}}$ is given by

$$h(t) := C_{\mathtt{r}} e^{A_{\mathtt{r}} t} B_{\mathtt{r}} = \sum_{\ell=0}^{\infty} \frac{t^{\ell}}{\ell!} C_{\mathtt{r}} A_{\mathtt{r}}^{\ell} B_{\mathtt{r}}.$$

Therefore, it is clear that $C_{\mathtt{r}}(sI_{2\mathtt{n}} - A_{\mathtt{r}})^{-1} B_{\mathtt{r}} = 0$ if and only if $C_{\mathtt{r}} A_{\mathtt{r}}^{\ell} B_{\mathtt{r}} = 0$ for all $\ell \in \mathbb{N}$.
*(6)* $\Rightarrow$ *(7):* We prove this using induction.
*Base case:*$(k = 0)$ For $\ell = 1$, from Statement *(6)* we have

$$C_{\mathtt{r}} A_{\mathtt{r}} B_{\mathtt{r}} = 0 \Rightarrow \begin{bmatrix} 0 & B_1^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2 \widehat{R}^{-1} B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = 0 \Rightarrow B_1^T \widetilde{Q} B_1 = 0.$$

From Statement *(1)* of Lemma 4.1, we know that $\widetilde{Q} = Q - S_2 \widehat{R}^{-1} S_2^T \geqslant 0$. Hence, using the property of positive-semidefinite matrices we have $B_1^T \widetilde{Q} B_1 = 0 \Rightarrow \widetilde{Q} B_1 = 0$.

*Induction step:* Assume $\widetilde{Q}\widetilde{A}^i B_1 = 0$ for $0 \leqslant i \leqslant (k-1)$. We prove that $\widetilde{Q}\widetilde{A}^k B_1 = 0$.

$$C_{\mathtt{r}}A_{\mathtt{r}}^{(2k+1)}B_{\mathtt{r}} = \begin{bmatrix} 0 & B_1^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}^{2k-1} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(\widetilde{Q}B_1)^T & -(\widetilde{A}B_1)^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}^{2k-1} \begin{bmatrix} \widetilde{A}B_1 \\ -\widetilde{Q}B_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(\widetilde{A}B_1)^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}^{2k-3} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} \widetilde{A}B_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(\widetilde{Q}\widetilde{A}B_1)^T & (\widetilde{A}^2B_1)^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}^{2k-3} \begin{bmatrix} \widetilde{A}^2B_1 \\ -\widetilde{Q}\widetilde{A}B_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (\widetilde{A}^2B_1)^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}^{2k-3} \begin{bmatrix} \widetilde{A}^2B_1 \\ 0 \end{bmatrix}. \tag{4.36}$$

Proceeding in a similar way and using the assumption that $\widetilde{Q}\widetilde{A}^i B_1 = 0$ for all $0 \leqslant i \leqslant (k-1)$, we infer from equation (4.36) that

$$C_{\mathtt{r}}A_{\mathtt{r}}^{(2k+1)}B_{\mathtt{r}} = \begin{bmatrix} 0 & (-1)^k(\widetilde{A}^kB_1)^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} (-1)^k(\widetilde{A}^kB_1) \\ 0 \end{bmatrix} = -(\widetilde{A}^kB_1)^T\widetilde{Q}(\widetilde{A}^kB_1). \tag{4.37}$$

We know from Statement *(6)* that $C_{\mathtt{r}}A_{\mathtt{r}}^{(2k+1)}B_{\mathtt{r}} = 0$. Therefore, from equation (4.37) we get $(\widetilde{A}^kB_1)^T\widetilde{Q}(\widetilde{A}^kB_1) = 0$. From Statement *(1)* of Lemma 4.1 we know that $\widetilde{Q} \geqslant 0$ and hence, from equation (4.37) we get $\widetilde{Q}\widetilde{A}^kB_1 = 0$. This completes the proof.
*(7) ⇒ (6):* We first claim that $A_{\mathtt{r}}^{\ell}B_{\mathtt{r}} = \mathtt{col}\left(\widetilde{A}^{\ell}B_1, 0\right)$. We prove this using induction.
*Base case:* $(\ell = 0)$ $B_{\mathtt{r}} = \mathtt{col}\,(B_1, 0)$ is true trivially.
*Induction step:* Assume $A_{\mathtt{r}}^{\ell}B_{\mathtt{r}} = \mathtt{col}(\widetilde{A}^{\ell}B_1, 0)$, we prove that $A_{\mathtt{r}}^{\ell+1}B_{\mathtt{r}} = \mathtt{col}(\widetilde{A}^{\ell+1}B_1, 0)$.

$$A_{\mathtt{r}}^{\ell+1}B_{\mathtt{r}} = \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix}^{\ell+1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{A} & -B_2\widehat{R}^{-1}B_2^T \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} \widetilde{A}^{\ell}B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{A}^{\ell+1}B_1 \\ -\widetilde{Q}\widetilde{A}^{\ell}B_1 \end{bmatrix}. \tag{4.38}$$

Since, $\widetilde{Q}\widetilde{A}^kB_1 = 0$ for all $k \in \mathbb{N}$. Therefore, from equation (4.38) we get $A_{\mathtt{r}}^{\ell+1}B_{\mathtt{r}} = \mathtt{col}(\widetilde{A}^{\ell+1}B_1, 0)$. Hence, by mathematical induction we infer that $A_{\mathtt{r}}^{\ell}B_{\mathtt{r}} = \mathtt{col}(\widetilde{A}^{\ell}B_1, 0)$ for all $\ell \in \mathbb{N}$. Therefore, for all $\ell \in \mathbb{N}$, we have $C_{\mathtt{r}}A_{\mathtt{r}}^{\ell}B_{\mathtt{r}} = \begin{bmatrix} 0 & B_1^T \end{bmatrix} \begin{bmatrix} \widetilde{A}^{\ell}B_1 \\ 0 \end{bmatrix} = 0$. ■

Statement *(4)* of Theorem 4.8 leads to a necessary condition for the solvability of a CGCARE that reveals interesting system-theoretic interpretations about the systems that admit CGCARE solutions. We present this necessary condition as a corollary next. This corollary would be crucially used to prove the second main result of this chapter (Theorem 4.16).

A necessary condition for the solvability of CGCARE

**Corollary 4.9.** *Consider the singular LQR Problem 2.1 with its corresponding Hamiltonian pencil pair $(E,H)$ as defined in equation* (4.2)*. If the corresponding CGCARE* (4.1) *admits a solution, then* $\det(sE-H)$ *is the zero polynomial.*

*Proof:* Since $(sE-H) \in \mathbb{R}[s]^{(2\mathtt{n}+\mathtt{m})\times(2\mathtt{n}+\mathtt{m})}$ and $\mathtt{rank}(R) < \mathtt{m}$, from Statement *(4)* of Theorem 4.8 we must have $\mathtt{det}(sE-H) = 0$. ■

Theorem 4.8 and Corollary 4.9 bring out some interesting facts pertaining to the optimal trajectories of singular LQR problems and hence, we briefly discuss these facts next.

Note that for a single-input controllable system we have $\mathtt{rank}\begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n}-1}b \end{bmatrix} = \mathtt{n}$. However, from Theorem 4.8 we know that CGCARE is solvable if and only if $QA^ib = 0$ for all $i \in \mathbb{N}$. Thus, we must have $Q = 0$. Hence, for single-input singular LQR problems CGCARE is solvable *only* for the trivial case when $Q = 0$. Hence, we have the following corollary

Single-input singular LQR problems do not admit CGCARE solution

**Corollary 4.10.** *Consider the singular LQR Problem 2.21 with $Q \neq 0$. Then, the corresponding CGCARE does not admit any solution.*

*Proof:* The proof follows from the discussion above this corollary. ■

As motivated earlier the DAEs in equation (4.2) arise on application of PMP to the LQR problem. It follows from PMP that, for the regular case, the optimal solutions of the LQR problem are nothing but suitably chosen trajectories of the Hamiltonian system (these are the optimal trajectories corresponding to a Lambda-set $\Lambda$ of the Hamiltonian pencil such that $\sigma(\Lambda) \subsetneq \mathbb{C}_-$). For the singular case, since the Hamiltonian system becomes a singular descriptor system, PMP becomes applicable only to the smooth trajectories of the Hamiltonian system. Let us assume that $(x^*, z^*, u^*) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{2\mathtt{n}+\mathtt{p}})$ be a trajectory in the Hamiltonian system. All such trajectories are called the stationary trajectories. For a regular LQR problem, since $\mathtt{det}(sE-H) \neq 0$, from Proposition 2.17 we infer that the Hamiltonian system is autonomous. Further, it is known that the trajectories of an autonomous system are smooth but not compactly supported (exponential). Hence, the stationary trajectories in case of regular LQR problems are all smooth but not compactly supported. To the contrary, from Lemma 4.4 it is clear that the condition $\mathtt{det}(sE-H) = 0$ from Corollary 4.9 means that the transfer function $G(s) = \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B}$ is not invertible as a rational matrix. Hence, by Proposition 2.17 we infer that the Hamiltonian system is *non-autonomous*. Hence, the stationary trajectories corresponding to such problems are compactly supported and smooth. However, this non-autonomy of the Hamiltonian system is only necessary, and not sufficient, for CGCARE solvability. Accordingly, the necessary and sufficient condition in this regard can be inferred from Theorem 4.9, which is as follows. The non-autonomy of the Hamiltonian system implies that it admits inputs. Theorem 4.9 reveals that CGCARE solvability is equivalent to the input cardinality of the Hamiltonian system being precisely equal to $\mathtt{m}-\mathtt{rank}(R)$ (see [TMR09] for more on non-autonomous systems and input cardinality).

Now that we have found the solvability conditions for a CGCARE we present an interesting property of the solutions of a CGCARE. It is clear from the definition of CGCARE that for a symmetric matrix $K$ to be a solution of CGCARE (4.5), $K$ must satisfy the linear matrix equation $KB_1 = 0$. Interestingly, apart from such linear matrix equations any solution $K$ of the CGCARE (4.5) must satisfy certain other algebraic relations also; this is the content of the next lemma.

Algebraic relations satisfied by the solutions of a CGCARE

**Lemma 4.11.** *Consider the singular LQR Problem 4.2 with the corresponding reduced Hamiltonian system be as given in equation* (4.10). *Let* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *be a solution to the corresponding CGCARE* (4.5). *Then,*

$$K\widetilde{A}^i B_1 = 0, \ \textit{for all } i \in \mathbb{N}. \tag{4.39}$$

*Proof:* We use mathematical induction for the proof.
*Base case:* $(i = 0)$ Since $K$ is a solution of CGCARE (4.5), it is evident that $KB_1 = 0$.
*Induction step:* Assume $K\widetilde{A}^i B_1 = 0$. We prove that $K\widetilde{A}^{i+1} B_1 = 0$. Note that solution $K$ of the CGCARE must satisfy

$$\begin{aligned}
& A^T K + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^T K + S_2^T) = 0 \\
\Rightarrow\ & A^T K + KA + Q - KB_2\widehat{R}^{-1}B_2^T K - KB_2\widehat{R}^{-1}S_2^T - S_2\widehat{R}^{-1}B_2^T K - S_2\widehat{R}^{-1}S_2^T = 0 \\
\Rightarrow\ & (A - B_2\widehat{R}^{-1}S_2^T)^T K + K(A - B_2\widehat{R}^{-1}S_2^T) + (Q - S_2\widehat{R}^{-1}S_2^T) - KB_2\widehat{R}^{-1}B_2^T K = 0 \\
\Rightarrow\ & \widetilde{A}^T K + K\widetilde{A} + \widetilde{Q} - KB_2\widehat{R}^{-1}B_2^T K = 0.
\end{aligned} \tag{4.40}$$

Post-multiplying equation (4.40) by $\widetilde{A}^i B_1$, we have

$$\widetilde{A}^T K\widetilde{A}^i B_1 + K\widetilde{A}^{i+1} B_1 + \widetilde{Q}\widetilde{A}^i B_1 - KB_2\widehat{R}^{-1}B_2^T K\widetilde{A}^i B_1 = 0. \tag{4.41}$$

Since CGCARE (4.5) admits a solution, from Statement *(7)* of Theorem 4.8 it is evident that $\widetilde{Q}\widetilde{A}^i B_1 = 0$. In addition to this using the induction hypothesis $K\widetilde{A}^i B = 0$ on equation (4.41), we have $K\widetilde{A}^{i+1} B_1 = 0$. This completes the proof of this lemma. ■

Recall that all the results in Chapter 2 and Chapter 3 are true only for the case when $\mathtt{det}(sE - H) \neq 0$. However, from Corollary 4.9 we know that a necessary condition for the solvability of a CGCARE is $\mathtt{det}(sE - H) = 0$. Therefore, the results in the preceding chapters (Chapters 2 and 3) cannot be applied to the case when a singular LQR problem admits CGCARE solutions. Further, for the single-input case, the reduced Hamiltonian system (4.10) and the transformed Hamiltonian system in equation (4.9) is the same. This leads to some interesting comparison between the results of this section and some of the results obtained in Chapter 2. We present this in Table 4.1.

Now we provide examples to demonstrate the results derived in this section. The first example demonstrates that $\mathtt{det}(sE - H)$ being a zero polynomial is a necessary condition for

| **Results from Chapter 2** | | **Results from Chapter 4** (adapted to the single-input case) | |
|---|---|---|---|
| Assumption: $\sigma(E,H)\cap j\mathbb{R}=\emptyset$ and $\texttt{det}(sE-H)\neq 0$. | | Necessary condition (Corollary 4.9): $\texttt{det}(sE-H)=0$ | |
| Lemma 2.36 | $\widehat{C}\widehat{A}^k\widehat{b}=0$ for $k\in\{0,1,\dots,2(\mathtt{n_f}-1)\}$ | Theorem 4.8 | $\widehat{C}\widehat{A}^k\widehat{b}=0$ for all $k\in\mathbb{N}$ |
| | $QA^\ell b=0$ for $\ell\in\{0,1,\dots,\mathtt{n_f}-2\}$ | | $QA^\ell b=0$ for all $\ell\in\mathbb{N}$ |
| Lemma 2.37 | $KA^ib=0$ for $i\in\{0,1,\dots,\mathtt{n_f}-1\}$, where $K$ satisfies LQR LMI (2.9) | Lemma 4.11 | $KA^ib=0$ for all $i\in\mathbb{N}$, where $K$ satisfies CGCARE (4.1) |

Table 4.1: Comparison of the results in Chapter 2 with the results from Chapter 4 adapted to single-input systems.

CGCARE to admit a solution. We use the example used in [NF19, Example 6.4] to demonstrate Theorem 4.9.

**Example 4.12.** *Consider the singular LQR problem with*

$$Q=\begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix},\ S=\begin{bmatrix}0&1\\0&0\\0&-1\end{bmatrix},\ \text{and } R=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

*The system dynamics is* $\frac{d}{dt}x=\begin{bmatrix}1&1&1\\-3&1&0\\1&0&0\end{bmatrix}x+\begin{bmatrix}2&0\\0&0\\0&1\end{bmatrix}u$. *Here* $B_1=\begin{bmatrix}2\\0\\0\end{bmatrix}$, $B_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $S_1=\begin{bmatrix}0\\0\\0\end{bmatrix}$, $S_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ *and* $\widehat{R}=1$. *The solutions to the CGCARE* (4.1) *for this problem are* $K_1=\texttt{diag}(0,0,0)$ *and* $K_2=\texttt{diag}(0,0,2)$. *On simple computation it can be verified that corresponding to this problem the Hamiltonian pencil is singular, i.e.,* $\texttt{det}(sE-H)=0$.

*Note that* $\widetilde{A}=A-B_2\widehat{R}^{-1}S_2^T=\begin{bmatrix}1&1&1\\-3&1&0\\0&0&0\end{bmatrix}$. *Therefore,* $\widetilde{Q}=Q-S_2\widehat{R}^{-1}S_2^T=0_{3,3}$. *It is evident that for all* $i\in\mathbb{N}$, *we have in this example* $K_1\widetilde{A}^iB_1=0$, $K_2\widetilde{A}^iB_1=0$, *and* $\widetilde{Q}\widetilde{A}^iB_1=0$.

The next example shows that $\texttt{det}(sE-H)=0$ is *not* a sufficiency condition for the CGCARE to admit a solution.

**Example 4.13.** *Consider the singular LQR problem with*

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \text{and } R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

*The system dynamics is* $\frac{d}{dt}x = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u$. *For this problem it can be verified that* $\mathtt{det}(sE - H) = 0$. *Here* $B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

*The constrained equation of the CGCARE corresponding to this problem is* $K \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$. *The solution to this equation is* $K = \mathtt{diag}(0,0,k)$ *for all* $k \in \mathbb{R}$. *Therefore, the ARE corresponding to the CGCARE for this problem with* $K = \mathtt{diag}(0,0,k)$ *is given by* $A^T K + KA + Q - KB_2\widehat{R}^{-1}B_2^T K = 0$, *i.e.,*

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix} = 0_{3,3}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -k^2 + 2k + 1 \end{bmatrix} = 0_{3,3}.$$

*Thus, no value of k satisfies the above equation. Therefore, the CGCARE does not admit a solution. Note that here* $\widetilde{Q} = Q$ *and* $\widetilde{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Evidently,* $\widetilde{Q}\widetilde{A}B_1 = \begin{bmatrix} 1 & 4 \\ 1 & 4 \\ 0 & 0 \end{bmatrix} \neq 0$.

## 4.4 Genericity of CGCARE insolubility among all singular LQR problems

In this section we answer the second question we raised in Section 4.1: how often are the necessary and sufficient conditions for CGCARE solvability satisfied?, i.e., how often are CGCAREs solvable? We show in this section that the CGCARE (4.1), corresponding to the singular LQR

Problem (2.1), is generically unsolvable. Note that there are several formal notions of genericity in the literature, which are not necessarily equivalent [HK10], [SP76]. Genericity often involves the idea that a property over a set holds with probability 1, where the probability is suitably defined. From this perspective, genericity of a property over a Euclidean space is often defined in terms of an algebraic variety. However, the difficulty that we face in this section is that the set over which the property is defined, i.e., the set from where the samples are collected (sample space), is itself a measure zero set in a Euclidean space. Hence, for the problem in this section, adapting the definition of genericity based on the notion of algebraic varieties presents a significant challenge. We, therefore, consider a definition of genericity that is often used in the literature, but is somewhat weaker than the one that uses algebraic varieties. We formally define this notion of genericity in Definition 4.15. Intuitively, this definition tells that a property is generic if

(i) for every data-point, where the property holds, there exist arbitrarily small perturbations so that the property no longer holds, and

(ii) for every data-point, where the property does not hold, there exists a suitably chosen ball around that point over which the property continues to not hold.

The formal definition of a property over a set and its genericity is presented next.

**Definition 4.14.** [Won85, Section 0.16] *Consider a set $\Theta \subseteq \mathbb{R}^{\mathtt{N}}$. A property $\Gamma$ over $\Theta$ is a function $\Gamma : \Theta \to \{0,1\}$, where $\Gamma(p) = 1$ means $\Gamma$ holds at $p \in \Theta$ and $\Gamma(p) = 0$ means $\Gamma$ does not hold at $p \in \Theta$.*

**Definition 4.15.** [SP76, Section II] *A property $\Gamma$ over a set $\Theta \subseteq \mathbb{R}^{\mathtt{N}}$ is said to be generic, if there exists a set $\mathbb{V} \subseteq \Theta$ such that the following statements hold:*

*(S1)* $\mathtt{ker}(\Gamma) \subseteq \mathbb{V}$ *and* $\mathbb{V} \neq \Theta$.

*(S2) For every $p^* \in \Theta \setminus \mathbb{V}$, there exist $\varepsilon > 0$ such that $\{p \in \Theta \,|\, \|p - p^*\| \leqslant \varepsilon\} \subseteq \Theta \setminus \mathbb{V}$.*

*(S3) For every $\widetilde{p} \in \mathbb{V}$ and for all $\varepsilon > 0$, there exists $p \in \Theta \setminus \mathbb{V}$ such that $\|p - \widetilde{p}\| \leqslant \varepsilon$.*

Using the above notion of genericity, we present the second main result of this chapter.

Hamiltonian matrix pencil is generically regular among all singular LQR problems

**Theorem 4.16.** *Let $\mathtt{n},\mathtt{m} \in \mathbb{N}, \mathtt{d} := \mathtt{n}+\mathtt{m}$. Consider all singular LQR problems with the parameter matrices coming from*

$$\mathscr{P}_{\mathtt{n},\mathtt{m}} := \{(A,B,\mathscr{Q}) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \times \mathbb{R}^{\mathtt{n}\times\mathtt{m}} \times \mathbb{R}^{\mathtt{p}\times\mathtt{p}} \,|\, \mathscr{Q} := \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0, \\ Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}, \mathtt{det}(R) = 0\}. \quad (4.42)$$

*Let the corresponding Hamiltonian pencil pair $(E,H)$ be as given in equation* (4.2). *Then, the property $\mathtt{det}(sE - H) \neq 0$ over the set $\mathscr{P}_{\mathtt{n},\mathtt{m}}$ holds generically.*

It is important to note here that the property $\mathtt{det}(sE-H)\neq 0$ among all LQR problems is generically true based on the notion of algebraic varieties. However, the nontrivial part of Theorem 4.16 is the claim that $\mathtt{det}(sE-H)\neq 0$ among *all singular* LQR problems. As motivated previously the reason for this is the fact that the set of singular LQR problems is a set of measure zero in a Euclidean space. Now we prove Theorem 4.16 in the next section.

### 4.4.1 Hamiltonian matrix pencil is generically regular

In this section we prove Theorem 4.16 which states that among all singular LQR problems, the set of corresponding Hamiltonian matrix pencils are generically regular, i.e., the property $\mathtt{det}(sE-H)\neq 0$ over the all singular LQR problems holds generically. In order to prove Theorem 4.16, we first define the set

$$\mathscr{N} := \{(A,B,\mathscr{Q})\in\mathscr{P}_{\mathtt{n,m}} \,|\, \mathtt{det}(sE-H)=0\}. \tag{4.43}$$

We also define the property $\Gamma:\mathscr{P}_{\mathtt{n,m}}\to\{0,1\}$ such that

$$\Gamma(v)=\begin{cases}0, & \text{for } v\in\mathscr{N}\\ 1 & \text{otherwise.}\end{cases} \tag{4.44}$$

From the definition of $\Gamma$ it is clear that $\mathtt{ker}(\Gamma)=\mathscr{N}$. Therefore, to prove Theorem 4.16, it suffices to show that $\mathscr{N}$ satisfies the three statements on $\mathbb{V}$ defined in Definition 4.15. We prove these three statements on $\mathscr{N}$ one-by-one.

***Set $\mathscr{N}$ satisfies (S1):*** Clearly, $\mathtt{ker}(\Gamma)=\mathscr{N}$. Now consider

$$A=Q=I_{\mathtt{n}},\quad S=0_{\mathtt{nm}},\quad R=\mathtt{diag}(0_{\mathtt{m-r,m-r}},I_{\mathtt{r}}),\ \text{ and }\ B=\mathtt{col}(I_{\mathtt{m}},0_{\mathtt{n-m}}).$$

It is easy to verify that $\mathtt{det}(sE-H)=(s^2-1)^{\mathtt{n-m}}\times(2-s^2)^{\mathtt{r}}\neq 0$. Thus, $(A,B,\mathscr{Q})\in\mathscr{P}_{\mathtt{n,m}}\setminus\mathscr{N}$ and therefore $\mathscr{N}\neq\mathscr{P}_{\mathtt{n,m}}$. ■

***Set $\mathscr{N}$ satisfies (S2):*** Let $\mathtt{d}:=\mathtt{n}+\mathtt{m}$. Now, we define the vector space

$$\Omega_\ell := \left\{(A,B,\mathscr{Q})\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}\times\mathbb{R}^{\mathtt{n}\times\mathtt{m}}\times\mathbb{R}^{\mathtt{d}\times\mathtt{d}} \,|\, \mathscr{Q}:=\begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}, Q\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}, R\in\mathbb{R}^{\mathtt{m}\times\mathtt{m}}\right\}. \tag{4.45}$$

We also define the following set:

$$\mathscr{N}_{\mathtt{ext}} := \left\{(A,B,\mathscr{Q})\in\Omega_\ell | \mathtt{det}(sE-H)=0, \text{ where } (E,H) \text{ is as defined in equation (4.2)}\right\}. \tag{4.46}$$

Note that $\Omega_\ell$ is isomorphic to $\mathbb{R}^\ell$, i.e., $\Omega_\ell\cong\mathbb{R}^\ell$. Hence, listing the elements of $(A,B,\mathscr{Q})$ in some arbitrary order, we regard an element $(A,B,\mathscr{Q})\in\Omega_\ell$ as a point $p\in\mathbb{R}^\ell$, where $\ell$ is a function of $\mathtt{n},\mathtt{m}$. (We also identify sets like $\mathscr{P}_{\mathtt{n,m}},\mathscr{N}$ as subsets of $\mathbb{R}^\ell$, which is understood via the isomorphism $\Omega_\ell\cong\mathbb{R}^\ell$.) In order to emphasize the dependence of $(E,H)$ on $p\in\Omega_\ell$, we write this $(E,H)$ pair as $(E(p),H(p))$ in the sequel. Now, $\mathtt{det}\,(sE(p)-H(p))=\sum_{i=0}^{2\mathtt{n}+\mathtt{m}} a_i(p)s^i$, where each coefficient $a_i(p)$ is some polynomial in $p$ with real coefficients. Therefore, viewing $\mathscr{N}_{\texttt{ext}}$ as a subset of $\mathbb{R}^\ell$ we can write

$$\mathscr{N}_{\texttt{ext}} = \left\{ p \in \mathbb{R}^\ell \,|\, \textstyle\sum_{i=0}^{2\texttt{n}+\texttt{m}} a_i(p)s^i \equiv 0 \right\} = \left\{ p \in \mathbb{R}^\ell \,|\, a_0(p) = a_1(p) = \cdots = a_{2\texttt{n}+\texttt{m}}(p) = 0 \right\}.$$

Consequently, $\mathscr{N}_{\texttt{ext}}$ is a variety in $\mathbb{R}^\ell$, and $\mathbb{R}^\ell \setminus \mathscr{N}_{\texttt{ext}}$ is a Zariski[1] open-set in $\mathbb{R}^\ell$. Therefore, for all points $\widehat{p} \in \mathbb{R}^\ell \setminus \mathscr{N}_{\texttt{ext}}$, there exists an $\varepsilon > 0$ such that

$$\mathscr{B}_{\widehat{p},\varepsilon} := \{ p \in \mathbb{R}^\ell \,|\, \|p - \widehat{p}\| \leqslant \varepsilon \} \subseteq \mathbb{R}^\ell \setminus \mathscr{N}_{\texttt{ext}}. \tag{4.47}$$

Now, given an arbitrary point $p^* \in \mathscr{P}_{\texttt{n,m}} \setminus \mathscr{N}$, we can infer from the definitions of $\mathscr{N}$ and $\mathscr{N}_{\texttt{ext}}$ (Equation (4.43) and equation (4.46), respectively) that $p^* \in \mathbb{R}^\ell \setminus \mathscr{N}_{\texttt{ext}}$. It then follows from equation (4.47) that there exists an $\varepsilon > 0$ such that $\mathscr{B}_{p^*,\varepsilon} \subseteq \mathbb{R}^\ell \setminus \mathscr{N}_{\texttt{ext}}$. Consequently,

$$\mathscr{B}_{p^*,\varepsilon} \cap \mathscr{P}_{\texttt{n,m}} \subseteq (\mathbb{R}^\ell \setminus \mathscr{N}_{\texttt{ext}}) \cap \mathscr{P}_{\texttt{n,m}} = \mathscr{P}_{\texttt{n,m}} \setminus \mathscr{N} \Rightarrow \{ p \in \mathscr{P}_{\texttt{n,m}} \,|\, \|p - p^*\| \leqslant \varepsilon \} \subseteq \mathscr{P}_{\texttt{n,m}} \setminus \mathscr{N}.$$

Since $p^* \in \mathscr{P}_{\texttt{n,m}} \setminus \mathscr{N}$ was taken arbitrarily, it follows that for all $p^* \in \mathscr{P}_{\texttt{n,m}} \setminus \mathscr{N}$, there exists an $\varepsilon > 0$ such that

$$p \in \mathscr{P}_{\texttt{n,m}} \,|\, \|p - p^*\| \leqslant \varepsilon \} \subseteq \mathscr{P}_{\texttt{n,m}} \setminus \mathscr{N}.$$

This proves that $\mathscr{N}$ satisfies $(S2)$ of Definition 4.15. ■

***Set $\mathscr{N}$ satisfies (S3):*** Let $(A_{\texttt{t}}, B_{\texttt{t}}, \mathscr{Q}_{\texttt{t}})$ be an arbitrary point in $\mathscr{N}$, where $\mathscr{Q}_{\texttt{t}} = \left[\begin{smallmatrix} Q_{\texttt{t}} & S_{\texttt{t}} \\ S_{\texttt{t}}^T & R_{\texttt{t}} \end{smallmatrix}\right]$, and $\varepsilon > 0$ be arbitrary. From Lemma 4.1 recall that there exists an orthogonal matrix $U \in \mathbb{R}^{\texttt{m}\times\texttt{m}}$ such that $U^T R_{\texttt{t}} U = \texttt{diag}(0, \widehat{R})$, where $\widehat{R} \in \mathbb{R}^{\texttt{r}\times\texttt{r}}$ and $\texttt{det}(\widehat{R}) \neq 0$. We fix this $U$ to define the map $\mathscr{W}_U : \Omega_\ell \to \Omega_\ell$ as $(A, B, \mathscr{Q}) \mapsto (A, BU, V^T \mathscr{Q} V)$, where $V := \texttt{diag}(I_{\texttt{n}}, U)$. Clearly, $\mathscr{W}_U$ is a linear map. It can be checked that $\mathscr{P}_{\texttt{n,m}}, \mathscr{N}$ are invariant under $\mathscr{W}_U$. Further, since $U$ is invertible, $\mathscr{W}_U$ is a bijection from $\Omega_\ell$ to $\Omega_\ell$. Consequently, $\mathscr{W}_U|_{\mathscr{P}_{\texttt{n,m}}}$ is also a bijection from $\mathscr{P}_{\texttt{n,m}}$ to $\mathscr{P}_{\texttt{n,m}}$. Likewise, $\mathscr{W}_U|_{\mathscr{N}}$ is a bijection from $\mathscr{N}$ to $\mathscr{N}$. Let the induced norm of $\mathscr{W}_U^{-1}$ be $\rho$, i.e., $\|\mathscr{W}_U^{-1}\| =: \rho$. In order to proceed further, we need to define the following set:

$$\begin{aligned} \Theta_{\texttt{n,m,r}} := \{ (A, B, \mathscr{Q}) \in \mathbb{R}^{\texttt{n}\times\texttt{n}} \times \mathbb{R}^{\texttt{n}\times\texttt{m}} \times \mathbb{R}^{\texttt{p}\times\texttt{p}} \,|\, \mathscr{Q} \geqslant 0, & \\ \mathscr{Q} := \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix}, Q \in \mathbb{R}^{\texttt{n}\times\texttt{n}}, \widehat{R} \in \mathbb{R}^{\texttt{r}\times\texttt{r}}, \texttt{det}(\widehat{R}) \neq 0 \}. & \end{aligned} \tag{4.48}$$

Note that $\Theta_{\texttt{n,m,r}} \subsetneq \mathscr{P}_{\texttt{n,m}} \subsetneq \Omega_\ell$ where $\Omega_\ell$ is as defined in equation (4.45). Recall $(\widehat{E}, \widehat{H})$ from equation (4.9) to define

$$\mathscr{Z} := \{ (A, B, \mathscr{Q}) \in \Theta_{\texttt{n,m,r}} \,|\, \texttt{det}(s\widehat{E} - \widehat{H}) = 0 \}. \tag{4.49}$$

Clearly $\mathscr{Z} = \mathscr{N} \cap \Theta_{\texttt{n,m,r}}$. Lemma 4.17 will be of crucial importance in the sequel.

[1]In Zariski topology, a closed set is defined to be the set of zeros of polynomial equations.

Genericity of the property $\mathtt{det}(s\widehat{E}-\widehat{H}) \neq 0$ among all singular LQR problems

**Lemma 4.17.** *Consider the sets* $\Theta_{\mathtt{n,m,r}}$ *and* $\mathscr{Z}$ *as defined in equation* (4.48) *and equation* (4.49), *respectively. Then, for every* $z \in \mathscr{Z}$ *and for all* $\varepsilon > 0$, *there exists* $p \in \Theta_{\mathtt{n,m,r}} \setminus \mathscr{Z}$ *such that* $\|p-z\| \leqslant \varepsilon$.

In order to prove Lemma 4.17 we need to a few auxiliary results which we present next. We need to define the following sets in addition to the sets $\Omega_\ell$ and $\Theta_{\mathtt{n,m,r}}$ defined in equation (4.45) and equation (4.48), respectively. Fix an $\mathtt{r} \in \mathbb{N}$, $\mathtt{r} < \mathtt{m}$. The set $\Omega_{\mathtt{N}}$ then is defined as

$$\Omega_{\mathtt{N}} := \{(A,B,C,D,S_2) \,|\, A,C \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, D \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}, S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}\}. \tag{4.50}$$

Note that $\Omega_{\mathtt{N}} \cong \mathbb{R}^{\mathtt{N}}$, where $\mathtt{N}$ is a function of $\mathtt{n},\mathtt{m},\mathtt{r}$. Hence, an element $(A,B,C,D,S_2) \in \Omega_{\mathtt{N}}$ is represented as a vector $\xi \in \mathbb{R}^{\mathtt{N}}$ in the sequel. Further, we define

$$\Theta^*_{\mathtt{n,m,r}} := \{(A,B,C,D,S_2) \in \Omega_{\mathtt{N}} \,|\, \mathtt{det}(D) \neq 0\}. \tag{4.51}$$

Define the map $\Psi : \Theta^*_{\mathtt{n,m,r}} \to \Omega_\ell$ as $\Psi(A,B,C,D,S_2) := (A,B,\mathscr{Q})$ where $\mathscr{Q} := \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix}$, $Q := C^TC + S_2(D^TD)^{-1}S_2^T$, and $\widehat{R} := D^TD$.

Range-space of $\Psi$ is the set $\Theta_{\mathtt{n,m,r}}$

**Lemma 4.18.** *Let* $\Psi : \Theta^*_{\mathtt{n,m,r}} \to \Omega_\ell$ *be as defined above, and* $\Theta_{\mathtt{n,m,r}}$ *be as defined in equation* (4.48). *Then,* $\mathtt{img}\,\Psi = \Theta_{\mathtt{n,m,r}}$.

*Proof:* Note that $\mathtt{det}(D) \neq 0 \Rightarrow \widehat{R} = D^TD > 0$. Since $Q = C^TC + S_2(D^TD)^{-1}S_2^T$, Schur complement of $\mathscr{Q}$ with respect to $\widehat{R}$ gives

$$\mathscr{Q}_{\mathtt{Schur}} := \mathtt{diag}(Q - S_2\widehat{R}^{-1}S_2^T, 0, \widehat{R}) = \mathtt{diag}(C^TC, 0, D^TD) \geqslant 0.$$

Clearly, $\mathscr{Q}_{\mathtt{Schur}} \geqslant 0 \Rightarrow \mathscr{Q} \geqslant 0$. Thus, $\mathtt{img}\,\Psi \subseteq \Theta_{\mathtt{n,m,r}}$.

For the converse, let $(A,B,\mathscr{Q}) \in \Theta_{\mathtt{n,m,r}}$ be given, where $\mathscr{Q} = \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0$. In order to show $\Theta_{\mathtt{n,m,r}} \subseteq \mathtt{img}\,\Psi$, it suffices to show there exist $C \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$, $\mathtt{det}(D) \neq 0$ such that $Q := C^TC + S_2(D^TD)^{-1}S_2^T$, and $\widehat{R} := D^TD$. Such a $D$ clearly exists by taking a Cholesky decomposition of $\widehat{R}$. For the desired $C$, note that $Q - S_2\widehat{R}^{-1}S_2^T \geqslant 0$ since $\mathscr{Q} \geqslant 0$ (follows by taking the Schur complement wit respect to $\widehat{R}$). This admits the factorization $Q - S_2\widehat{R}^{-1}S_2^T = C^TC$ to get the desired $C$. ∎

Next we define the map

$$\Phi : \Theta_{\mathtt{n,m,r}} \to \mathbb{R}[s] \text{ such that } \Phi(A,B,\mathscr{Q}) := \mathtt{det}(s\widehat{E}-\widehat{H}),$$

where $(\widehat{E},\widehat{H})$ is as given in equation (4.5) with $B =: \mathtt{col}(B_1,B_2)$, and $B_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. Define $\Phi^* := \Phi \circ \Psi$. Note that $\mathscr{Z} = \mathtt{ker}\,\Phi$, where $\mathscr{Z}$ is as defined in equation (4.49). The commutative diagram of the maps defined here is as follows:

$$\begin{tikzcd} \Theta_{\mathtt{n,m,r}} \arrow[r, "\Phi"] & \mathbb{R}[s] \\ \Theta^*_{\mathtt{n,m,r}} \arrow[u, two heads, "\Psi"] \arrow[ur, "\Phi^*"] & \end{tikzcd}$$

Figure 4.1: A commutative diagram involving the maps $\Psi$, $\Phi$ and $\Phi^*$.

Restriction of map $\Psi$ to $\mathscr{Z}^*$ is the set $\mathscr{Z}$

**Lemma 4.19.** *Define* $\mathscr{Z}^* := \mathtt{ker}\,\Phi^*$. *Then,* $\Psi(\mathscr{Z}^*) = \mathscr{Z}$.

*Proof:* $\Psi(\mathscr{Z}^*) \subseteq \mathscr{Z}$ follows trivially from the definitions of $\Psi, \Phi$, and $\Phi^*$. The converse, i.e., $\Psi(\mathscr{Z}^*) \supseteq \mathscr{Z}$ follows from Lemma 4.18. Indeed, suppose $\alpha \in \mathscr{Z}$. By Lemma 4.18 there exists $\beta \in \Theta^*_{\mathtt{n,m,r}}$ such that $\Psi(\beta) = \alpha$. However, $\Phi^*(\beta) = \Phi(\Psi(\beta)) = \Phi(\alpha) = 0$ because $\alpha \in \mathscr{Z}$. Therefore, $\beta \in \mathscr{Z}^*$. ■

Continuity of the map $\Psi$

**Lemma 4.20.** $\Psi : \Theta^*_{\mathtt{n,m,r}} \to \Omega_\ell$ *is locally Lipschitz continuous.*

*Proof:* It follows from the fact that $\Psi$ is continuously differentiable. ■

$\mathscr{Z}^*$ is a "thin set" in $\Theta^*$

**Lemma 4.21.** *For every* $z^* \in \mathscr{Z}^*$ *and for all* $\varepsilon > 0$ *there exists* $p^* \in \Theta^*_{\mathtt{n,m,r}} \setminus \mathscr{Z}^*$ *such that* $\|p^* - z^*\| \leqslant \varepsilon$.

*Proof:* We first show that $\mathscr{Z}^* \neq \Theta^*_{\mathtt{n,m,r}}$. Take

$$A = C = I_{\mathtt{n}}, \quad D = I_{\mathtt{r}}, \quad S_2 = 0_{\mathtt{n,r}}, \quad B = \mathtt{col}(I_{\mathtt{m}}, 0_{\mathtt{n-m,m}}) =: \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

where $B_1 \in \mathbb{R}^{\mathtt{n}\times(\mathtt{m}-\mathtt{r})}$, $B_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. It is easy to verify that $\mathtt{det}(s\widehat{E} - \widehat{H}) \neq 0$. Thus, $\mathscr{Z}^* \neq \Theta^*_{\mathtt{n,m,r}}$.

Let $z^* \in \mathscr{Z}^*$ and $\varepsilon > 0$ be arbitrary. Recall that $\Theta^*_{\mathtt{n,m,r}} \subseteq \Omega_{\mathtt{N}} \cong \mathbb{R}^{\mathtt{N}}$. Let $\xi$ be a typical point in $\mathbb{R}^{\mathtt{N}}$, we write $\mathtt{det}(D)$ as a polynomial $d(\xi)$. From the definition of $\mathscr{Z}^*$ it follows that there exist polynomials $a_0(\xi), a_1(\xi), \ldots, a_{2\mathtt{n}+\mathtt{m}}(\xi)$ such that

$$\begin{aligned} \mathscr{Z}^* &= \{\xi \in \Theta^*_{\mathtt{n,m,r}} \,|\, \textstyle\sum_{i=0}^{2\mathtt{n}+\mathtt{m}} a_i(\xi)s^i \equiv 0\} \\ &= \{\xi \in \Theta^*_{\mathtt{n,m,r}} \,|\, a_0(\xi) = a_1(\xi) = \cdots = a_{2\mathtt{n}+\mathtt{m}}(\xi) = 0\} \\ &= \{\xi \in \mathbb{R}^{\mathtt{N}} \,|\, a_0(\xi) = \cdots = a_{2\mathtt{n}+\mathtt{m}}(\xi) = 0, d(\xi) \neq 0\}. \end{aligned} \tag{4.52}$$

Define $Y := \{\xi \in \mathbb{R}^{\mathtt{N}} \,|\, a_0(\xi) = \cdots = a_{2\mathtt{n}+\mathtt{m}}(\xi) = 0\}$. From equation (4.51), $\Theta^*_{\mathtt{n,m,r}} = \{\xi \in \mathbb{R}^{\mathtt{N}} \,|\, d(\xi) \neq 0\}$. Clearly, $\mathscr{Z}^* = Y \cap \Theta^*_{\mathtt{n,m,r}}$. Hence, $z^* \in \mathscr{Z}^* \Rightarrow z^* \in Y$ and $z^* \in \Theta^*_{\mathtt{n,m,r}}$. Since $Y$

is a variety in $\mathbb{R}^{\mathtt{N}}$,

$$\text{for all } \varepsilon' > 0 \text{ there exists } p \in \mathbb{R}^{\mathtt{N}} \setminus Y \text{ such that } \|p - z^*\| \leqslant \varepsilon'. \tag{4.53}$$

Further, since $\Theta^*_{\mathtt{n,m,r}}$, too, is a Zariski open-set in $\mathbb{R}^{\mathtt{N}}$, there exists $\delta > 0$ such that

$$\{x \in \mathbb{R}^{\mathtt{N}} \mid \|x - z^*\| \leqslant \delta\} \subseteq \Theta^*_{\mathtt{n,m,r}}.$$

Therefore, defining $\varepsilon^* := \mathtt{min}(\delta, \varepsilon)$, it follows from equation (4.53) that, there exists $p^* \in \mathbb{R}^{\mathtt{N}} \setminus Y$ such that $\|p^* - z^*\| \leqslant \varepsilon^*$. Clearly, $p^* \in \Theta^*_{\mathtt{n,m,r}}$ because $\|p^* - z^*\| \leqslant \varepsilon^* \leqslant \delta$. Also, $p^* \notin \mathscr{Z}^*$ since $p^* \notin Y$ and $\mathscr{Z}^* \subseteq Y$. Thus, $p^* \in \Theta^*_{\mathtt{n,m,r}} \setminus \mathscr{Z}^*$ and $\|p^* - z^*\| \leqslant \varepsilon^* \leqslant \varepsilon$. Since $z^* \in \mathscr{Z}^*$ and $\varepsilon > 0$ were arbitrary, it follows that for every $z^* \in \mathscr{Z}^*$ and for all $\varepsilon > 0$ there exists $p^* \in \Theta^*_{\mathtt{n,m,r}} \setminus \mathscr{Z}^*$ such that $\|p^* - z^*\| \leqslant \varepsilon$. ∎

*Proof of Lemma 4.17:* For a given $z \in \mathscr{Z} \subsetneq \Theta_{\mathtt{n,m,r}}$, by Lemma 4.18, there exists $z^* \in \Theta^*_{\mathtt{n,m,r}}$ such that $\Psi(z^*) = z$. From Lemma 4.19, $z^* \in \mathscr{Z}^*$. Now, since $\Theta^*_{\mathtt{n,m,r}} \subseteq \mathbb{R}^{\mathtt{N}}$ is Zariski open, there exists $\delta > 0$ such that $\mathscr{B}_{z^*,\delta} := \{w \in \mathbb{R}^{\mathtt{N}} \mid \|w - z^*\| \leqslant \delta\} \subseteq \Theta^*_{\mathtt{n,m,r}}$. From Lemma 4.20, it follows that there exists $L_\delta > 0$ (Lipschitz constant) such that

$$\|\Psi(\zeta) - \Psi(\eta)\| \leqslant L_\delta \|\zeta - \eta\| \;\; \forall \;\zeta, \eta \in \mathscr{B}_{z^*,\delta}. \tag{4.54}$$

Now for a given $\varepsilon > 0$, define $\varepsilon' := \frac{\varepsilon}{L_\delta}$. Let $\widehat{\varepsilon} := \mathtt{min}(\varepsilon', \delta)$. From Lemma 4.21, there exists $p^* \in \Theta^*_{\mathtt{n,m,r}} \setminus \mathscr{Z}^*$ such that

$$\|p^* - z^*\| \leqslant \widehat{\varepsilon}. \tag{4.55}$$

Since $\|p^* - z^*\| \leqslant \widehat{\varepsilon} \leqslant \delta$, $p^* \in \mathscr{B}_{z^*,\delta}$. Hence, using inequality (4.54) with $\zeta = p^*$ and $\eta = z^*$ and the fact that $\Psi(z^*) = z$, we have

$$\|\Psi(p^*) - \Psi(z^*)\| \leqslant L_\delta \|p^* - z^*\| \Rightarrow \|\Psi(p^*) - z\| \leqslant L_\delta \|p^* - z^*\|. \tag{4.56}$$

Using inequality (4.55) in inequality (4.56), we have $\|\Psi(p^*) - z\| \leqslant L_\delta \|p^* - z^*\| \leqslant L_\delta \widehat{\varepsilon} \leqslant L_\delta \varepsilon' = L_\delta \frac{\varepsilon}{L_\delta} = \varepsilon$. Define $p := \Psi(p^*)$. Since $p^* \in \Theta^*_{\mathtt{n,m,r}} \setminus \mathscr{Z}^*$, by Lemma 4.19 it follows that $p \in \Theta_{\mathtt{n,m,r}} \setminus \mathscr{Z}$. Thus, for every $z \in \mathscr{Z}$ and for all $\varepsilon > 0$, there exists $p \in \Theta_{\mathtt{n,m,r}} \setminus \mathscr{Z}$ such that $\|p - z\| \leqslant \varepsilon$. ∎

Let $\widetilde{p} \in \mathbb{R}^\ell$ be the vector that represents $(A_{\mathtt{t}}, B_{\mathtt{t}}, \mathscr{Q}_{\mathtt{t}}) \in \mathscr{N}$. Note that, $(A_{\mathtt{t}}, B_{\mathtt{t}}U, U^T \mathscr{Q}_{\mathtt{t}} U) \in \Theta_{\mathtt{n,m,r}}$. Hence, $y := \mathscr{W}_U(\widetilde{p}) \in \Theta_{\mathtt{n,m,r}}$. Since $\widetilde{p} \in \mathscr{N}$, it follows from Lemma 4.3 and equation (4.49) that $y \in \mathscr{Z}$. From Lemma 4.17, we know that there exists $y^* \in \Theta_{\mathtt{n,m,r}} \setminus \mathscr{Z}$ such that

$$\|y^* - y\| \leqslant \frac{\varepsilon}{\rho}. \tag{4.57}$$

Define $p := \mathscr{W}_U^{-1}(y^*)$. Utilizing the basic inequality of the norm of the linear map $\mathscr{W}_U^{-1}$ and inequality (4.57) we get

$$\|p - \widetilde{p}\| = \left\|\mathscr{W}_U^{-1}(y^*) - \mathscr{W}_U^{-1}(y)\right\| \leqslant \rho \|y^* - y\| \leqslant \rho \frac{\varepsilon}{\rho} = \varepsilon.$$

Moreover, since $y^* \in \Theta_{\mathtt{n,m,r}} \setminus \mathscr{Z}$ we must have $p \in \mathscr{P}_{\mathtt{n,m}} \setminus \mathscr{N}$. Indeed, if $p \in \mathscr{N}$ then due to invariance of $\mathscr{N}$ under $\mathscr{W}_U$ we have $y^* = \mathscr{W}_U(p) \in \mathscr{N} \cap \Theta_{\mathtt{n,m,r}} = \mathscr{Z}$ ($y^* \in \Theta_{\mathtt{n,m,r}}$ by definition). Thus, for every $\widetilde{p} \in \mathscr{N}$ and for all $\varepsilon > 0$, there exists $p \in \mathscr{P}_{\mathtt{n,m}} \setminus \mathscr{N}$ such that $\|p - \widetilde{p}\| \leqslant \varepsilon$. This shows that $\mathscr{N}$ satisfies $S3$ in Definition 4.15. ■.

### 4.4.2 CGCARE is generically unsolvable

We present the third main result of this chapter in this section.

CGCARE is generically unsolvable

**Theorem 4.22.** *Consider all singular LQR problems of the form given in Problem* (2.1). *Let the corresponding CGCARE be as given in equation* (4.1). *Then, the CGCARE is generically unsolvable.*

*Proof:* From Theorem 4.8, it is clear that for the CGCARE to have a solution, a necessary condition is that the corresponding $\mathtt{det}(sE - H)$ must be zero. From Theorem 4.16, we know that $\mathtt{det}(sE - H) \neq 0$ is a generic property among all singular LQR problems. Therefore, the corresponding CGCARE is generically unsolvable. ■

From [FN14] and [FN16] it is known that for a singular infinite-horizon LQR problem to admit an impulse-free solution, a necessary and sufficient condition is that the corresponding CGCARE (4.1) must admit a solution. However, we have shown in Theorem 4.22 that the CGCARE is unsolvable generically. Thus, we infer that all singular LQR problems generically do not admit impulse-free solutions. This further means that unlike the regular LQR problem, singular infinite-horizon LQR problems generically cannot be solved using static state-feedback optimal control law. Informally, this means that given any singular LQR problem that can be solved using static state-feedback optimal control law, an arbitrarily small perturbation in the parameter matrices converts it to another singular LQR problem that is unsolvable using static state-feedback optimal control law.

The results in this chapter reveals that the theory of CGCARE is applicable only if the Hamiltonian system is non-autonomous. On the other hand, the theory that we have developed in Chapter 3 had the assumption that $\mathtt{det}(sE - H) \neq 0$, i.e., the Hamiltonian system is autonomous. This leads to the natural question: How do we solve a singular LQR problem that admits a non-autonomous Hamiltonian system with no CGCARE solution? Note that the theory developed in Chapter 3 is applicable to LQR problems with the underlying system being single-input. We have established in Lemma 3.16 that for the single-input case the Hamiltonian system is always autonomous for non-trivial $Q$. Hence, there exists no single-input LQR problem, for non-trivial $Q$, that admits a CGCARE solution. Thus, the theory developed in Chapter 3 is applicable to all single-input LQR problems with the exception of problems that admit Hamitlomian systems with imaginary-axis eigenvalues. However, for the multi-input case there exist LQR problems that admit non-autonomous Hamiltonian systems and do not admit a CGCARE solution: see Example 4.13. Although the set of such a class of system is thin among the set of singular LQR problems (by Theorem 4.16) yet it is a question worth investigating. This is a matter of future research and hence we do not dwell on such a class of system further.

## 4.5 Summary

In this chapter, we showed that the CGCARE corresponding to a singular LQR problem is generically unsolvable (Theorem 4.22). We obtained this result in three steps: we first deduced necessary and sufficient conditions for solvability of the CGCARE (Theorem 4.8). Using these conditions we derived a necessary condition for solvability of the CGCARE (Corollary 4.9) and then finally we showed that this necessary condition generically fails to hold in the sample space of all infinite-horizon singular LQR problems (Theorem 4.16). It has been shown in the literature that solvability of CGCARE is a necessary and sufficient condition for singular LQR problems to admit an impulse-free solution that is implementable as a static state-feedback law. Using this fact, in conjunction with Theorem 4.22 of this chapter, we can then infer that singular infinite-horizon LQR problems generically do not admit solutions by static state-feedback optimal control law. As a matter of fact for single-input systems CGCARE is solvable if and only if $Q = 0$. This makes the singular LQR problem for the single-input case trivial. Hence, we infer that a nontrivial singular LQR problem, for a single input system, that admits a regular Hamiltonian pencil needs to be solved using PD state-feedback control law that we have proposed in Theorem 3.12.

# Part II

# Passive systems

# Chapter 5

# Storage functions of singularly passive SISO systems

## 5.1 Introduction

Theory of passive systems has been a cornerstone in network theory, systems theory and control. Passivity theory has benefited immensely from the celebrated Kalman-Yakubovich-Popov (KYP) lemma that made the theory useful for large scale systems by providing a linear matrix inequality (LMI) based formulation of passivity [Kal63], [Yak62], [Pop64]. The KYP lemma states that a bounded-input bounded-output (BIBO) stable system, with minimal i/s/o representation

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \tag{5.1}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$, is passive if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $K \geqslant 0$, such that $K$ satisfies the following LMI:

$$\begin{bmatrix} A^TK + KA & KB - C^T \\ B^TK - C & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{5.2}$$

We call this LMI the *KYP LMI*. Upon assuming that $(D + D^T)$ is positive definite, the KYP LMI becomes equivalent to the following quadratic matrix inequality, known as the algebraic Riccati inequality (ARI):

$$A^TK + KA + (KB - C^T)(D + D^T)^{-1}(B^TK - C) \leqslant 0. \tag{5.3}$$

In analysis and synthesis of passive systems, it is often required to obtain the rank-minimizing solutions of the KYP LMI, especially the maximal and minimal among the rank-minimizing solutions – these special rank-minimizing solutions are called the *extremal solutions*. These extremal solutions are obtained by solving the matrix *equation* corresponding to the ARI, the algebraic Riccati equation (ARE):

$$A^TK + KA + (KB - C^T)(D + D^T)^{-1}(B^TK - C) = 0. \tag{5.4}$$

Apart from these extremal solutions, all the other solutions of the ARE (5.4) are also known to be the rank-minimizing solutions of the corresponding KYP LMI. Thus, the ARE/ARI based techniques have become one of the most widely used methods to solve the KYP LMI. However, these methods are rendered unusable when $D+D^T$ becomes singular. Hence, the condition of $D+D^T$ being invertible is the *feed-through regularity* condition for the KYP LMI (5.2). The class of systems that do not satisfy this feed-through regularity condition is what we are concerned with in this chapter. We call such systems *singularly passive systems*, and the KYP LMI corresponding to such systems the *singular KYP LMI* (see Definition 5.1 for a formal definition).

As mentioned earlier, rank-minimizing solutions of the KYP LMI are used in various domains of systems and control theory. In network theory, such solutions reveal the trajectories of optimal-charging and optimal-discharging in an RLC circuit [Wil72], [WT98, Remark 5.14], [CSMB14]. Numerous model order reduction techniques have been developed recently that rely on the rank-minimizing solutions, in particular the extremal solutions, of the KYP LMI: see [GA04], [TMR09] and references therein. In all of these problems, satisfaction of the feed-through regularity condition remains a standing assumption [GA04], [TMR09]. A version of the regularity condition appears in $H_2/H_\infty$-optimal control also (see [DGKF89], [Sch89]). However, in many situations, such a condition may turn out to be too restrictive. Indeed, many RLC networks that do not satisfy the feed-through regularity condition can be readily constructed. An example of such is shown in Figure 5.1. In this chapter, we present a method to compute rank-minimizing solutions of the KYP LMI that arises out of a passive system which need not necessarily satisfy the feed-through regularity condition.

Relaxation of the feed-through regularity condition from the KYP lemma has been an active area of research (see [CS92], [WWS94], [Rei11]). Our approach in this chapter, to tackle the problem of solving singular KYP LMI, is somewhat closer to the one taken in [Rei11]; to the extent that, just like in [Rei11], the central object in our analysis that leads to the solution, is the *Hamiltonian pencil* (see equation (5.7) for the definition). However, unlike [Rei11], our approach does not depend on the notion of neutral deflating subspaces. Rather our approach is similar to the one taken in Chapter 2 for singular LQR problems. Although the approach is the same as that in Chapter 2, however the auxiliary results required to get to the rank-minimizing solutions of singular KYP LMI are not exactly similar to that of the singular LQR problem. Further, these auxiliary results reveal new insights into passive systems, as well. Similar to the LQR LMI for a singular LQR problem, our solution to the singular KYP LMI is obtained by providing a simple extension to the conventional method of solving the regular KYP LMI by computing eigenspaces of the Hamiltonian pencil. It is known that, for singular KYP LMIs, the Hamiltonian pencil shows a deficit in the dimension of the required eigenspaces (see Example 5.5). The above-mentioned exten-

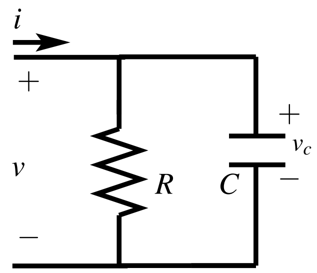


Figure 5.1: A singularly passive RC circuit

sion presented in this chapter, comes in the form of compensating these eigenspaces by adding new basis vectors coming from a suitable arrangement of the controllability and observability matrices. A remarkable outcome of this extension is that, once this compensation is done, a rank-minimizing solution of the KYP LMI can then be found by following a method exactly same as the conventional one. This is the main result in this chapter (Theorem 5.7). Perhaps the most crucial observation that enables us in arriving at this result is an interesting property satisfied by the Markov parameters of a singularly passive SISO system (Lemma 5.14). The relative simplicity of construction of the rank-minimizing solutions of a singular KYP LMI, as provided by Theorem 5.7, results in an easy-to-implement Algorithm 5.16. As a special case of this algorithm, we can retrieve the already known algorithm to compute rank-minimizing solutions of KYP LMI for passive systems that admit the ARE (Section 5.4). A well-known computational approach towards solving the KYP LMI involves using semi-definite programming techniques [VBW+05]. However, such methods require $\mathscr{O}(\mathrm{n}^6)$ floating point operations (flops) [VBW+05], while exploitation of certain structures in the problem may lead to an improvement up to $\mathscr{O}(\mathrm{n}^{4.5})$ [VBW+05]. The method we propose in this chapter has a flop count of $\mathscr{O}(\mathrm{n}^3)$ (Table 5.3). Note that the main result of this chapter is currently formulated and proved for SISO systems.

## 5.2 Preliminaries

In this section we review the preliminary concepts required to develop the theory behind the main results of this chapter.

### 5.2.1 Controller canonical form

Though the controller canonical form is standard, we include it for completeness. Consider a system with a strictly proper transfer function $G(s) = \frac{n(s)}{d(s)}$ where $n(s) = b_{\mathrm{n}-1}s^{\mathrm{n}-1} + b_{\mathrm{n}-2}s^{\mathrm{n}-2} + \cdots + b_0$ and $d(s) = s^{\mathrm{n}} + a_{\mathrm{n}-1}s^{\mathrm{n}-1} + \cdots + a_1 s + a_0$. Define the controller canonical form state-space representation of the system $\frac{d}{dt}x = Ax + Bu$ and $y = Cx$, where $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and $B, C^T \in \mathbb{R}^{\mathrm{n}}$ with $A, B, C$ as

$$A := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{\mathrm{n}-2} & -a_{\mathrm{n}-1} \end{bmatrix},\ B := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C := \begin{bmatrix} b_0 \\ \vdots \\ b_{\mathrm{n}-2} \\ b_{\mathrm{n}-1} \end{bmatrix}^T.$$

### 5.2.2 Passivity

The LMI (5.2) originates from a more fundamental law that passive systems satisfy known as the *dissipation inequality*: a system with minimal i/s/o representation (5.1) is passive if and only if there exists a $K = K^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ with $K \geqslant 0$ such that for every $\mathtt{col}(x,u,y) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathrm{n}+2\mathrm{p}})$ that satisfies equation (5.1), we must have (see [Wil72])

$$\frac{d}{dt}\left(x^T Kx\right) \leqslant 2u^T y \text{ for all } t \in \mathbb{R}. \tag{5.5}$$

It can be shown that $K$ satisfies the dissipation inequality (5.5) if and only if $K$ is a solution of the KYP LMI (5.2). The quadratic form $\left(x^T Kx\right)$ that appears on the left-hand side of inequality (5.5) is known as a *storage function* of the system. Thus, every solution to the KYP LMI (5.2) is uniquely associated with a storage function of the system. In the sequel, once a state-space is specified, we often ignore the distinction between the matrix $K$ and the quadratic form it induces, and, hence, use the term 'storage function' and 'solution matrix $K$ of the corresponding KYP LMI' interchangeably.

It is known that a system with transfer function matrix $G(s)$ is passive if and only if it is *positive real*: i.e., $G(s)$ is BIBO stable, and $G(i\omega)+G(-i\omega)^T \geqslant 0$ for all $\omega \in \mathbb{R}$ [AV06, Chapter 5]. In this chapter, we deal with passive systems for which $\mathtt{det}\left(G(i\omega)+G(-i\omega)^T\right) \to 0$ as $\omega \to \infty$. Note that, this is true if and only if the rational function $\mathtt{det}\left(G(s)+G(-s)^T\right)$ is strictly proper. In terms of the i/s/o representation, this is equivalent to $D+D^T$ being singular. Clearly, in this situation, obtaining storage functions by solving the KYP LMI via ARE/ARI is ruled out. Such systems are of central importance in this chapter. Due to the singularity of the corresponding feed-through term $D+D^T$, we call these systems *singularly passive* systems (see Definition 5.1 below for an exact definition). Similar to Chapter 2, in this chapter, we present an easy-to-implement algorithm that provides explicit solution formulae to KYP LMI of singularly passive SISO systems. Although the underlying principles that make the algorithm work are in stark difference from the conventional ARE/ARI-based methods, there is, however, a close parallel between the two. In order to put our analysis in perspective with respect to the conventional ARE/ARI approach, we formally define below both singularly passive systems and those passive systems, as well, for which ARE/ARI approach works: in contrast to singularly passive systems, we call these systems *regularly passive*. In the sequel, we denote the numerator of a rational function $p(s)$ by the symbol $\mathtt{num}(p(s))$. We use the symbols $\mathtt{roots}(q(s))$ and $\mathtt{rootnum}(p(s))$ to denote the set of the roots of $q(s)$ and $\mathtt{num}(p(s))$, respectively, where $q(s) \in \mathbb{R}[s]$ and $p(s) \in \mathbb{R}(s)$. Here a root is included in the set $\mathtt{roots}(q(s))$ and $\mathtt{rootnum}(p(s))$ as many times as it appears as a root of $q(s)$ and $\mathtt{num}(p(s))$, respectively. Further, we denote the degree of the polynomials $q(s)$ and $\mathtt{num}(p(s))$ with $\mathtt{deg}\,(q(s))$ and $\mathtt{degnum}(p(s))$, respectively.

**Definition 5.1.** *A SISO system with transfer function $G(s) \in \mathbb{R}(s)$ and a minimal i/s/o representation given by equation* (5.1) *is said to be* <u>regularly passive</u> *if it satisfies each of the following four conditions:*

1. *The system is BIBO stable, i.e., all the poles of $G(s)$ lie in $\mathbb{C}_-$.*
2. *There exists a solution $K = K^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ to the corresponding KYP LMI* (5.2).
3. $\mathtt{rootnum}\,(G(s)+G(-s)) \cap j\mathbb{R} = \emptyset$.

*4.* $\texttt{degnum}\,(G(s)+G(-s)) = 2\mathtt{n}$.

*In contrast, a SISO system with transfer function* $G(s) \in \mathbb{R}(s)$ *and a minimal i/s/o representation given by equation* (5.1) *is said to be* singularly passive *if the system satisfies properties (1), (2) and (3) above, but, instead of property (4), it satisfies* $0 \leqslant \texttt{degnum}\,(G(s)+G(-s)) < 2\mathtt{n}$.

For a SISO system with transfer function $G(s)$, the rational function $G(s)+G(-s)$ is known as the *Popov function*. Further, the roots of the numerator of the Popov function are also known as the *spectral zeros* of the system $\Sigma$: see [TMR09]. The following property of the poles and spectral zeros of a passive SISO system is crucially used in this chapter.

A passive and BIBO stable SISO system does not share spectral zeros and poles

**Lemma 5.2.** *Consider a SISO system* $\Sigma$ *with transfer function* $G(s) := \frac{n(s)}{d(s)}$*, where* $n(s), d(s) \in \mathbb{R}[s]$ *are coprime. Define* $q(s) := n(s)d(-s) + d(s)n(-s)$*. Let* $\lambda \in \texttt{roots}\,(d(s))$*. Then,* $\lambda \in \texttt{roots}\,(q(s)) \cap \texttt{roots}\,(d(s))$ *if and only if* $\lambda \in \texttt{roots}\,(d(-s))$*. In particular, if* $\Sigma$ *is BIBO stable, then the following statements are true*

*(1)* $\texttt{roots}(q(s)) \cap \texttt{roots}(d(s)) = \emptyset$.

*(2)* $\texttt{num}\,(G(s)+G(-s)) = q(s)$.

*Proof:* **If:** Given $d(\lambda) = d(-\lambda) = 0$. Therefore, $q(\lambda) = n(\lambda)d(-\lambda) + n(-\lambda)d(\lambda) = 0$. Thus, $\lambda \in \texttt{roots}\,(q(s)) \cap \texttt{roots}\,(d(s))$.
**Only if:** Given $d(\lambda) = q(\lambda) = 0$. Thus, $q(\lambda) = n(\lambda)d(-\lambda) + n(-\lambda)d(\lambda) = n(\lambda)d(-\lambda) = 0$. Since $n(s)$ and $d(s)$ are coprime, $n(\lambda) \neq 0$. Therefore, $d(-\lambda) = 0$, i.e., $\lambda \in \texttt{roots}\,(d(-s))$.
*(1):* Clearly, if $G(s)$ is BIBO stable then $\lambda \in \texttt{roots}\,(d(s))$ implies that $\lambda \notin \texttt{roots}\,(d(-s))$. Therefore, $\lambda \notin \texttt{roots}\,(q(s)) \cap \texttt{roots}\,(d(s))$. Since this is true for all roots of $d(s)$, we must have $\texttt{roots}\,(q(s)) \cap \texttt{roots}\,(d(s)) = \emptyset$.
*(2):* From Statement *(1)* it is clear that $q(s)$ and $d(s)$ are coprime. We claim that $q(s)$ and $d(-s)$ are coprime, as well. To the contrary, assume that $q(s)$ and $d(-s)$ are not coprime. Let $\lambda_1 \in \texttt{roots}\,(q(s)) \cap \texttt{roots}\,(d(-s))$. Then, $\lambda_1 \in \texttt{roots}(d(-s)) \Rightarrow -\lambda_1 \in \texttt{roots}(d(s))$. Further, $q(\lambda_1) = n(\lambda_1)d(-\lambda_1) + n(-\lambda_1)d(\lambda_1) = 0 \Rightarrow n(-\lambda_1)d(\lambda_1) = 0 \Rightarrow d(\lambda_1) = 0$. Therefore, $\lambda_1 \in \texttt{roots}(d(s))$. However, since $\Sigma$ is BIBO stable, $\pm\lambda_1 \in \texttt{roots}(d(s))$ is not possible. Therefore, we must have $\texttt{roots}\,(q(s)) \cap \texttt{roots}\,(d(-s)) = \emptyset$. Thus, $q(s)$ and $d(-s)$ are coprime, as well. Therefore, $q(s)$ and $d(s)d(-s)$ are coprime. This implies that $\texttt{num}\,(G(s)+G(-s)) = \texttt{num}\left(\frac{q(s)}{d(s)d(-s)}\right) = q(s)$. ∎

Of crucial importance in the sequel is the degree of the numerator of the Popov function, as well. Note that the numerator of $G(s)+G(-s)$ is an *even-degree polynomial*. Therefore, $\texttt{degnum}\,(G(s)+G(-s))$ is a non-negative even integer. We denote this number by $2\mathtt{n_s}$, i.e., $\mathtt{n_s} := (\texttt{degnum}\,(G(s)+G(-s)))/2$, where $0 \leqslant \mathtt{n_s} < \mathtt{n}$. In this case, the system $\Sigma$ is said to be a *singularly passive SISO system of order* $\mathtt{n_s}$. It is important to note that we do not include, in the class of singularly passive SISO systems, those systems for which $G(s)+G(-s) = 0$.

Such systems, called *lossless*, do not admit an ARE approach as well. The theory presented in this chapter, do not directly apply to these systems. See [AV06, Section 6.5] and Chapter 7 for alternative methods for constructing storage functions of lossless systems. However, the method of computation of storage functions, presented in this chapter, although is different in principle from the one presented in [AV06, Section 6.5] for lossless systems, structurally the two methods are the same (see Section 7.3 of Chapter 7).

### 5.2.3 Hamiltonian matrix and Hamiltonian pencil

One of the most widely used methods to compute solutions of the KYP LMI of regularly passive systems is via finding solutions of the corresponding algebraic Riccati equation (ARE). This, in turn, is done using suitably chosen bases of invariant subspaces of the corresponding Hamiltonian matrix of the form:

$$\mathscr{H} := \begin{bmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -\left(A - B(D+D^T)^{-1}C\right)^T \end{bmatrix}. \tag{5.6}$$

Note that existence of the Hamiltonian matrix crucially depends on the nonsingularity of $D + D^T$. This renders the method of ARE/Hamiltonian matrix for singularly passive systems unusable. This problem can be circumvented if, instead of considering the Hamiltonian matrix, one considers the corresponding Hamiltonian *pencil*.

Even when $D+D^T$ is invertible, the presence of the inverse in the Hamiltonian matrix causes numerical issues with floating point arithmetic. It has been shown in [vD81] that this explicit inversion of $D+D^T$ can be avoided by using a special matrix pencil having the following form:

$$s\underbrace{\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & (D+D^T) \end{bmatrix}}_{H}. \tag{5.7}$$

Analogous to Chapter 2, we call the matrix pair $(E, H)$, the *Hamiltonian matrix pair*. Clearly, owing to the absence of the explicit inverse of $D+D^T$ in the expression for the Hamiltonian matrix pair, this pencil exists even for singularly passive systems. In what follows, we describe how this pencil can be exploited to obtain explicit solution formulae for storage functions of singularly passive SISO systems. It is important to note here that, the problem of finding solutions to the KYP LMI, when the corresponding ARE does not exist, using the Hamiltonian pencil has also been worked out in [Rei11]. It has been shown how solutions to the KYP LMI can be obtained by looking at maximal neutral deflating subspaces of the Hamiltonian pencil. The solution formulae that we present in this chapter, however, do not use the notion of deflating subspaces. Our solution follows from a crucial observation regarding Markov parameters of the system (Lemma 5.14) together with a few basic results regarding the structure of the eigenspaces of the Hamiltonian matrix pair (5.7). Similar to singular LQR problems, $(E,H)$ can be associated with a system of the form

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & (D+D^T) \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix}. \tag{5.8}$$

We call this the Hamiltonian system and use the symbol $\Sigma_{\text{Ham}}$ to represent it.

As mentioned above, eigenvalues and eigenvectors of the matrix pair $(E,H)$ plays a crucial role in the sequel. Hence, we establish the relation between the spectral zeros of a passive SISO system and the eigenvalues of the corresponding Hamiltonian matrix pair $(E,H)$ in the next lemma.

Eigenvalues of the Hamiltonian matrix pair are the spectral zeros of the system

**Lemma 5.3.** *Consider a BIBO stable, SISO system $\Sigma$ with transfer matrix $G(s)$ and a minimal i/s/o representation as given in equation* (5.1). *Let $(sE-H)$ with $E$ and $H$, as defined in equation* (5.7) *above, be the corresponding Hamiltonian pencil. Then,* $\mathtt{det}(sE-H) = -\mathtt{num}\,(G(s)+G(-s))$. *In particular,* $\sigma(E,H) = \mathtt{rootnum}\,(G(s)+G(-s))$.

*Proof:* Let $G(s) =: n(s)/d(s)$, where $n(s),d(s) \in \mathbb{R}[s]$ are coprime. The eigenvalues of $(E,H)$ are given by the roots of the characteristic polynomial $\mathtt{det}(sE-H)$. Using the procedure known as Schur complement, we get the following identity of rational functions:

$$\begin{aligned}
\det(sE-H) &= -\mathtt{det}\,(sI-A)\,\mathtt{det}\left(sI+A^T\right) \\
&\qquad \mathtt{det}\left((D+D^T) + \begin{bmatrix} C & -B^T \end{bmatrix} \begin{bmatrix} (sI-A)^{-1} & 0 \\ 0 & (sI+A^T)^{-1} \end{bmatrix} \begin{bmatrix} B \\ C^T \end{bmatrix}\right) \\
&= -\mathtt{det}\,(sI-A)\,\mathtt{det}\left(sI+A^T\right)\left[\left(D+C(sI-A)^{-1}B\right) + \left(D^T - B^T(sI+A^T)^{-1}C^T\right)\right] \\
&= -d(s)d(-s)\left(\frac{n(s)}{d(s)} + \frac{n(-s)}{d(-s)}\right) \\
&= -\left(n(s)d(-s) + n(-s)d(s)\right)
\end{aligned}$$

Since $\Sigma$ is BIBO stable, from Lemma 5.2 it follows that $\mathtt{num}\,(G(s)+G(-s)) = n(s)d(-s)+n(-s)d(s)$. Therefore, $\mathtt{det}(sE-H) = -\mathtt{num}\,(G(s)+G(-s))$. Hence, we infer that $\sigma(E,H) = \mathtt{rootnum}\,(G(s)+G(-s))$. ∎

Because of Lemma 5.3, henceforth, we refer to the eigenvalues of $(E,H)$, too, as the spectral zeros of the system.

A comparative summary among the various subclasses of passive systems defined in this chapter is presented in the form of a table in Table 5.1. Note that, in the lossless case, i.e., when $\mathtt{det}\left(G(s)+G(-s)^T\right) = 0$, the degree of the numerator of $\mathtt{det}\left(G(s)+G(-s)^T\right)$ is taken to be $-\infty$ as a matter of convention.

| Properties | Passive systems | | |
|---|---|---|---|
| | Lossless | Singularly passive | Regularly passive |
| Degree of $\mathtt{det}(sE-H)$ | $-\infty$ | $\{0,2,4,\cdots,2(\mathtt{n}-1)\}$ | $2\mathtt{n}$ |
| KYP LMI | exists | exists | exists |
| Feed-through term $D+D^T$ | singular | singular | nonsingular |
| ARE and Hamiltonian matrix | do not exist | do not exist | exist |
| Set of spectral zeros | entire $\mathbb{C}$-plane | cardinality is $\mathtt{degnum}\left(G(s)+G(-s)^T\right) =: 2\mathtt{n_s} < 2\mathtt{n}$ | cardinality is $\mathtt{degnum}\left(G(s)+G(-s)^T\right) = 2\mathtt{n}$ |

Table 5.1: Properties of passive systems

### 5.2.4 Solution to the KYP LMI: regularly passive systems

Before getting to the main results of this chapter, which provide methods for solving the KYP LMI for singularly passive SISO systems, it would be worthwhile looking at the regularly passive case first. Written next is a brief review of the method to compute rank-minimizing solutions of the KYP LMI corresponding to a regularly passive system using the eigenspaces of the Hamiltonian pencil. Similar to Chapter 2, this will help us in highlighting the similarities/dissimilarities between the existing method for regularly passive systems and the one presented in this chapter (the main result Theorem 5.7) for their singular counter-part.

Proposition 5.4 below summarizes the well-known method of computing rank-minimizing solutions of the KYP LMI for regularly passive systems. This result, in various different forms, can be found in several earlier works, for example, [Wil71], [Cop74], [Wim84], [Kuč91]. We have collated these results and paraphrased them in Proposition 5.4 below. Before we present the proposition, it is important to note the following property of the Hamiltonian pencil corresponding to a regularly passive SISO system. Let $\Sigma$ be a regularly passive SISO system with transfer function $G(s)$ and a minimal i/s/o representation given by (5.1). From Lemma 5.3 and Definition 5.1 it follows that, for the corresponding Hamiltonian matrix pair $(E,H)$, we must have $\mathtt{det}\,(sE-H)$ to be an even-degree polynomial with no roots on the imaginary axis. Thus, $\mathtt{det}\,(sE-H)$ for a regularly passive system must admit a Lambda-set (see Definition 2.18 for a definition of Lambda-sets). This fact is pivotal for Proposition 5.4 below.

**Proposition 5.4.** *Consider a regularly passive system $\Sigma$ with a minimal i/s/o representation as given in equation* (5.1) *and corresponding Hamiltonian matrix pair $(E,H)$ given by equation* (5.7)*. Assume $\Lambda$ to be a Lambda-set of $\mathtt{det}(sE-H)$ with cardinality $\mathtt{n}$. Let $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $V_{3\Lambda} \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$ be such that the columns of $V_{\mathtt{e}\Lambda} := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ form a basis of the $\mathtt{n}$-dimensional eigenspace of $(E,H)$ corresponding to the eigenvalues of $(E,H)$ in $\Lambda$. Then, the following statements hold.*

*(1) $V_{1\Lambda}$ is invertible.*

*(2) $K := V_{2\Lambda}V_{1\Lambda}^{-1}$ is symmetric.*

*(3) $K$ is a solution of the ARE: $A^TK+KA+(KB-C^T)(D+D^T)^{-1}(B^TK-C)=0$.*

(4) *$K$ is a rank-minimizing solution of the KYP LMI* (5.2).

(5) *$K$ is positive semi-definite, i.e., $K \geqslant 0$.*

(6) *$x^T Kx$ is a storage function of the system $\Sigma$, i.e., $\frac{d}{dt}\left(x^T Kx\right) \leqslant 2u^T y$ for all $\mathtt{col}(x,u,y) \in \mathfrak{C}^{\infty}\left(\mathbb{R},\mathbb{R}^{\mathtt{n}+1+1}\right)$ that satisfy equation* (5.1).

It is crucial to note here that although there seems to be no requirement for satisfiability of the feed-through regularity condition in this method, for $V_{1\Lambda}$ to be a square matrix, nonsingularity of $D+D^T$ can be shown to be a necessary condition. Thus, this method cannot be used to compute the storage functions of systems that are not regularly passive. The significance of this proposition is the striking similarity between this and the singularly passive case in Theorem 5.7. In the next section, we first show with an example the reason for the failure of Proposition 5.4 when applied to singularly passive systems. Then we state the first main result of this chapter (Theorem 5.7) that presents a method to compute storage functions of a singularly passive system, in particular the rank-minimizing solutions of the corresponding KYP LMI.

## 5.3 Rank-minimizing solutions of the KYP LMI: SISO case

In this section we present the first main result of this chapter, Theorem 5.7. Since we are dealing with only singularly passive SISO systems, a minimal i/s/o representation of the system takes the following simpler form:

$$\frac{d}{dt}x = Ax + bu, \quad y = cx, \quad \text{where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, b, c^T \in \mathbb{R}^{\mathtt{n}}. \tag{5.9}$$

Let $\Sigma$ be a singularly passive SISO system of order $\mathtt{n_s}$ with a minimal i/s/o representation of $\Sigma$ as given in equation (5.9). Therefore, the KYP LMI (5.2) for $\Sigma$ takes the following form:

$$\begin{bmatrix} A^T K + KA & Kb - c^T \\ b^T K - c & 0 \end{bmatrix} \leqslant 0. \tag{5.10}$$

This is the singular KYP LMI corresponding to the singularly passive SISO system $\Sigma$. Similarly, the Hamiltonian pencil for the singularly passive SISO system $\Sigma$ takes the form

$$s\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} A & 0 & b \\ 0 & -A^T & c^T \\ c & -b^T & 0 \end{bmatrix}}_{H} = s\begin{bmatrix} I_{2\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \widehat{A} & \widehat{b} \\ \widehat{c} & 0 \end{bmatrix}, \tag{5.11}$$

where $\widehat{A} := \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times 2\mathtt{n}}$, $\widehat{b} := \begin{bmatrix} b \\ c^T \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times 1}$ and $\widehat{c} := \begin{bmatrix} c & -b^T \end{bmatrix} \in \mathbb{R}^{1\times 2\mathtt{n}}$. Recall from Lemma 5.3 that $\mathtt{det}(sE - H) = -\mathtt{num}(G(s) + G(-s))$. Therefore, $\Sigma$ being singularly passive of order $\mathtt{n_s}$ implies that $\mathtt{degdet}\,(sE-H) = 2\mathtt{n_s}$. Hence, the cardinality of $\sigma(E,H)$ is $2\mathtt{n_s}$ (counted with multiplicity). For the rest of the chapter, we define $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$.

Next, using an example of a singularly passive SISO system, we motivate the reason of non-applicability of Proposition 5.4 in computation of the rank-minimizing solutions of the singular KYP LMI (5.10). Failure of Proposition 5.4 to compute rank-minimizing solutions of the singular KYP LMI in the next example will also lead to certain crucial questions whose answers are provided by the main result (Theorem 5.7) of this chapter.

**Example 5.5.** *Consider the transfer function* $G(s) = \dfrac{3s^2+12s+11}{s^3+6s^2+11s+6}$ *of system* $\Sigma$*. A minimal i/s/o representation of* $\Sigma$ *is as follows.*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 11 & 12 & 3 \end{bmatrix} x.$$

*The system* $\Sigma$ *is a singularly passive system of order* 2*. The spectral zeros of* $\Sigma$*, i.e., zeros of the determinant of the corresponding Hamiltonian pencil* $(sE-H)$ *are*

$$\left\{-\sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}, \sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\}.$$

*Let us consider any one of the four possible Lambda-sets of* $\mathtt{det}(sE-H)$*. Let a Lambda-set* $\Lambda$ *be given by* $\Lambda = \left\{-\sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\}$*. The eigenspace of* $(E,H)$ *corresponding to* $\Lambda$ *is given by the column-span of*

$$V_{\mathtt{e}\Lambda} = \begin{bmatrix} 0.34 & -0.85 & 2.12 & -0.35 & -0.36 & -0.09 & 0.13 \\ 0.01 & 0.01 & 0.01 & 2.27 & 1.35 & 0.17 & 0.18 \end{bmatrix}^T,$$

*where the columns are the eigenvectors of the eigenvalues of* $(E,H)$ *in* $\Lambda$*. Suitably partitioning* $V_{\mathtt{e}\Lambda}$*, as in Proposition 5.4, gives*

$$V_{1\Lambda} = \begin{bmatrix} 0.34 & -0.85 & 2.12 \\ 0.01 & 0.01 & 0.01 \end{bmatrix}^T \quad V_{2\Lambda} = \begin{bmatrix} -0.35 & -0.36 & -0.09 \\ 2.27 & 1.35 & 0.17 \end{bmatrix}^T$$

*Clearly,* $V_{1\Lambda}$ *is a non-square matrix, and hence, a solution of the KYP LMI of the form* $K = V_{2\Lambda}V_{1\Lambda}^{-1}$ *does not exist. This shows that Proposition 5.4 cannot be directly used to compute the storage functions of singularly passive systems.*

From Example 5.5, it is clear that, similar to the singular LQR problems, the primary reason for the failure of Proposition 5.4 in case of singularly passive SISO systems is the fact that the degree of $\mathtt{det}\,(sE-H)$ is *strictly less* than $2\mathtt{n}$. This fall in the degree causes a deficit in the cardinality of possible Lambda-sets of $\mathtt{det}\,(sE-H)$. Indeed, a Lambda set of $\mathtt{det}\,(sE-H)$ can now have cardinality only $\mathtt{n_s}$, which is strictly less than $\mathtt{n}$. Consequently, the eigenspace of $(E,H)$ corresponding to such a Lambda-set would also show a deficit in its dimension from being $\mathtt{n}$ in the regular case. This deficit in the dimension of the eigenspace is required to be compensated by $\mathtt{n}-\mathtt{n_s}$ suitable vectors. Of course, this compensation cannot be done by arbitrary vectors. Our first main result, Theorem 5.7, shows exactly how this compensation is to be done. However, before we address the issue of choosing suitable vectors to append to $V_{\mathrm{e}\Lambda}$, another fundamental question needs to be answered first: do Lambda-sets of $\mathtt{det}\,(sE-H)$ exist for the case of singularly passive SISO systems? The next lemma answers this question.

Singularly passive SISO systems admit Lambda-sets

**Lemma 5.6.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with transfer function $G(s)$ and a minimal i/s/o representation as given by equation* (5.9). *Let the corresponding Hamiltonian pencil be as defined in equation* (5.11). *Assume $\mathtt{det}(sE-H)\notin\mathbb{R}$. Then, there exists a non-empty Lambda-set of $\mathtt{det}(sE-H)$ with cardinality $\mathtt{n_s}$.*

*Proof:* Let $G(s)=\frac{n(s)}{d(s)}$. From Lemma 5.2 we have, $\mathtt{num}\,(G(s)+G(-s))=n(s)d(-s)+d(s)n(-s)=:q(s)$. Since $q(s)\notin\mathbb{R}$, there exists $\lambda\in\mathbb{C}$ such that $q(\lambda)=0$. Based on the following three arguments, we infer that $\mathtt{roots}\,(q(s))$ are mirrored about $j\mathbb{R}$.

1. Since $q(s)=q(-s)$, $\lambda\in\mathtt{roots}\,(q(s))\Rightarrow-\lambda\in\mathtt{roots}\,(q(s))$.
2. Since $q(s)\in\mathbb{R}[s]$, $\lambda\in\mathtt{roots}\,(q(s))\Rightarrow\bar{\lambda}\in\mathtt{roots}\,(q(s))$.
3. Since $\Sigma$ is singularly passive, from Definition 5.1 it follows that $\mathtt{roots}\,(q(s))\cap j\mathbb{R}=\emptyset$.

From Lemma 5.2 and Lemma 5.3, we have $\sigma(E,H)=\mathtt{rootnum}\,(G(s)+G(-s))=\mathtt{roots}\,(q(s))$, therefore the elements of $\sigma(E,H)$ are mirrored about $j\mathbb{R}$. Therefore, if $\mathtt{degdet}(sE-H)=2\mathtt{n_s}$ ($\mathtt{n_s}<\mathtt{n}$), then $\sigma(E,H)$ can be decomposed into two disjoint subsets $\Lambda_1,\Lambda_2\subsetneq\sigma(E,H)$ such that $\Lambda_1\cup\Lambda_2=\sigma(E,H)$ and $\Lambda_1\subsetneq\mathbb{C}_+$, $\Lambda_2\subsetneq\mathbb{C}_-$. Note that

1. Since $\Lambda_1\subsetneq\mathbb{C}_+$ and $q(s)\in\mathbb{R}[s]$, $\Lambda_1=\bar{\Lambda}_1$.
2. From $\Lambda_1\subsetneq\mathbb{C}_+$ and $\Lambda_2\subsetneq\mathbb{C}_-$, it follows that $\Lambda_2=-\Lambda_1$. Therefore, $\Lambda_1\cap(-\Lambda_1)=\emptyset$.
3. The cardinality of $\Lambda_1$ and $\Lambda_2$ are $\mathtt{n_s}$ each. Therefore, $\Lambda_1\cup(-\Lambda_1)=\Lambda_1\cup\Lambda_2=\sigma(E,H)=\mathtt{roots}(q(s))$.

Using the above three arguments and Definition 2.18, we infer that $\Lambda_1$ is a Lambda-set of $\mathtt{det}(sE-H)$ with cardinality $\mathtt{n_s}$. This completes the proof of Lemma 5.6. ■

Now that we have proved the existence of Lambda-sets in singularly passive SISO systems, the crucial questions left to be answered are:

1. For singularly passive SISO systems, are there $(\mathtt{n}-\mathtt{n_s})$ independent vectors that can be appended to $V_{\mathrm{e}\Lambda}$ (the matrix whose columns span the eigenspace of $(E,H)$ corresponding to a Lambda-set of $\mathtt{det}\,(sE-H)$) so as to replicate the method described in Proposition 5.4? If such vectors exist, then how can we find them?

2. In particular, for singularly passive SISO systems of order 0, i.e., $\mathtt{det}(sE-H)\in\mathbb{R}$, there exists no Lambda-set since $\mathtt{n_s}=0$. Is it possible to find such vectors for these systems also?

The answer to all these questions are provided by the next theorem. This is the first main result of this chapter.

A method to compute the rank-minimizing solutions of a KYP LMI

**Theorem 5.7.** *Consider a singularly passive SISO system* $\Sigma$ *of order* $\mathtt{n_s}$ *with a minimal i/s/o representation as in equation* (5.9). *Let the corresponding Hamiltonian pencil be as defined in equation* (5.11). *Define* $\mathtt{deg}\,\mathtt{det}\,(sE-H)=:2\mathtt{n_s}$. *Assume* $\Lambda$ *to be a Lambda-set of* $\mathtt{det}(sE-H)$ *with cardinality* $\mathtt{n_s}$. *Define* $\mathtt{n_f}:=\mathtt{n}-\mathtt{n_s}$. *Let* $V_{1\Lambda},V_{2\Lambda}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ *and* $V_{3\Lambda}\in\mathbb{R}^{1\times\mathtt{n_s}}$ *be such that the columns of* $\mathtt{col}(V_{1\Lambda},V_{2\Lambda},V_{3\Lambda})=:V_{\mathtt{e}\Lambda}$ *form a basis of the* $\mathtt{n_s}$*-dimensional eigenspace of* $(E,H)$ *corresponding to* $\Lambda$*, i.e.,*

$$\begin{bmatrix} A & 0 & b\\ 0 & -A^T & c^T\\ c & -b^T & 0\end{bmatrix}\begin{bmatrix} V_{1\Lambda}\\ V_{2\Lambda}\\ V_{3\Lambda}\end{bmatrix}=\begin{bmatrix} I_{\mathtt{n}} & 0 & 0\\ 0 & I_{\mathtt{n}} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} V_{1\Lambda}\\ V_{2\Lambda}\\ V_{3\Lambda}\end{bmatrix}\Gamma_\Lambda,\ \text{where } \Gamma_\Lambda\in\mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}} \text{ and } \sigma(\Gamma_\Lambda)=\Lambda. \quad (5.12)$$

*Define* $V_\Lambda:=\begin{bmatrix} V_{1\Lambda}\\ V_{2\Lambda}\end{bmatrix}\in\mathbb{R}^{2\mathtt{n}\times\mathtt{n_s}}$ *and* $W:=\begin{bmatrix}\widehat{b} & \widehat{Ab} & \ldots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b}\end{bmatrix}\in\mathbb{R}^{2\mathtt{n}\times\mathtt{n_f}}$. *Partition* $\begin{bmatrix} V_\Lambda & W\end{bmatrix}\in\mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$ *as*

$$\begin{bmatrix} V_\Lambda & W\end{bmatrix}=:\begin{bmatrix} X_{1\Lambda}\\ X_{2\Lambda}\end{bmatrix}\ \text{where } X_{1\Lambda},X_{2\Lambda}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}. \quad (5.13)$$

*Then, the following statements hold.*

*(1)* $X_{1\Lambda}$ *is invertible.*

*(2)* $K:=X_{2\Lambda}X_{1\Lambda}^{-1}$ *is symmetric.*

*(3)* $K$ *is a solution to LMI* (5.10)*, i.e.,* $Kb-c^T=0$ *and* $A^TK+KA\leqslant 0$.

*(4)* $K$ *is a rank-minimizing solution of LMI* (5.10).

*(5)* $K$ *is positive semi-definite, i.e.,* $K\geqslant 0$.

*(6)* $x^TKx$ *is a storage function of the system* $\Sigma$*, i.e.,* $\frac{d}{dt}\left(x^TKx\right)\leqslant 2u^Ty$ *for all* $\mathtt{col}(x,u,y)\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}+1+1})$ *that satisfy the i/s/o representation of* $\Sigma$.

A striking fact about the above result is the similarity between Theorem 5.7 and the regularly passive case (see Proposition 5.4). However, the procedure and concepts involved in obtaining $K$ is very different. In Theorem 5.7, the columns of the matrix $W$ supply the additional $\mathtt{n_f}$

number of independent vectors that completes the matrix $V_{1\Lambda}$ to a square nonsingular matrix $X_{1\Lambda}$. Recall that the existence of $W$ has been alluded to in Example 5.5 and the discussion thereafter.

Note that Theorem 5.7 is similar to Theorem 2.30 in Chapter 2. However the proof of Theorem 2.30, especially Statement *(1)*, requires auxiliary results that are not similar to the ones required to prove Theorem 2.30. Therefore, in the next section we present all the auxiliary results required to prove Theorem 5.7.

## 5.3.1 Auxiliary results to prove Theorem 5.7

For the ease of referencing, we summarize the results required for the proof of Theorem 5.7 in the form of a table in Table 5.2.

Recall the definitions of the following matrices, namely, $(A,b,c),(\widehat{A},\widehat{b},\widehat{c}),V_{\mathtt{e}\Lambda},(V_{1\Lambda},V_{2\Lambda})$ from Theorem 5.7. The following notational convention is required for the auxiliary results. Let $T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, nonsingular, be such that, under the similarity transformation induced by $T$, the system matrix $A$ transforms to $A_{\mathtt{t}}:=T^{-1}AT$. It is known that under this transformation, matrices $b$ and $c$ are transformed to $b_{\mathtt{t}}:=T^{-1}b$ and $c_{\mathtt{t}}:=cT$, respectively. We assume that $(A_{\mathtt{t}},b_{\mathtt{t}},c_{\mathtt{t}})$ is in the controller canonical form (see Section 5.2.1). Let $(E_{\mathtt{t}},H_{\mathtt{t}})$ be the Hamiltonian matrix pair formed using the matrices $(A_{\mathtt{t}},b_{\mathtt{t}},c_{\mathtt{t}})$. Let $X_{1\Lambda}$ and $X_{1\Lambda_{\mathtt{t}}}$ be constructed as defined in Theorem 5.7 using Hamiltonian matrix pair $(E,H)$ and $(E_{\mathtt{t}},H_{\mathtt{t}})$, respectively.

| Lemma | Result | Remarks/Conclusion |
|---|---|---|
| Lemma 5.8 | $\sigma(E,H)\cap\sigma(A)=\emptyset$ | Singularly passive SISO systems do not share spectral zeros and poles. |
| Lemma 5.9 | $X_{1\Lambda}$ is invertible $\Leftrightarrow$ $X_{1\Lambda_{\mathtt{t}}}$ is invertible | Nonsingularity of $X_{1\Lambda}$ is invariant under change of basis. |
| Lemma 5.10 | Existence of $V_{\mathtt{e}\Lambda}=\mathtt{col}(V_{1\Lambda},V_{2\Lambda},V_{3\Lambda})$ with $V_{1\Lambda},V_{2\Lambda}\in\mathbb{C}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda}\in\mathbb{C}^{1\times\mathtt{n_s}}$ that satisfies equation (5.12) with $\Gamma_\Lambda\in\mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ in Jordan canonical form. | The structure of $V_{1\Lambda}$ is crucially used in the proof of Statement *(1)* of Theorem 5.7. This structure has been elaborated in equation (5.18); see also equation (5.28). |
| Lemma 5.12 | $V_{1\Lambda}^TV_{2\Lambda}=V_{2\Lambda}^TV_{1\Lambda}$ | Relation between the left- and right-eigenvectors of $(E,H)$ corresponding to $\sigma(\Lambda)$ and $\sigma(-\Lambda)$. |
| Lemma 5.13 | $\widehat{c}\left(sI_{2\mathtt{n}}-\widehat{A}\right)^{-1}\widehat{b}=G(s)+G(-s)$ | Popov function is the transfer function of a Hamiltonian system. |
| Lemma 5.14 | First $2(\mathtt{n_f}-1)$ moments of Popov function are zero, i.e., $\widehat{c}\widehat{A}^k\widehat{b}=0$ for $k\in\{0,1,\ldots,2(\mathtt{n_f}-1)\}$. | A property of the Markov parameters of the Hamiltonian system. |
| Lemma 5.15 | $\begin{bmatrix}-V_{2\Lambda}^T & V_{1\Lambda}^T\end{bmatrix}\widehat{A}^k\widehat{b}=0$ for $k\in\{0,1,\ldots,2(\mathtt{n_f}-1)\}$ | An identity relating the eigenvectors of $(E,H)$ and the system matrices of $\Sigma_{\mathtt{Ham}}$. |

Table 5.2: Table with a summary of lemmas used to prove Theorem 5.7

Next we state each of the lemmas in Table 5.2 and prove them one-by-one. The first lemma reveals an interesting fact about all BIBO stable SISO systems. It establishes that a BIBO stable SISO system never admits common spectral zeros and poles.

A BIBO stable SISO system does not share spectral zeros and poles

**Lemma 5.8.** *Consider a BIBO stable SISO system* $\Sigma$ *with a minimal i/s/o representation as in equation* (5.1)*. Let* $(E,H)$ *be the corresponding Hamiltonian pencil as defined in equation* (5.7)*. Then* $\sigma(E,H)\cap\sigma(A)=\emptyset$.

*Proof:* Define $G(s) := \frac{n(s)}{d(s)}$, where $n(s),d(s)\in\mathbb{R}[s]$ are coprime. Since $\Sigma$ is BIBO stable, from Lemma 5.2 we have $\texttt{num}\,(G(s)+G(-s)) = n(s)d(-s)+n(-s)d(s) =: q(s)$. Recall from Lemma 5.3 that for a BIBO stable system, $\sigma(E,H)=\texttt{rootnum}\,(G(s)+G(-s))=\texttt{roots}\,(q(s))$. Since $\Sigma$ is BIBO stable, from Lemma 5.2 we have $\texttt{roots}\,(q(s))\cap\texttt{roots}\,(d(s))=\emptyset$. Thus, $\sigma(E,H)\cap\sigma(A)=\emptyset$. ■

Since singularly passive SISO systems are BIBO stable (Statement *(1)* of Definition 5.1), from Lemma 5.8 it is evident that such systems have no common poles and spectral zeros.

The next lemma establishes the relation between the eigenvalues of the Hamiltonian matrix pair $(E,H)$ constructed using $(A,b,c)$ as given in equation (5.6), and the transformed Hamiltonian matrix pair $(E_{\texttt{t}},H_{\texttt{t}})$ constructed using $(A_{\texttt{t}},b_{\texttt{t}},c_{\texttt{t}})$.

Nonsingularity of $X_{1\Lambda}$ is invariant under change of basis

**Lemma 5.9.** *Consider a singularly passive SISO system* $\Sigma$ *with a minimal i/s/o representation as in equation* (5.9)*. Let the corresponding Hamiltonian matrix pair be* $(E,H)$ *as constructed in equation* (5.11)*. Let a controller canonical form i/s/o representation of* $\Sigma$ *be* $\frac{d}{dt}x = A_{\texttt{t}}x+b_{\texttt{t}}u$ *and* $y=c_{\texttt{t}}x$*. Let the Hamiltonian matrix pair constructed using* $(A_{\texttt{t}},b_{\texttt{t}},c_{\texttt{t}})$ *be* $(E_{\texttt{t}},H_{\texttt{t}})$*. Then,*

$$\sigma(E,H)=\sigma(E_{\texttt{t}},H_{\texttt{t}}).$$

*Further, let* $X_{1\Lambda},X_{2\Lambda}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *be constructed as defined in equation* (5.13) *of Theorem 5.7 using system matrices* $(A,b,c)$ *and Hamiltonian matrix pair* $(E,H)$ *corresponding to a Lambda-set* $\Lambda$*. Similarly, let* $X_{1\Lambda\texttt{t}},X_{2\Lambda\texttt{t}}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *be constructed using equation* (5.13)*, system matrices* $(A_{\texttt{t}},b_{\texttt{t}},c_{\texttt{t}})$ *and the Hamiltonian matrix pair* $(E_{\texttt{t}},H_{\texttt{t}})$ *corresponding to a Lambda-set* $\Lambda$*. Then,*

$X_{1\Lambda}$ *is invertible if and only if* $X_{1\Lambda\texttt{t}}$ *is invertible.*

*Proof:* Let $T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be a nonsingular matrix such that $T^{-1}AT=A_{\texttt{t}}, T^{-1}b=b_{\texttt{t}}$, and $cT=c_{\texttt{t}}$. Define $\widehat{A}_{\texttt{t}} := \texttt{diag}(A_{\texttt{t}},-A_{\texttt{t}}^T)$, $\widehat{b}_{\texttt{t}} = \texttt{col}(b_{\texttt{t}},c_{\texttt{t}}^T)$, and $\widehat{c}_{\texttt{t}} := \begin{bmatrix} c_{\texttt{t}} & -b_{\texttt{t}}\end{bmatrix}$. Further, define $\widehat{T} :=$

$\mathtt{diag}(T, T^{-T})$ and $\widetilde{T} := \mathtt{diag}(\widehat{T}, I_{\mathtt{p}})$, where $T^{-T} := (T^{-1})^T$. Then, we have the following

$$\widetilde{T}^{-1} H \widetilde{T} = \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} A & 0 & b \\ 0 & -A^T & c^T \\ c & -b^T & 0 \end{bmatrix} \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix} = \begin{bmatrix} A_{\mathtt{t}} & 0 & b_{\mathtt{t}} \\ 0 & -A_{\mathtt{t}}^T & c_{\mathtt{t}}^T \\ c_{\mathtt{t}} & -b_{\mathtt{t}}^T & 0 \end{bmatrix} = H_{\mathtt{t}},$$

$$\widetilde{T}^{-1} E \widetilde{T} = \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_{\mathtt{t}}$$

Therefore, the Hamiltonian matrix pair $(E, H)$ and $(E_{\mathtt{t}}, H_{\mathtt{t}})$ are equivalent[1]. By the property of equivalent matrix pencils, $\sigma(E, H) = \sigma(E_{\mathtt{t}}, H_{\mathtt{t}})$.

Define $V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda}$ as in Theorem 5.7 corresponding to a Lambda-set $\Lambda$ of $\mathtt{det}(sE - H)$. Similarly, Define $V_{1\Lambda\mathtt{t}}, V_{2\Lambda\mathtt{t}}, V_{3\Lambda\mathtt{t}}$ as in Theorem 5.7 corresponding to a Lambda-set $\Lambda$ of $\mathtt{det}(sE_{\mathtt{t}} - H_{\mathtt{t}})$. Then, from equation (5.13) we have

$$\begin{bmatrix} A & 0 & b \\ 0 & -A^T & c^T \\ c & -b^T & 0 \end{bmatrix} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma \text{ and } \begin{bmatrix} A_{\mathtt{t}} & 0 & b_{\mathtt{t}} \\ 0 & -A_{\mathtt{t}}^T & c_{\mathtt{t}}^T \\ c_{\mathtt{t}} & -b_{\mathtt{t}}^T & 0 \end{bmatrix} \begin{bmatrix} V_{1\Lambda\mathtt{t}} \\ V_{2\Lambda\mathtt{t}} \\ V_{3\Lambda\mathtt{t}} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda\mathtt{t}} \\ V_{2\Lambda\mathtt{t}} \\ 0 \end{bmatrix} \Gamma, \text{ where } \sigma(\Gamma) = \Lambda.$$

Replacing $(A, b, c)$ by $(TA_{\mathtt{t}}T^{-1}, Tb_{\mathtt{t}}, c_{\mathtt{t}}T^{-1})$ in the above equation, we have

$$\begin{bmatrix} TA_{\mathtt{t}}T^{-1} & 0 & Tb_{\mathtt{t}} \\ 0 & -(TA_{\mathtt{t}}T^{-1})^T & (c_{\mathtt{t}}T^{-1})^T \\ c_{\mathtt{t}}T^{-1} & -(Tb_{\mathtt{t}})^T & 0 \end{bmatrix} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma, \text{ where } \sigma(\Gamma) = \Lambda$$

$$\begin{bmatrix} A_{\mathtt{t}} & 0 & b_{\mathtt{t}} \\ 0 & -A_{\mathtt{t}}^T & c_{\mathtt{t}}^T \\ c_{\mathtt{t}} & -b_{\mathtt{t}}^T & 0 \end{bmatrix} \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma. \tag{5.14}$$

Therefore, from equation (5.14) it is clear that $\widehat{T}^{-1}\mathtt{col}(V_{1\Lambda}, V_{2\Lambda}) = \mathtt{col}(V_{1\Lambda\mathtt{t}}, V_{2\Lambda\mathtt{t}})$. Further, it is easy to verify that

$$W = \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix} = \widehat{T} \begin{bmatrix} \widehat{b}_{\mathtt{t}} & \widehat{A}_{\mathtt{t}}\widehat{b}_{\mathtt{t}} & \cdots & \widehat{A}_{\mathtt{t}}^{\mathtt{n_f}-1}\widehat{b}_{\mathtt{t}} \end{bmatrix}.$$

Defining $V_\Lambda := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda})$ and $V_{\Lambda_{\mathtt{t}}} := \mathtt{col}(V_{1\Lambda\mathtt{t}}, V_{2\Lambda\mathtt{t}})$, we therefore have

$$\begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix} = \begin{bmatrix} V_\Lambda & \widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix} = \widehat{T} \begin{bmatrix} V_{\Lambda_{\mathtt{t}}} & \widehat{b}_{\mathtt{t}} & \widehat{A}_{\mathtt{t}}\widehat{b}_{\mathtt{t}} & \cdots & \widehat{A}_{\mathtt{t}}^{\mathtt{n_f}-1}\widehat{b}_{\mathtt{t}} \end{bmatrix} = \begin{bmatrix} T & \\ & T^{-T} \end{bmatrix} \begin{bmatrix} X_{1\Lambda\mathtt{t}} \\ X_{2\Lambda\mathtt{t}} \end{bmatrix}.$$

[1] Two matrix pairs $(A_1, A_2)$ and $(B_1, B_2)$ are equivalent if there exist nonsingular matrices $P$ and $Q$ such that $P(sA_1 - A_2)Q = (sB_1 - B_2)$. Note that $\mathtt{det}(sB_1 - B_2) = \mathtt{det}(P)\mathtt{det}(Q)\mathtt{det}(sA_1 - A_2)$. Therefore, characteristic polynomials of $(A_1, A_2)$ and $(B_1, B_2)$ are the same (up to to scaling), i.e., $\sigma(A_1, A_2) = \sigma(B_1, B_2)$.

Thus, we have $X_{1\Lambda} = TX_{1\Lambda_{\mathtt{t}}}$. Since $T$ is nonsingular, $X_{1\Lambda}$ is nonsingular if and only if $X_{1\Lambda_{\mathtt{t}}}$ is nonsingular. ■

Note that in the proof of Lemma 5.9 we have not used the fact that $A_{\mathtt{t}}$, $b_{\mathtt{t}}$, and $c_{\mathtt{t}}$ are in controller canonical form. This indicates that the lemma holds true for any change in basis of the state-space $\mathbb{R}^{\mathtt{n}}$ and hence, the title of the lemma: $X_{1\Lambda}$ is invariant under change of basis on the state-space.

The next lemma shows the existence and the structure of the eigenvectors corresponding to the spectral zeros of a singularly passive SISO system. The structure of the eigenvectors is crucially used in the proof of Statement *(1)* of Theorem 5.7.

Existence of eigenvectors corresponding to an eigenvalue of $\sigma(E,H)$

**Lemma 5.10.** *Consider a singularly passive SISO system* $\Sigma$ *of order* $\mathtt{n_s}$ *with a minimal i/s/o representation as in equation* (5.9) *and with the Hamiltonian pencil as defined in equation* (5.11). *Let* $(A,b,c)$ *be in the controller canonical form. Assume* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $(E,H)$ *with algebraic multiplicity* $\mathtt{m}$. *Let* $J_\lambda := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{\mathtt{m}\times\mathtt{m}}$ *be the complex Jordan block of size* $\mathtt{m}$. *Then, there exists* $S,P \in \mathbb{C}^{\mathtt{n}\times\mathtt{m}}$ *and* $Q \in \mathbb{C}^{1\times\mathtt{m}}$ *such that*

$$H\begin{bmatrix} S \\ P \\ Q \end{bmatrix} = E\begin{bmatrix} S \\ P \\ Q \end{bmatrix} J_\lambda, \text{ i.e., } \begin{bmatrix} A & 0 & b \\ 0 & -A^T & c^T \\ c & -b^T & 0 \end{bmatrix}\begin{bmatrix} S \\ P \\ Q \end{bmatrix} = \begin{bmatrix} S \\ P \\ 0 \end{bmatrix} J_\lambda. \tag{5.15}$$

*Proof:* Let the characteristic polynomial of $A$ be $\mathscr{X}_A(s) := \mathtt{det}(sI_{\mathtt{n}} - A)$. Construct

$$Q := \begin{bmatrix} \mathscr{X}_A(\lambda) & \mathscr{X}_A^{(1)}(\lambda) & \mathscr{X}_A^{(2)}(\lambda) & \cdots & \mathscr{X}_A^{(\mathtt{m}-1)}(\lambda) \end{bmatrix} \tag{5.16}$$

where $\mathscr{X}_A^{(i)}(\lambda) := \frac{d^i}{ds^i}\left(\mathscr{X}_A(s)\right)|_{s=\lambda}$. We need to find $S,P$ such that $AS + bQ = SJ_\lambda$, $-A^TP + c^TQ = PJ_\lambda$ and $cS - b^TP = 0$. Note that the equation $AS + bQ = SJ_\lambda$, after re-arrangement reduces to

$$-AS + SJ_\lambda = bQ, \tag{5.17}$$

which is a Sylvester equation in the unknown $S$. By construction, we know that $\lambda$ is the eigenvalue of $J_\lambda$, i.e., $\lambda \in \sigma(E,H)$. Owing to the fact that $\sigma(E,H)$ has a reflection symmetry with respect to the imaginary axis (see proof of Lemma 5.6), we get $-\lambda \in \sigma(E,H)$. Since $\Sigma$ is singularly passive, and equation (5.9) is a minimal i/s/o representation of $\Sigma$, the system matrix $A$ must be Hurwitz. Therefore, by Lemma 5.2, $-\lambda \notin \sigma(A)$. Hence $\sigma(J_\lambda) \cap \sigma(-A) = \emptyset$. Therefore, there exists a unique $S$ that satisfies equation (5.17) ([Ant05, Proposition 6.2]). It can be verified that this unique $S$ for $Q$ defined in equation (5.16) is given by the following

Vandermonde matrix:

$$S := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda & 1 & \cdots & 0 \\ \lambda^2 & 2\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{\mathtt{n}-1} & \binom{\mathtt{n}-1}{1}\lambda^{\mathtt{n}-2} & \cdots & \binom{\mathtt{n}-1}{\mathtt{m}-1}\lambda^{\mathtt{n}-\mathtt{m}} \end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times\mathtt{m}}. \tag{5.18}$$

Note that the $i$-th column of $S$, i.e., $s_i$ can also be found using the following formula

$$s_i = \sum_{\ell=1}^{i} (-1)^{\ell+1}(\lambda I - A)^{-\ell} b \left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right), \text{ for } i \in \{1,2,\ldots,\mathtt{m}\}. \tag{5.19}$$

In equation (5.19), we have used the fact that $\lambda \notin \sigma(A)$ because $\lambda \in \sigma(E,H)$ (see Lemma 5.8). Similarly, the equation involving $P$, i.e., $-A^T P + c^T Q = PJ_\lambda$ can be transformed to the Sylvester equation $A^T P + PJ_\lambda = c^T Q$ in the unknown $P$. Arguing like before, this Sylvester equation can be shown to admit a unique solution because $\sigma(J_\lambda) \cap \sigma(A^T) = \emptyset$. Like before, the $i$-th column of $P$, say $p_i$, can be found using the following formula:

$$p_i = \sum_{\ell=1}^{i} (-1)^{\ell+1}(\lambda I + A^T)^{-\ell} c^T \left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right), \text{ for } i \in \{1,2,\ldots,\mathtt{m}\}. \tag{5.20}$$

We have used the fact that $\lambda \in \sigma(E,H)$ implies $-\lambda \in \sigma(E,H)$ and therefore by Lemma 5.8, $-\lambda \notin \sigma(A) \implies \lambda \notin \sigma(-A^T)$. In order to show that $S$, $P$, thus constructed, satisfies $cS - b^T P = 0$, we note that the $i$-th column of $cS - b^T P$ is given by

$$\begin{aligned} [cS - b^T P]_i &= \sum_{\ell=1}^{i} (-1)^{\ell+1} \left(c(\lambda I - A)^{-\ell} b - b^T (sI + A^T)^{-\ell} c^T\right) \left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right) \\ &= \sum_{\ell=1}^{i} \left(\frac{d^{(\ell-1)}}{ds^{(\ell-1)}} \left(G(s) + G(-s)\right)\Big|_{s=\lambda}\right) \left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right). \end{aligned} \tag{5.21}$$

Since $\lambda$ has algebraic multiplicity $\mathtt{m}$, and $\mathtt{det}(sE - H) = \mathtt{num}(G(s) + G(-s))$ (Lemma 5.3), we have
$\frac{d^{(\ell-1)}}{ds^{(\ell-1)}} \left(G(s) + G(-s)\right)\Big|_{s=\lambda} = 0$, for $\ell \in \{1,2,\cdots,\mathtt{m}\}$. Therefore, the ride hand side of equation (5.21) evaluates to zero for all $\ell \in \{1,2,\cdots,\mathtt{m}\}$; we thus infer that $cS - b^T P = 0$. ■

The next proposition is well-known in the literature. However, we present its proof here for the sake of completeness. This proposition is used to prove our next lemma (Lemma 5.12) that establishes a relation between the left- and right-vectors of the Hamiltonian matrix pair $(E,H)$ corresponding to a Lambda-set.

**Proposition 5.11.** *Consider a matrix pencil $(sP_2 - P_1) \in \mathbb{R}[s]^{\mathtt{m}\times\mathtt{m}}$, where $P_2 = P_2^T, P_1 \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ and $\mathtt{det}(sP_2 - P_1) \neq 0$. Let $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}_1}$ and $L \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}_2}$ be such that $P_1 U = P_2 U \Gamma_1$ and $L^T P_1 = \Gamma_2 L^T P_2$, where $\Gamma_1 \in \mathbb{R}^{\mathtt{m}_1\times\mathtt{m}_1}$ and $\Gamma_2 \in \mathbb{R}^{\mathtt{m}_2\times\mathtt{m}_2}$. Suppose $\sigma(\Gamma_1) \cap \sigma(\Gamma_2) = \emptyset$. Then, $L^T P_2 U = 0$.*

*Proof:* Clearly, $P_1U = P_2U\Gamma_1 \Rightarrow L^TP_1U = L^TP_2U\Gamma_1$ and $L^TP_1 = \Gamma_2L^TP_2 \Rightarrow L^TP_1U = \Gamma_2L^TP_2U$. Subtracting one equation from the other and defining $T := L^TP_2U$, we get the Sylvester equation $T\Gamma_1 - \Gamma_2T = 0$. Since, by assumption, $\sigma(\Gamma_1)\cap\sigma(\Gamma_2) = \emptyset$, there exists a unique $T$ that solves this Sylvester equation. Note that $T = 0$ is a solution to this equation. Therefore, by uniqueness argument, $T = 0$ is the only solution. Therefore, $T = L^TP_2U = 0$. ■

Loosely speaking, Proposition 5.11 states that the eigenspace of $(P_2,P_1)$ spanned by the right-eigenvector of $(P_2,P_1)$ corresponding to an eigenvalue $\lambda$ is always $P_2$-orthogonal with the subspace spanned by a left-eigenvector of $(P_2,P_1)$ corresponding to an eigenvalue other than $\lambda$. Utilizing this in the next lemma we establish a relation between the left- and right-eigenvectors of the Hamiltonian matrix pair $(E,H)$.

Relation between the left- and right-eigenvectors of $(E,H)$

**Lemma 5.12.** *Consider the Hamiltonian matrix pencil* $(sE-H)\in\mathbb{R}[s]^{(2\mathtt{n}+1)\times(2\mathtt{n}+1)}$ *(see equation* (5.11)*) corresponding to a singularly passive SISO system of order* $\mathtt{n_s}$ *having a minimal i/s/o representation as in equation* (5.9)*. Let* $\Lambda$ *be a Lambda-set of* $\mathtt{det}(sE-H)$ *with cardinality* $\mathtt{n_s}$*. Define* $R := \mathtt{col}\,(V_{1\Lambda},V_{2\Lambda},V_{3\Lambda})$ *such that* $HR = ER\Gamma_\Lambda$*, where* $\Gamma_\Lambda\in\mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$*,* $V_{1\Lambda},V_{2\Lambda}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$*,* $V_{3\Lambda}\in\mathbb{R}^{1\times\mathtt{n_s}}$ *and* $\sigma(\Gamma_\Lambda)=\Lambda$*. Then,* $V_{2\Lambda}^TV_{1\Lambda} = V_{1\Lambda}^TV_{2\Lambda}$.

*Proof:* Define $L := \mathtt{col}\,(V_{2\Lambda},-V_{1\Lambda},-V_{3\Lambda})$. Then,

$$L^TH = \begin{bmatrix} V_{2\Lambda}^TA - V_{3\Lambda}^Tc & V_{1\Lambda}^TA^T + V_{3\Lambda}^Tb^T & V_{2\Lambda}^Tb - V_{1\Lambda}^Tc^T\end{bmatrix} \tag{5.22}$$

Since $HR = ER\Gamma_\Lambda$, from equation (5.12), we have

$$AV_{1\Lambda} + bV_{3\Lambda} = V_{1\Lambda}\Gamma_\Lambda \tag{5.23}$$
$$-A^TV_{2\Lambda} + c^TV_{3\Lambda} = V_{2\Lambda}\Gamma_\Lambda \tag{5.24}$$
$$cV_{1\Lambda} - b^TV_{2\Lambda} = 0 \tag{5.25}$$

Using equations (5.23) - (5.25) in equation (5.22), we have $L^TH = \begin{bmatrix}-\Gamma_\Lambda^TV_{2\Lambda}^T & \Gamma_\Lambda^TV_{1\Lambda}^T & 0\end{bmatrix} = -\Gamma_\Lambda^TL^TE$. Note that $\sigma(-\Gamma_\Lambda) = -\Lambda$. Since $\Lambda$ is a Lambda-set (see Definition 2.18), $\Lambda\cap(-\Lambda) = \emptyset$. Therefore, by Proposition 5.11 we conclude that $L^TER = 0$. Expanding $L^TER$ we get $V_{2\Lambda}^TV_{1\Lambda} = V_{1\Lambda}^TV_{2\Lambda}$. ■

The next lemma shows that $G(s)+G(-s)$ is the transfer function corresponding to the Hamiltonian system $\Sigma_{\text{Ham}}$ defined in equation 5.8.

Popov function is the transfer function of a Hamiltonian system

**Lemma 5.13.** *Consider a singularly passive SISO system* $\Sigma$ *of order* $\mathtt{n_s}$ *with transfer function* $G(s)$ *and a minimal i/s/o representation as in equation* (5.9)*. Let the corresponding Hamiltonian pencil be as defined in equation* (5.11)*. Then,*

$$\widehat{c}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{b} = G(s) + G(-s).$$

*Proof:* Note that since we are dealing with a SISO system, we can write $G(-s)$ as $G(-s)^T$. Therefore, $G(s) = c(sI_{\mathtt{n}} - A)^{-1}b \Rightarrow G(-s) = -b^T(sI_{\mathtt{n}} + A^T)c^T$. Thus,

$$\begin{aligned} G(s) + G(-s)^T &= c(sI - A)^{-1}b - b^T(sI + A^T)c^T \\ &= \begin{bmatrix} c & -b^T \end{bmatrix} \begin{bmatrix} sI_{\mathtt{n}} - A & 0 \\ 0 & sI_{\mathtt{n}} + A^T \end{bmatrix}^{-1} \begin{bmatrix} b \\ c^T \end{bmatrix} = \widehat{c}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{b} \end{aligned}$$

This completes the proof of the lemma. ■

One of the crucial properties that leads to the main result (Theorem 5.7) of this chapter is the fact that the first $\mathtt{n_f}$ Markov parameters of the Hamiltonian system $\Sigma_{\text{Ham}}$ are all zero. We prove this in the next lemma.

A property of the Markov parameters of the Hamiltonian system

**Lemma 5.14.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with a minimal i/s/o representation as in equation* (5.9). *Let the corresponding Hamiltonian pencil be as defined in equation* (5.11). *Define $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$. Then,*

$$\widehat{c}\widehat{A}^k\widehat{b} = 0, \qquad \textit{for } k \in \{0, 1, \dots, 2(\mathtt{n_f} - 1)\}.$$

*Proof:* Defining $P := \widehat{A}$, $L := \widehat{b}$, $M := \widehat{c}$ in Lemma 2.23. Further, note that here $\mathtt{N} = 2\mathtt{n}$ and $\mathtt{N_s} = 2\mathtt{n_s}$. Therefore, $\mathtt{N_f} = \mathtt{N} - \mathtt{N_s} = 2\mathtt{n} - 2\mathtt{n_s} = 2\mathtt{n_f}$. Then, it is evident from Lemma 2.23 that $\widehat{c}\widehat{A}^k\widehat{b} = 0$, for $k \in \{0, 1, \dots, 2(\mathtt{n_f} - 1)\}$. ■

An identity relating the eigenvectors of $(E,H)$ and the system matrices of $\Sigma_{\text{Ham}}$

**Lemma 5.15.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with a minimal i/s/o representation as in equation* (5.9). *Let the corresponding Hamiltonian pencil be as defined in equation* (5.11). *Let $\Lambda, \mathtt{n_s}, \mathtt{n_f}, V_{1\Lambda}, V_{2\Lambda}$, and $V_{3\Lambda}$ be as defined in Theorem 5.7. Then,*

$$\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^k\widehat{b} = 0, \textit{ for } k \in \{0, 1, \dots, 2(\mathtt{n_f} - 1)\}.$$

*Proof:* We use induction for this proof.
**Base step:** $(k = 0)$ We have $\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{b} = (cV_{1\Lambda} - b^TV_{2\Lambda})^T$. Using equation (5.25), we infer $\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{b} = 0$.
**Inductive step:** Suppose $i \leqslant 2(\mathtt{n_f} - 1) - 1$. Assume $\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^i\widehat{b} = 0$. We prove that $\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^{i+1}\widehat{b} = 0$. Note that

$$\begin{aligned} \begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^{i+1}\widehat{b} &= -V_{2\Lambda}^T A^{i+1}b + (-1)^{i+1}V_{1\Lambda}^T A^{(i+1)T}c^T \\ &= (-A^TV_{2\Lambda})^T A^i b + (-1)^{i+1}(AV_{1\Lambda})^T(cA^i)^T \end{aligned} \tag{5.26}$$

Since $HR = ER\Gamma_\Lambda$, using equations (5.23) and (5.24) in equation (5.26), we have

$$\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^{i+1}\widehat{b} = (V_{2\Lambda}\Gamma_\Lambda - c^T V_{3\Lambda})^T A^i b + (-1)^{i+1}(V_{1\Lambda}\Gamma_\Lambda - bV_{3\Lambda})^T (cA^i)^T.$$

Opening the brackets and cancelling terms with opposite signs, we get

$$\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^{i+1}\widehat{b} = -\Gamma_\Lambda^T \begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^i\widehat{b} - V_{3\Lambda}^T \widehat{c}\widehat{A}^i\widehat{b}.$$

Hence, from the inductive hypothesis and Lemma 5.14, we have $\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{A}^{i+1}\widehat{b} = 0$. ■

### 5.3.2 Proof of Theorem 5.7

Now that we have proved all the auxiliary results, we prove Theorem 5.7.
*Proof of Statement (1) of Theorem 5.7:* We prove this in two-steps.
*Step 1 (Construction of $V_{\mathtt{e}\Lambda} \in \mathbb{R}^{(2\mathtt{n}+1)\times \mathtt{n_s}}$ that satisfies equation* (5.12)*):* Lemma 5.9 implies that $(A,B,C)$ can be assumed to be in controller canonical form as given in Section 5.2.1 without loss of generality. From the definition of Lambda-sets (Definition 2.18) we know that if $\lambda \in \Lambda$ then, $\bar{\lambda} \in \Lambda$. Thus, without loss of generality, we assume that there are $\alpha$ number of complex-conjugate pairs in $\Lambda$ and $\beta$ number of real elements in $\Lambda$ such that each distinct element $\lambda_i$ in $\Lambda$ has an algebraic multiplicity $\mathtt{m}_{\lambda_i}$. Thus for a Lambda-set with cardinality $\mathtt{n_s}$, we have $\sum_{i=1}^{2\alpha+\beta} \mathtt{m}_{\lambda_i} = \mathtt{n_s}$.

Now, we associate a matrix $S_{\lambda_i} \in \mathbb{C}^{\mathtt{n}\times \mathtt{m}_{\lambda_i}}$ with each distinct element $\lambda_i \in \Lambda$. These matrices $S_{\lambda_i}$ have a structure as defined in equation (5.18) of the proof of Lemma 5.10, i.e.,

$$S_{\lambda_i} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda_i & 1 & \cdots & 0 \\ \lambda_i^2 & 2\lambda_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_i^{\mathtt{n}-1} & \binom{\mathtt{n}-1}{1}\lambda_i^{\mathtt{n}-2} & \cdots & \binom{\mathtt{n}-1}{\mathtt{m}_{\lambda_i}-1}\lambda_i^{\mathtt{n}-\mathtt{m}_{\lambda_i}} \end{bmatrix} \tag{5.27}$$

Note that since $E$, $H$ are real matrices, the algebraic multiplicities of $\lambda_i \in \sigma(E,H)$ and $\bar{\lambda}_i \in \sigma(E,H)$ are the same. Further, from the structure of $S_{\lambda_i}$ it is evident that $S_{\bar{\lambda}_i} = \bar{S}_{\lambda_i}$, where $\bar{S}_{\lambda_i}$ is the complex-conjugate matrix of $S_{\lambda_i}$. Now we define a matrix $V_{1\Lambda}^{\mathbb{C}}$ as follows:

$$V_{1\Lambda}^{\mathbb{C}} := \begin{bmatrix} S_{\lambda_1} & \bar{S}_{\lambda_1} & \cdots & S_{\lambda_\alpha} & \bar{S}_{\lambda_\alpha} & S_{\lambda_{2\alpha+1}} & \cdots & S_{\lambda_{2\alpha+\beta}} \end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n_s}} \tag{5.28}$$

$V_{1\Lambda}^{\mathbb{C}}$ in equation (5.28) is constructed such that the matrices $S_{\lambda_i}$ and $\bar{S}_{\lambda_i}$ corresponding to each of the complex-conjugate pairs in $\Lambda$ are appended consecutively and this is followed by the matrices associated with the real elements in $\Lambda$. Using Lemma 5.10, we infer that corresponding to $V_{1\Lambda}^{\mathbb{C}}$ in equation (5.28) there exists $V_{2\Lambda}^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda}^{\mathbb{C}} \in \mathbb{C}^{1\times\mathtt{n_s}}$ such that $V_{\mathtt{e}\Lambda}^{\mathbb{C}} :=$

$\mathtt{col}(V^{\mathbb{C}}_{1\Lambda}, V^{\mathbb{C}}_{2\Lambda}, V^{\mathbb{C}}_{3\Lambda})$ satisfies $HV^{\mathbb{C}}_{\mathrm{e}\Lambda} = EV^{\mathbb{C}}_{\mathrm{e}\Lambda}J^{\mathbb{C}}$, where $J^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ is a block diagonal matrix with each block being a complex Jordan block and $\sigma(J^{\mathbb{C}}) = \Lambda$. Now we construct a matrix $V_{1\Lambda}$ such that $V_{1\Lambda} := \begin{bmatrix}\mathtt{Re}(S_{\lambda_1}) & \mathtt{Im}(S_{\lambda_1}) & \cdots & \mathtt{Re}(S_{\lambda_\alpha}) & \mathtt{Im}(S_{\lambda_\alpha}) & S_{\lambda_{2\alpha+1}} & \cdots & S_{\lambda_{2\alpha+\beta}}\end{bmatrix} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$, where $\mathtt{Re}(S_{\lambda_i})$ and $\mathtt{Im}(S_{\lambda_i})$ denotes the real-part and imaginary-part of the matrix $S_{\lambda_i}$, respectively. It can be verified that there exists a nonsingular matrix $L \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ such that $V^{\mathbb{C}}_{1\Lambda}L = V_{1\Lambda}$, where $V_{1\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$. Using this nonsingular matrix $L$, we now define $V_{\mathrm{e}\Lambda} := V^{\mathbb{C}}_{\mathrm{e}\Lambda}L$. It is easy to verify that $V_{\mathrm{e}\Lambda} \in \mathbb{R}^{(2\mathtt{n}+1)\times\mathtt{n_s}}$. Thus, $HV^{\mathbb{C}}_{\mathrm{e}\Lambda} = EV^{\mathbb{C}}_{\mathrm{e}\Lambda}J^{\mathbb{C}} \implies HV_{\mathrm{e}\Lambda} = EV_{\mathrm{e}\Lambda}\Gamma_\Lambda$, where $\Gamma_\Lambda := L^{-1}J^{\mathbb{C}}L \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$. Importantly, $\sigma(\Gamma_\Lambda) = \sigma(J^{\mathbb{C}}) = \Lambda$, albeit unlike the matrix $J^{\mathbb{C}}$, matrix $\Gamma_\Lambda$ is in real Jordan form. This shows the existence of a matrix $V_{\mathrm{e}\Lambda}$ that satisfies equation (5.12).

*Step 2 ($X_{1\Lambda}$ is nonsingular):* Conforming to the partition of $V_{\mathrm{e}\Lambda}$ in equation (5.12), we partition $V_{\mathrm{e}\Lambda}$ as $V_{\mathrm{e}\Lambda} := \mathtt{col}\,(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ and define $V_\Lambda := \mathtt{col}\,(V_{1\Lambda}, V_{2\Lambda})$, where $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda} \in \mathbb{R}^{1\times\mathtt{n_s}}$. Similarly, partition $W$, defined in the statement of the theorem, as follows: $W := \mathtt{col}\,(W_1, W_2)$ , where $W_1, W_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_f}}$. Recall from the theorem that $\begin{bmatrix}V_\Lambda & W\end{bmatrix} = \mathtt{col}\,(X_{1\Lambda}, X_{2\Lambda})$, where $X_{1\Lambda}, X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Therefore, $X_{1\Lambda} = \begin{bmatrix}V_{1\Lambda} & W_1\end{bmatrix}$. Recall that $V^{\mathbb{C}}_{1\Lambda}L = V_{1\Lambda}$. We crucially use this in the next step of the proof.

In order to prove the invertibility of $X_{1\Lambda}$, we partition $X_{1\Lambda}$ as $X_{1\Lambda} := \begin{bmatrix}V_{11} & W_{11}\\ V_{12} & W_{12}\end{bmatrix}$, where $V_{11} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$, $V_{12} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_s}}$, $W_{11} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_f}}$ and $W_{12} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}$. Conforming to this partition, we partition $V^{\mathbb{C}}_{1\Lambda}$, as well: $V^{\mathbb{C}}_{1\Lambda} = \begin{bmatrix}V^{\mathbb{C}}_{11}\\ V^{\mathbb{C}}_{12}\end{bmatrix}$, where $V^{\mathbb{C}}_{11} \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$, $V^{\mathbb{C}}_{12} \in \mathbb{C}^{\mathtt{n_f}\times\mathtt{n_s}}$. Since $V^{\mathbb{C}}_{1\Lambda}L = V_{1\Lambda}$, clearly $V_{11} = V^{\mathbb{C}}_{11}L$. From the structure of $V^{\mathbb{C}}_{1\Lambda}$ shown in equation (5.28), it is evident that $V^{\mathbb{C}}_{11} \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ is a Vandermonde matrix of the form:

$$V^{\mathbb{C}}_{11} = \begin{bmatrix} 1 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\ \lambda_1 & \cdots & 0 & \lambda_2 & \cdots & \lambda_{\mathtt{k}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{\mathtt{n_s}-1} & \cdots & \begin{pmatrix}\mathtt{n_s}-1\\ \mathtt{m}_{\lambda_1}-1\end{pmatrix}\lambda_1^{\mathtt{n_s}-\mathtt{m}_{\lambda_1}} & \lambda_2^{\mathtt{n_s}-1} & \cdots & \lambda_{\mathtt{k}}^{\mathtt{n_s}-1} & \cdots & \begin{pmatrix}\mathtt{n_s}-1\\ \mathtt{m}_{\lambda_{2\alpha+\beta}}-1\end{pmatrix}\lambda_{\mathtt{k}}^{\mathtt{n_s}-\mathtt{m}_{\lambda_{2\alpha+\beta}}} \end{bmatrix}.$$

Since $V^{\mathbb{C}}_{11}$ is a Vandermonde matrix with $2\alpha+\beta$ distinct $\lambda_i$s such that their multiplicities add up to the size of the matrix, $V^{\mathbb{C}}_{11}$ must be invertible [Ber08, Fact 5.16.5]. Thus, $V_{11}$ is the product of two nonsingular matrices $V^{\mathbb{C}}_{11}$ and $L$. Therefore, $V_{11}$ is nonsingular, as well. Now, we concentrate on the structure of $W_1$. First, recall that

$$W = \begin{bmatrix}\widehat{b} & \widehat{Ab} & \cdots & \widehat{A^{\mathtt{n_f}-1}b}\end{bmatrix} = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \\ c^T & -(cA)^T & \cdots & (-1)^{\mathtt{n_f}-1}(cA^{\mathtt{n_f}-1})^T \end{bmatrix}. \tag{5.29}$$

Hence, $W_1 = \begin{bmatrix}b & Ab & \cdots & A^{\mathtt{n_f}-1}b\end{bmatrix} = \begin{bmatrix}W_{11}\\ W_{12}\end{bmatrix}$. Since $(A,b,c)$ is in the controller canonical form, $W_{11}$ is a zero matrix, i.e., $W_1 := \begin{bmatrix}0\\ W_{12}\end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n_f}}$, where $W_{12} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}$ has the following structure

$W_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & \star \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & \star & \star \\ 1 & \star & \cdots & \star & \star \end{bmatrix}$ with $\star$ denoting possibly nonzero entries. Clearly, $W_{12}$ is nonsingular.

Thus, $X_{1\Lambda}$ has the following structure

$$X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix} =: \begin{bmatrix} V_{11} & 0 \\ V_{12} & W_{12} \end{bmatrix}, \tag{5.30}$$

Thus, $X_{1\Lambda}$ is a block lower-triangular matrix with the diagonal blocks being $V_{11}$ and $W_{12}$. Since $V_{11}$ and $W_{12}$ are nonsingular matrices, $X_{1\Lambda}$ is nonsingular. ■

*Proof of Statement (2) of Theorem 5.7:* In order to prove Statement (2) of Theorem 5.7 we need to show that $K := X_{2\Lambda}X_{1\Lambda}^{-1}$ is symmetric, i.e., $X_{2\Lambda}X_{1\Lambda}^{-1} = (X_{2\Lambda}X_{1\Lambda}^{-1})^T$. This is clearly equivalent to showing $X_{1\Lambda}^T X_{2\Lambda} = X_{2\Lambda}^T X_{1\Lambda}$. With $V_{1\Lambda}, V_{2\Lambda}$ as defined in equation (5.12) and

$$W_1 := \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix}, \text{ and } W_2 := \begin{bmatrix} c^T & -(cA)^T & \cdots & (-1)^{\mathtt{n_f}-1}(cA^{\mathtt{n_f}-1})^T \end{bmatrix},$$

we have

$$X_{1\Lambda}^T X_{2\Lambda} - X_{2\Lambda}^T X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda}^T V_{2\Lambda} - V_{2\Lambda}^T V_{1\Lambda} & V_{1\Lambda}^T W_2 - V_{2\Lambda}^T W_1 \\ -(V_{1\Lambda}^T W_2 - V_{2\Lambda}^T W_1)^T & W_1^T W_2 - W_2^T W_1 \end{bmatrix}. \tag{5.31}$$

We now show that in this $2 \times 2$ block representation of $X_{1\Lambda}^T X_{2\Lambda} - X_{2\Lambda}^T X_{1\Lambda}$, every block is equal to the zero matrix. For the top left block, we get directly from Lemma 5.12 that $V_{1\Lambda}^T V_{2\Lambda} - V_{2\Lambda}^T V_{1\Lambda} = 0$. For the off-diagonal blocks we notice the following:

$$V_{1\Lambda}^T W_2 - V_{2\Lambda}^T W_1 = \begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix}. \tag{5.32}$$

Using Lemma 5.15 in equation (5.32), it is clear that $V_{1\Lambda}^T W_2 - V_{2\Lambda}^T W_1 = 0$. It is now left to show that the bottom right block, i.e., $W_1^T W_2 - W_2^T W_1 = 0$. For this purpose, we define $J := \begin{bmatrix} 0 & -I_{\mathtt{n}} \\ I_{\mathtt{n}} & 0 \end{bmatrix}$ and recall that $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$. Now,

$$W_1^T W_2 - W_2^T W_1 = \begin{bmatrix} W_1^T & W_2^T \end{bmatrix} \begin{bmatrix} W_2 \\ -W_1 \end{bmatrix} = \begin{bmatrix} W_1^T & W_2^T \end{bmatrix} \begin{bmatrix} 0 & I_{\mathtt{n}} \\ -I_{\mathtt{n}} & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = -W^T J W. \tag{5.33}$$

It is easy to verify that

$$W^T J = \mathtt{col}\left(\widehat{c}, -\widehat{c}\widehat{A}, \ldots, (-1)^{\mathtt{n_f}-1}\widehat{c}\widehat{A}^{\mathtt{n_f}-1}\right). \tag{5.34}$$

Therefore, using equation (5.34) in equation (5.33), we have

$$W_1^T W_2 - W_2^T W_1 = -W^T J W = -\mathtt{col}\left(\widehat{c}, -\widehat{c}\widehat{A}, \ldots, (-1)^{\mathtt{n_f}-1}\widehat{c}\widehat{A}^{\mathtt{n_f}-1}\right) W$$
$$=: [p_{ki}]_{k,i\in\{1,2,\ldots,\mathtt{n_f}\}}, \text{ where } p_{ki} \in \mathbb{R}.$$

Clearly, $p_{ki} = (-1)^k \widehat{c}\widehat{A}^{k+i-2}\widehat{b}$ for $k,i \in \{1,2,\ldots,\mathtt{n_f}\}$. Therefore, using Lemma 5.14, we infer that $p_{ki} = 0$ for all $k,i \in \{1,2,\ldots,\mathtt{n_f}\}$. Thus, we have $W_1^T W_2 - W_2^T W_1 = 0$. ∎

*Proof of Statements (3), (4), (5) and (6) of Theorem 5.7:* Note that $Kb - c^T = X_{2\Lambda}X_{1\Lambda}^{-1}b - c^T$. By construction, $X_{1\Lambda}^{-1}b = e_{\mathtt{n_s}+1}$, where $e_{\mathtt{n_s}+1}$ is a vector in $\mathbb{R}^{\mathtt{n}}$ with all elements zero except the $(\mathtt{n_s}+1)$-st one which is 1. Therefore, $X_{2\Lambda}X_{1\Lambda}^{-1}b = X_{2\Lambda}e_{\mathtt{n_s}+1} = c^T$. This proves that $Kb - c^T = 0$.

Next we prove that $A^T K + KA \leqslant 0$. For the sake of brevity, define $\mathscr{L}(K) := A^T K + KA$. Now, in order to prove that $\mathscr{L}(K) \leqslant 0$, we first evaluate $X_{1\Lambda}^T \mathscr{L}(K) X_{1\Lambda}$.

$$\begin{bmatrix} V_{1\Lambda}^T \\ W_1^T \end{bmatrix} \mathscr{L}(K) \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix} = \begin{bmatrix} V_{1\Lambda}^T \mathscr{L}(K) V_{1\Lambda} & V_{1\Lambda}^T \mathscr{L}(K) W_1 \\ W_1^T \mathscr{L}(K) V_{1\Lambda} & W_1^T \mathscr{L}(K) W_1 \end{bmatrix}. \tag{5.35}$$

First we prove that $V_{1\Lambda}^T \mathscr{L}(K) V_{1\Lambda} = 0$. Note that $KV_{1\Lambda} = X_{2\Lambda}X_{1\Lambda}^{-1}V_{1\Lambda} = V_{2\Lambda}$ (by construction). Therefore,

$$V_{1\Lambda}^T \mathscr{L}(K) V_{1\Lambda} = V_{1\Lambda}^T (A^T K + KA) V_{1\Lambda} = V_{1\Lambda}^T A^T V_{2\Lambda} + V_{2\Lambda}^T A V_{1\Lambda} = \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \end{bmatrix} \widehat{A} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}. \tag{5.36}$$

Recall that $V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda}$ satisfy equation (5.12). Using equation (5.12) in equation (5.36), we have

$$\begin{aligned} \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \end{bmatrix} \widehat{A} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} &= \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \end{bmatrix} \left( \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} \Gamma_\Lambda - \widehat{b} V_{3\Lambda} \right) \\ &= \left( V_{2\Lambda}^T V_{1\Lambda} - V_{1\Lambda}^T V_{2\Lambda} \right) \Gamma_\Lambda + \begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{b} V_{3\Lambda}. \end{aligned} \tag{5.37}$$

From Lemma 5.12, we have $V_{1\Lambda}^T V_{2\Lambda} - V_{2\Lambda}^T V_{1\Lambda} = 0$, and Lemma 5.14 customized to $k=0$ gives $\begin{bmatrix} -V_{2\Lambda}^T & V_{1\Lambda}^T \end{bmatrix} \widehat{b} = 0$. Hence from equation (5.37) we conclude

$$V_{1\Lambda}^T \mathscr{L}(K) V_{1\Lambda} = 0. \tag{5.38}$$

Next we prove that $V_{1\Lambda}^T (A^T K + KA) W_1 = 0$. Note that $V_{1\Lambda}^T (A^T K + KA) W_1$ can be rewritten as

$$-V_{1\Lambda}^T \begin{bmatrix} -K & I \end{bmatrix} \widehat{A} \begin{bmatrix} I \\ K \end{bmatrix} W_1 = \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \end{bmatrix} \widehat{A} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \end{bmatrix} \widehat{A} \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix},$$

where we have used the fact that $KW_1 = X_{2\Lambda}X_{1\Lambda}^{-1}W_1 = W_2$ (by construction). Using Lemma 5.15, we conclude that

$$V_{1\Lambda}^T \mathscr{L}(K) W_1 = 0. \tag{5.39}$$

It remains to investigate $W_1^T\mathscr{L}(K)W_1$, which is what we do next. Recall again that $KW_1 = W_2$.

$$W_1^T\mathscr{L}(K)W_1 = W_1^T(A^TK+KA)W_1 = W_1^TA^TW_2+W_2^TAW_1 = W^TJ\widehat{A}W. \tag{5.40}$$

Using equation (5.34) in equation (5.40), we have

$$\begin{aligned} W^TJ\widehat{A}W &= \texttt{col}\left(\widehat{c}, -\widehat{c}\widehat{A}, \cdots, (-1)^{\mathtt{n_f}-1}\widehat{c}\widehat{A}^{\mathtt{n_f}-1}\right)\widehat{A}\begin{bmatrix}\widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b}\end{bmatrix} \\ &=: [\ell_{ki}]_{k,i\in\{1,2,\ldots,\mathtt{n_f}\}}. \end{aligned} \tag{5.41}$$

Here $\ell_{ki} = (-1)^{k-1}\widehat{c}\widehat{A}^{k+i-1}\widehat{b}$ for $k,i\in\{1,2,\ldots,\mathtt{n_f}\}$. From Lemma 5.14 it is clear that except $\ell_{\mathtt{n_f}\mathtt{n_f}}$, rest all $\ell_{ki}=0$. Now we claim that $\ell_{\mathtt{n_f}\mathtt{n_f}} = (-1)^{\mathtt{n_f}-1}\widehat{c}\widehat{A}^{2\mathtt{n_f}-1}\widehat{b} \leqslant 0$. Note that a BIBO stable, passive SISO system with transfer function $G(s)$ satisfies $G(j\omega)+G(-j\omega)\geqslant 0$ for all $\omega\in\mathbb{R}$: see [AV06]. Therefore, the singularly passive SISO system $\Sigma$ being BIBO stable and passive satisfies $G(-j\omega)+G(j\omega)\geqslant 0$. This implies

$$\lim_{\omega\to\infty}\omega^{2\mathtt{n_f}}\left(G(-j\omega)+G(j\omega)\right)\geqslant 0 \tag{5.42}$$

Using Lemma 5.13 in equation (5.42), we have $\lim_{\omega\to\infty}\omega^{2\mathtt{n_f}}\widehat{c}(j\omega I_{2\mathtt{n}}-\widehat{A})^{-1}\widehat{b}\geqslant 0$. Expanding $(sI_{2\mathtt{n}}-\widehat{A})^{-1}$ about $s=\infty$, we have $\lim_{\omega\to\infty}(-1)^{\mathtt{n_f}}(j\omega)^{2\mathtt{n_f}}\sum_{i=0}^{\infty}\frac{1}{(j\omega)^{i+1}}\widehat{c}\widehat{A}^i\widehat{b}\geqslant 0$. With the help of Lemma 5.14 this inequality reduces to $\lim_{\omega\to\infty}(-1)^{\mathtt{n_f}}(j\omega)^{2\mathtt{n_f}}\sum_{i=2\mathtt{n_f}-1}^{\infty}\frac{1}{(j\omega)^{i+1}}\widehat{c}\widehat{A}^i\widehat{b}\geqslant 0$. Expanding the sum, we have

$$(-1)^{\mathtt{n_f}}\widehat{c}\widehat{A}^{2\mathtt{n_f}-1}\widehat{b}+(-1)^{\mathtt{n_f}}\lim_{\omega\to\infty}(j\omega)^{2\mathtt{n_f}}\sum_{i=2\mathtt{n_f}}^{\infty}\frac{1}{(j\omega)^{i+1}}\widehat{c}\widehat{A}^i\widehat{b} = (-1)^{\mathtt{n_f}}\widehat{c}\widehat{A}^{2\mathtt{n_f}-1}\widehat{b}\geqslant 0. \tag{5.43}$$

From equation (5.43), we have $(-1)^{\mathtt{n_f}-1}\widehat{c}\widehat{A}^{2\mathtt{n_f}-1}\widehat{b}\leqslant 0$. Hence, $W_1^T\mathscr{L}(K)W_1\leqslant 0$. Thus,

$$X_{1\Lambda}^T\mathscr{L}(K)X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda}^T\mathscr{L}(K)V_{1\Lambda} & V_{1\Lambda}^T\mathscr{L}(K)W_1 \\ W_1^T\mathscr{L}(K)V_{1\Lambda} & W_1^T\mathscr{L}(K)W_1\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & W_1^T\mathscr{L}(K)W_1\end{bmatrix}, \tag{5.44}$$

where the bottom right block, $W_1^T\mathscr{L}(K)W_1\leqslant 0$. Therefore, $X_{1\Lambda}^T\mathscr{L}(K)X_{1\Lambda}\leqslant 0$. Recall from Statement (1) of this theorem that $X_{1\Lambda}$ is invertible. Hence, by Sylvester's law of inertia [Ber08, Corollary 5.4.7], we infer that $\mathscr{L}(K) = A^TK+KA\leqslant 0$. This completes the proof of Statement (3) of Theorem 5.7.

From equation (5.44), we know that $\texttt{rank}(X_{1\Lambda}^T\mathscr{L}(K)X_{1\Lambda})\leqslant 1$ since $W_1^T\mathscr{L}(K)W_1\in\mathbb{R}$. Note that if $W_1^T\mathscr{L}(K)W_1 = 0$, then we have a $K$ such that $A^TK+KA=0$ and $Kb-c^T=0$. However, this is not possible since the system is not lossless. Therefore, $W_1^T\mathscr{L}(K)W_1\in\mathbb{R}\setminus\{0\}$. Thus, $\texttt{rank}(X_{1\Lambda}^T\mathscr{L}(K)X_{1\Lambda}) = 1$ and this is the minimum rank that can be attained by $X_{1\Lambda}^T\mathscr{L}(K)X_{1\Lambda}$. Since $X_{1\Lambda}^T$ is invertible, the rank of $\mathscr{L}(K)$ is 1, as well. Thus, $K$ is the rank-minimizing solution of the KYP LMI (5.10).

From Statement (3) of Theorem 5.7, we have $A^TK+KA\leqslant 0$. Since $A$ is Hurwitz, $A^TK+KA\leqslant 0$ implies that $K\geqslant 0$. This completes the proof of Statement (5) of Theorem 5.7.

Note that $b^T K - c = 0$ and $A^T K + KA \leqslant 0$ means that $K$ satisfies the singular KYP LMI (5.10). Therefore, $K$ induces a storage function of the system $\Sigma$. This completes the proof of Statement (6) of Theorem 5.7. ■

Now that we have proved Theorem 5.7 we revisit Example 5.5 which could not be solved using Proposition 5.4.

**Example 5.16.** *Recall that for the system in Example 5.5,* $\mathtt{n} = 3$ *and* $\mathtt{n_s} = 2$. *Therefore, from Theorem 5.7, we have* $\mathtt{n_f} = 1$. *The spectral zeros of* $\Sigma$ *is given by the set*

$$\left\{-\sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}, \sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\}.$$

*The four Lambda-sets of* $\Sigma$ *are:*
*(a)* $\Lambda_1 = \left\{-\sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}\right\}$, *(b)* $\Lambda_2 = \left\{-\sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\}$,
*(c)* $\Lambda_3 = \left\{\sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}\right\}$, *(d)* $\Lambda_4 = \left\{\sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\}$.
*We show the computation of storage function* $K_{\Lambda_2}$ *corresponding to Lambda-set* $\Lambda_2$ *in details now. Recall from Example 5.5, the eigenvectors of* $(E,H)$ *corresponding to eigenvalues* $-\sqrt{4+\sqrt{5}}$ *and* $\sqrt{4-\sqrt{5}}$ *are given by the columns of* $V_{\mathtt{e}\Lambda_2}$ *as*

$$V_{\mathtt{e}\Lambda_2} = \begin{bmatrix} 0.34 & -0.85 & 2.12 & -0.35 & -0.36 & -0.09 & 0.13 \\ 0.01 & 0.01 & 0.01 & 2.27 & 1.35 & 0.17 & 0.18 \end{bmatrix}^T.$$

*Further from Theorem 5.7, we have* $W = \widehat{b} = \mathtt{col}(b, c^T) = \begin{bmatrix} 0 & 0 & 1 & 11 & 12 & 3 \end{bmatrix}^T$. *Therefore as defined in Theorem 5.7, we must have*

$$\begin{bmatrix} V_{1\Lambda} & W \end{bmatrix} = \left[\begin{array}{ccc} 0.34 & 0.01 & 0 \\ -0.85 & 0.01 & 0 \\ 2.12 & 0.01 & 1 \\ \hline -0.35 & 2.27 & 11 \\ -0.36 & 1.35 & 12 \\ -0.09 & 0.17 & 3 \end{array}\right] = \left[\begin{array}{c} X_{1\Lambda} \\ \hline X_{2\Lambda} \end{array}\right] \Rightarrow K_{\Lambda_2} = X_{2\Lambda} X_{1\Lambda}^{-1} = \begin{bmatrix} 239.53 & 123.80 & 11 \\ 123.80 & 79.98 & 12 \\ 11 & 12 & 3 \end{bmatrix}.$$

*It is easy to verify that* $K_{\Lambda_2} b - c^T = 0$ *and* $A^T K_{\Lambda_2} + K_{\Lambda_2} A \leqslant 0$. *Similarly, the storage functions corresponding to the other three Lambda-sets are*

*(a) For* $\Lambda_1$: $K_{\Lambda_1} = \begin{bmatrix} 40.75 & 44.20 & 11 \\ 44.20 & 48.10 & 12 \\ 11 & 12 & 3 \end{bmatrix}$, *(b) For* $\Lambda_3$: $K_{\Lambda_3} = \begin{bmatrix} 146.47 & 123.80 & 11 \\ 123.80 & 108.03 & 12 \\ 11 & 12 & 3 \end{bmatrix}$,

*(c) For* $\Lambda_4$: $K_{\Lambda_4} = \begin{bmatrix} 345.25 & 44.20 & 11 \\ 44.20 & 139.90 & 12 \\ 11 & 12 & 3 \end{bmatrix}$

Example 5.5 shows that corresponding to different Lambda-sets we have different storage functions of a singularly passive SISO system. In Chapter 6, we revisit this example again to show that one of these storage functions is the maximal and one of them is the minimal among all the storage functions of the system $\Sigma$.

Note that similar to the singular LQR case, the subspaces spanned by $\mathtt{img}\, V_\Lambda$ and $\mathtt{img}\, W$ with $V_\Lambda$ and $W$ as defined in Theorem 5.7 have interesting system-theoretic interpretations. Using the parallel between the output-nulling representations of $\Sigma_{\mathtt{p}}$ (in equation (2.12)) and $\Sigma_{\mathtt{Ham}}$ (in equation (5.11)), we define $P := \widehat{A}$, $L := \widehat{b}$, $M := \widehat{c}$, $U_1 := E$ and $U_2 := H$. Further, we have $\mathtt{degdet}(sE - H) = 2\mathtt{n_s}$. Therefore, $\mathtt{N_s} = 2\mathtt{n_s}$ and $\mathtt{N_f} = \mathtt{N} - \mathtt{N_s} = 2\mathtt{n} - 2\mathtt{n_s} = 2\mathtt{n_f}$. Hence, Theorem 2.24, Theorem 2.25, and Lemma 2.26 can be directly applied to $\Sigma_{\mathtt{Ham}}$. On choosing a Lambda-set $\Lambda$ of $\mathtt{det}(sE - H)$ such that $\Lambda \subsetneq \mathbb{C}_-$ it is evident from Lemma 2.26 that $\mathtt{img}\, V_\Lambda$ is the largest good $(\widehat{A}, \widehat{b})$-invariant subspace inside the kernel of $\widehat{c}$. Hence, the good slow subspace of $\Sigma_{\mathtt{Ham}}$ is given by $\mathscr{O}_w = \mathtt{img}\, V_\Lambda$ if $\Lambda \subsetneq \mathbb{C}_-$. Further, it is evident from Theorem 2.24 that $\mathtt{img}\, W \subsetneq \mathscr{R}_s$, where $W$ is as defined in Theorem 5.7 and $\mathscr{R}_s$ is the fast subspace of $\Sigma_{\mathtt{Ham}}$. Thus, we have a direct-sum decomposition of the state-space $\mathbb{R}^{2\mathtt{n}}$ of the Hamiltonian system as illustrated in Figure 5.2.

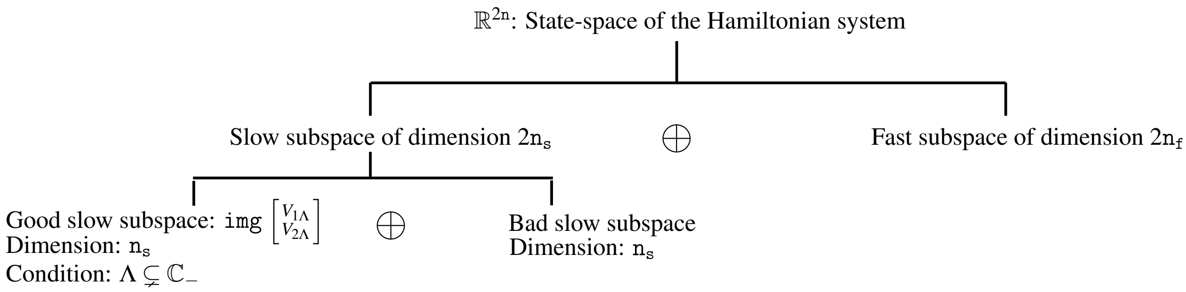



Figure 5.2: A direct-sum decomposition of the state-space of the Hamiltonian system $\Sigma_{\mathrm{Ham}}$

Further, from Theorem 5.7 and equation (5.29) it is evident that $X_{1\Lambda}$ is nonsingular and $X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix}$, where $W_1 = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix}$. Hence, the state-space $\mathbb{R}^{\mathtt{n}}$ of $\Sigma$ can be decomposed as:

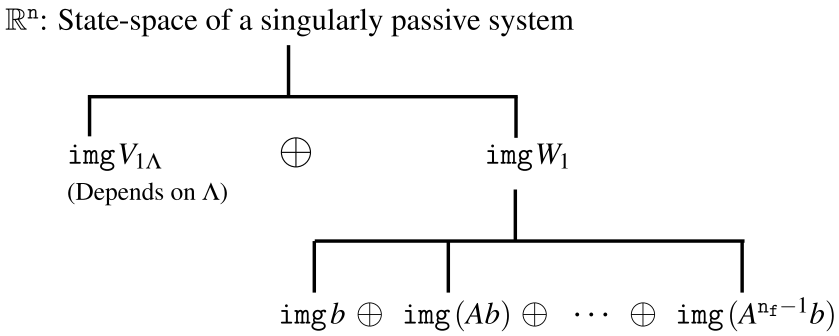



Figure 5.3: A direct-sum decomposition of the state-space of a singularly passive system

Interestingly, note that the subspace $\mathtt{img}\, V_{1\Lambda}$ depends on the choice of $\Lambda$. On the other hand, the subspace $\mathtt{img}\, W_1$ is independent of the choice of $\Lambda$, since it is spanned by the columns of a truncated controllability matrix that depends only on $A$, $b$, and $c$ matrices (see equation (5.29)). Hence, no matter what Lambda-set we choose for the computation of a rank-minimizing solution of the KYP LMI the subspace complementary to the eigenspace of $\Sigma_{\text{Ham}}$ corresponding to a Lambda-set always remains the same.

## 5.4 Algorithm to compute rank-minimizing solutions of a KYP LMI: SISO case

In this section we use Theorem 5.7 to propose an algorithm to compute rank-minimizing solutions of a singularly passive SISO system. Interestingly, note that one can retrieve the algorithm to compute solutions of ARE (rank-minimizing solutions of KYP LMI) corresponding to a regularly passive SISO system as a special case of Algorithm 5.16: we discuss this after presenting the algorithm. Hence Algorithm 5.16 is a generalized algorithm to compute rank-minimizing solutions of the KYP LMI (storage functions) corresponding to a passive SISO system provided such a system does not admit spectral zeros on the imaginary axis.

**Algorithm 5.16** Algorithm to compute rank-minimizing solutions of a KYP LMI.

**Input:** $(A,b,c)$ matrices corresponding to a passive SISO system $\Sigma$.

**Output:** $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Construct $(E,H)$ as defined in equation (5.7) and $\mathtt{n_s} := \mathtt{degdet}(sE-H)/2$.

Use generalized real-Schur decomposition algorithm on $(E,H)$ to compute basis of eigenspace corresponding to Lambda-set $\Lambda$ of $\mathtt{det}(sE-H)$. Let columns of $V_{\mathtt{e}\Lambda} \in \mathbb{R}^{(2\mathtt{n}+1)\times\mathtt{n_s}}$ be the basis.

Partition $V_{\mathtt{e}\Lambda} := \text{col}\,(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ where $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}, V_{3\Lambda} \in \mathbb{R}^{\mathtt{n_s}}$ and define $V_\Lambda := \text{col}(V_{1\Lambda}, V_{2\Lambda})$.

**if** $\mathtt{n_s} \neq \mathtt{n}$ **then**

  Construct $\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$ and $\widehat{b} = \begin{bmatrix} b \\ c^T \end{bmatrix}$

  Compute $\mathtt{n_f} = \mathtt{n} - \mathtt{n_s}$ and construct $W := \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \widehat{A}^2\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n_f}}$.

  Construct $X := \begin{bmatrix} V_\Lambda & W \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$

**else**

  Construct $X := V_\Lambda \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$

**end if**

Partition $X$ as $X =: \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix}$ where $X_{1\Lambda}, X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Compute the storage function: $K = X_{2\Lambda} X_{1\Lambda}^{-1} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Now, we list down a few special cases of Algorithm 5.16.

1. **Systems with $\mathtt{n_s} = \mathtt{n}$:** This is the case when we deal with regularly passive SISO systems. In this case, the regularity condition on $D + D^T$ is satisfied. Hence, they admit an ARE. The cardinality of the Lambda-sets in this case is equal to the system's order $\mathtt{n}$. This method of finding solutions of the ARE is exactly similar to the method proposed in [vD81] and Proposition 5.4.

2. **Systems with $\mathtt{n_f} = \mathtt{n}$:** This is the case when $\det(sE - H)$ is a nonzero constant, i.e., $\det(sE - H) \in \mathbb{R} \setminus \{0\}$. These are *singularly passive systems of order* 0. We also call these systems *strongly passive*. A more formal definition of a strongly passive system is as follows:

**Definition 5.17.** *A singularly passive SISO system with transfer function $G(s)$ (with no common poles and zeros) is called* strongly passive *if the numerator of $G(s) + G(-s)$ is a nonzero constant.*

This term "strongly" passive system stems from the fact that such a system does not admit any exponential *lossless* trajectories. We elaborate on this idea of lossless trajectories in Chapter 6. Note that for strongly passive systems, the Lambda-sets are empty. In this scenario, $X$ in Step (8) of Algorithm 5.16 is given by $W$. This answers the second question we posed after Lemma 5.6 in Section 5.3.

Now that we have an algorithm to compute the rank-minimizing solutions of a KYP LMI corresponding to a singularly passive SISO system, we list the flop-counts of each step of Algorithm 5.16 in Table 5.3.

| Step | Operations | Algorithm | Flops |
|---|---|---|---|
| 1 | Matrix concatenation | Merely bookkeeping | 0 |
| 2 | Eigenvector computation | QZ-algorithm | $\mathscr{O}(\mathtt{n}^3)$ |
| 3 | Matrix partitioning | Merely bookkeeping | 0 |
| 5 | Matrix concatenation | Merely bookkeeping | 0 |
| 6 | $(\mathtt{n_f} - 1)$ Matrix-vector multiplication | Normal matrix-vector multiplication | $\mathscr{O}\left(\mathtt{n_f^3}\right)$ |
| 7 - 11 | Matrix concatenation and partitioning | Merely bookkeeping | 0 |
| 12 | Matrix inversion | Cholesky, LU, Gaussian elimination | $\mathscr{O}\left(\mathtt{n}^3\right)$ |
| | Matrix-matrix multiplication | Normal matrix-matrix multiplication | $\mathscr{O}\left(\mathtt{n}^3\right)$ |

Table 5.3: Flop-count of each step in Algorithm 5.16

From Table 5.3 it is evident that the total flop count for Algorithm 5.16 is $\mathscr{O}\left(\mathtt{n}^3\right)$. A standard method to compute solutions of LMI is to use semi-definite programming (SDP) techniques. As discussed in the introduction of this chapter, it is known in the literature that solving an LMI using SDP techniques requires generically $\mathscr{O}\left(\mathtt{n}^6\right)$ flops, while exploitation of certain structures in the problem may lead to an improvement up to $\mathscr{O}\left(\mathtt{n}^{4.5}\right)$ flops [VBW+05]. Hence, Algorithm 5.16 is expected to perform faster when compared with SDP based optimization packages. To demonstrate this we compare the time required by Algorithm 5.16 to compute a solution of KYP LMI (5.2) to that required by two standard MATLAB based optimization packages viz., *CVX: MATLAB Software for Disciplined Convex Programming* (CVX) [GB13] and *Yet Another LMI Parser* (YALMIP) [Löf04]. Apart from these two packages, we also compare Algorithm 5.16 with the spectral factorization technique (SFT) described in [WT98]. We use a one-variable Euclidean division algorithm to implement this technique. We do not compare Algorithm 5.16 with the deflating subspace based method in [Rei11], [RRV15] due to the absence, to the best of our knowledge, of standard packages to implement it. The experimental setup for the comparison of Algorithm 5.16 with standard methods is as follows.

### 5.4.1 Experimental setup and procedure

The experiment has been carried out on an Intel(R) Xeon(R) computer operating at 3.50 GHz with 64 GB RAM using Ubuntu 16.04 LTS operating system. Numerical computational package MATLAB has been used to implement Algorithm 5.16 and the standard `tic-toc` command of MATLAB is used to record the computational time. Execution time for the Euclidean division based spectral factorization algorithm is also computed using the `tic-toc` command. The SDP solver used for both CVX and YALMIP is `sedumi`. The predefined numerical precision for the solver has been set to $10^{-12}$. The total computational time for CVX is obtained by the command `cvx_cputime`, which includes both CVX modelling time and solver time. Similarly, the field `yalmiptime` is used to obtain the total computational time, which includes modelling and solver time, for YALMIP.

Randomly generated transfer functions corresponding to singularly passive SISO systems of 5 different orders are used to compare the computational time of Algorithm 5.16 with that of CVX, YALMIP and SFT implemented in MATLAB. The computation time for each order has been averaged over fifteen randomly generated transfer functions. Further, in order to nullify the effect of CPU delays the computational time to calculate solutions of the KYP LMI (5.2) for each transfer function is further averaged over hundred iterations.

### 5.4.2 Experimental results

*Computational time:* Figure 5.4 demonstrates the time taken to compute the storage functions of singularly passive systems using CVX, YALMIP, SFT and Algorithm 5.16. From Figure 5.4, it is evident that Algorithm 5.16 is approximately $10^3$ times faster compared to CVX and YALMIP. Further, it is also clear that although the execution time for Algorithm 5.16 is better than that of SFT, it is comparable.

*Computational error:* Since SDP solvers have an inherent numerical tolerance associated with them, the solutions of LMI (5.2) found using CVX and YALMIP are within a predefined numerical precision. However, all the operations performed in Algorithm 5.16 are imple-

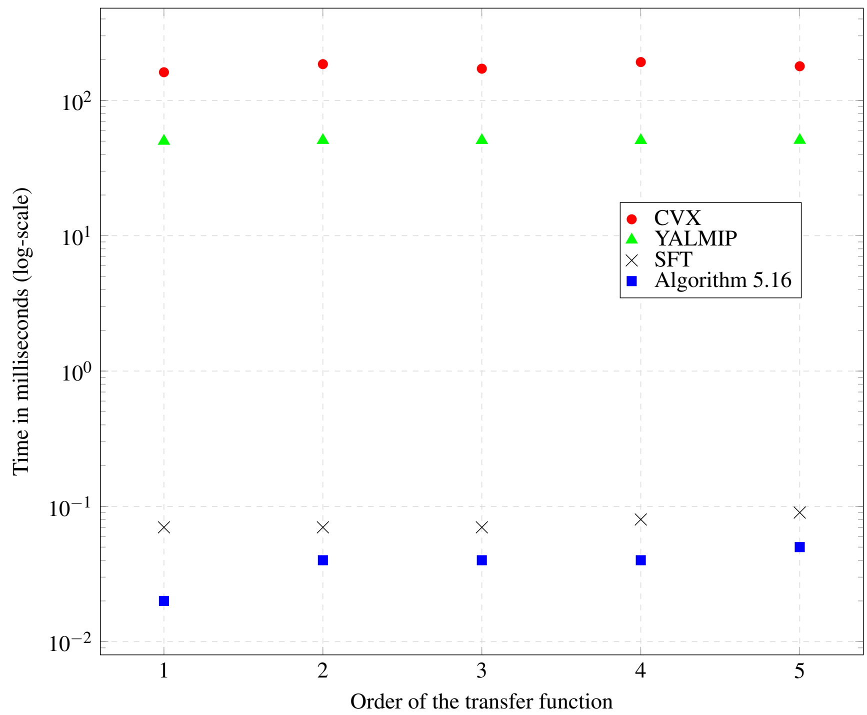



Figure 5.4: Plot of computational time to solve KYP LMI (5.2) for singularly passive SISO systems using CVX, YALMIP, SFT and Algorithm 5.16.

mentable using algorithms that are not only numerically stable [Wat02] but also do not admit any numerical tolerance. A few of such numerically stable algorithms are suggested in Table 5.3. Evidently, Algorithm 5.16 is better than CVX and YALMIP from a numerical precision viewpoint as well. On the other hand, since SFT based algorithms use matrix-matrix multiplication for its implementation, the error associated with SFT is comparable to that of Algorithm 5.16. However, one of the major drawbacks of SFT is that the solution obtained from the SFT algorithm corresponds to the KYP LMI (5.2) when the system matrices $(A,b,c)$ are in the controller canonical form. An advantage of the method we propose in this chapter (Algorithm 5.16) is that it leads to interesting system-theoretic interpretations that we reveal in the chapter that follows. Such system-theoretic interpretations might not be possible using the SFT method.

## 5.5 Summary

In this chapter, we presented a method to compute the rank-minimizing solutions of the KYP LMI corresponding to a singularly passive SISO system (Theorem 5.7). In order to derive Theorem 5.7, we crucially used the properties of the Markov parameters of the Hamiltonian system corresponding to a singularly passive SISO system (Lemma 5.14) and the fact that a singularly passive SISO system does not share spectral zeros and poles (Lemma 5.8). The method of computing the rank-minimizing solutions of the KYP LMI proposed in this chapter has a striking similarity to the method to compute the maximal solutions of LQR LMI in Chapter

2 (Theorem 2.30). We have shown that, similar to Theorem 2.30, augmenting a basis of the eigenspace of the Hamiltonian matrix pair $(E,H)$ corresponding to a Lambda-set with a basis of a subspace of the fast subspace of $\Sigma_{\text{Ham}}$ is the crucial idea that leads to Theorem 5.7.

Interestingly, it is known in the literature that a KYP LMI admits a maximal and a minimal solution. These solutions turn out to be rank-minimizing solutions of the KYP LMI as well. In the next chapter we show that such *extremal* solutions of the KYP LMI can be computed using the method we developed, in this chapter, to compute rank-minimizing solutions of the KYP LMI (Theorem 5.7). Further, for regularly passive systems it is known that the solutions of the corresponding ARE are related to the trajectories of minimal dissipation of the system, we call such trajectories *lossless trajectories*. Hence, to develop a generalized Riccati theory for the passive case, it is imperative that the notion of lossless trajectories is linked to the rank-minimizing solutions of the KYP LMI for the singularly passive SISO systems as well. We develop this theory in the next chapter.

# Chapter 6

# Lossless trajectories and extremal storage functions of passive systems

## 6.1 Introduction

It is well-known that the set of all storage functions of a regularly passive system admits a partial ordering by matrix semi-definiteness [Wil71]. Likewise the set of all storage functions of a singularly passive SISO system is also partially ordered. The likeness does not end here, however. Like in the case of regularly passive systems, in the case of singularly passive SISO systems also, there exist extremum storage functions. For the regularly passive case, these extremum storage functions happen to be maximal and minimal solutions of the corresponding ARE. We show in this section that, in absence of an ARE for singularly passive systems, these extremum storage functions are supplied by applying Theorem 5.7 (or, equivalently, Algorithm 5.16), with suitable choices of the Lambda-sets. This is the content of one of our main results Theorem 6.7. The extremal solutions of the KYP LMI corresponding to a passive system have interesting system-theoretic interpretations when viewed from a network-theoretic perspective. For example, for an RLC network the minimal energy required to change the states of the system from 0 to an arbitrary state $x_0 \in \mathbb{R}^{\mathtt{n}}$ is related to the maximal storage function $K_{\mathtt{max}}$ of the system:

$$x_0^T K_{\mathtt{max}} x_0 = \inf_{\mathtt{col}(u,y)\in\Sigma} \int_{-\infty}^{0} \left(2u^T y\right) dt. \tag{6.1}$$

For this reason, $x_0^T K_{\mathtt{max}} x_0$ is also called the *required supply* at $t = 0$ due to $\mathtt{col}(u,y)$ [WT98, Remark 5.14]. The input-output trajectories $\mathtt{col}(u,y)$ of the system that achieve this infimum are therefore called the *trajectories of optimal-charging* of the RLC circuit. Similarly, the maximum amount of energy that can be extracted from an RLC circuit when its state is changed from any arbitrary initial condition $x_0$ to 0 is related to the minimal storage function of $K_{\mathtt{min}}$ of the system:

$$x_0^T K_{\mathtt{min}} x_0 = \sup_{\mathtt{col}(u,y)\in\Sigma} \int_{0}^{+\infty} -\left(2u^T y\right) dt. \tag{6.2}$$

Hence, $x_0^T K_{\mathtt{min}} x_0$ is called the *available storage* due to $\mathtt{col}(u,y)$ [WT98, Remark 5.14]. The trajectories that achieve this supremum are called the *trajectories of optimal-discharging* for an RLC circuit. In this chapter we provide a method, similar to that in Chapter 3, to design state-feedback controllers that confine the set of system trajectories to its optimal-charging/discharging trajectories.

The main tool used in arriving at Theorem 6.7 is the dissipation inequality (5.5); especially, an extreme case of the same, when the inequality (5.5) reduces to an equality. A solution of the dynamical equations (5.1), for which the dissipation inequality is satisfied as an equation, is often called *lossless*. For such a solution, the input power gets entirely utilized in storage of energy. In the next section, we formally define this notion of lossless solutions and provide a method to construct such solutions for a given singularly passive SISO system (Lemmas 6.1 and 6.2). Thereafter, we utilize these solutions to find the extremal solutions of the KYP LMI and to design state-feedback controllers to confine the set of system trajectories to its optimal-charging/discharging trajectories.

## 6.2 Characterization of lossless trajectories

Recall from Figure 5.3 that the state-space $\mathbb{R}^{\mathtt{n}}$ of a singularly passive SISO system can be decomposed as $\mathbb{R}^{\mathtt{n}} = \mathtt{img}\, V_{1\Lambda} \oplus \mathtt{img}\, W_1$, where $V_{1\Lambda}$ and $W_1$ are as defined in Theorem 5.7 and equation (5.29), respectively. Hence, any arbitrary initial condition $x_0 \in \mathbb{R}^{\mathtt{n}}$ of the system $\Sigma$ can be decomposed as $x_0 = x_{0\mathtt{s}} + x_{0\mathtt{f}}$, where $x_{0\mathtt{s}} \in \mathtt{img}\, V_{1\Lambda}$ and $x_{0\mathtt{f}} \in \mathtt{img}\, W_1$. In what follows we present two lemmas, Lemma 6.1 and Lemma 6.2, that characterizes certain special trajectories of the system $\Sigma$ when the initial conditions of the system are from the subspace $\mathtt{img}\, V_{1\Lambda}$ and $\mathtt{img}\, W_1$, respectively. In the sequel, we show that these are the trajectories for which the input power gets entirely utilized in storage of energy. For reasons that would be clear after the lemmas we call $\mathtt{img}\, V_{1\Lambda}$ and $\mathtt{img}\, W_1$ the space of *regular* and *irregular* initial conditions of $\Sigma$, respectively.

Trajectories of a singularly passive system for regular initial conditions

**Lemma 6.1.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with a minimal i/s/o representation as in equation* (5.9)*. Assume $\Lambda, V_{1\Lambda}, V_{2\Lambda}$, and $V_{3\Lambda}$ be as defined in Theorem 5.7. Assume the initial condition of the system $\Sigma$ be $x_0 := V_{1\Lambda}\beta$, where $\beta \in \mathbb{R}^{\mathtt{n_s}}$. Define $z_0 := V_{2\Lambda}\beta$, $\bar{x}_{\mathtt{s}} := V_{1\Lambda} e^{\Gamma t}\beta$, $\bar{z}_{\mathtt{s}} := V_{2\Lambda} e^{\Gamma t}\beta$, $\bar{u}_{\mathtt{s}} := F V_{1\Lambda} e^{\Gamma t}\beta$ and $\bar{y}_{\mathtt{s}} := c\bar{x}_{\mathtt{s}}$, where $F : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}$ such that $V_{3\Lambda} = F V_{1\Lambda}$. Then, the following statements are true:*

*(1)* $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{z}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}) \in \Sigma_{\mathrm{Ham}}$ *corresponding to initial condition* $\mathtt{col}(x_0, z_0)$.

*(2)* $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}, \bar{y}_{\mathtt{s}}) \in \Sigma$ *corresponding to initial condition* $x_0$.

*Proof:* On using $\widehat{A}, \widehat{b}$ and $\widehat{c}$ as defined in equation (5.11) and the fact that $y = cx$, the proof of this lemma is exactly similar to the proof of Lemma 3.7. Hence, we do not repeat the proof here. ■

Trajectories of a singularly passive system for irregular initial conditions

**Lemma 6.2.** *Consider a singularly passive SISO system* $\Sigma$ *of order* $\mathtt{n_s}$ *with a minimal i/s/o representation as in equation* (5.1). *Let the corresponding Hamiltonian system* $\Sigma_{\text{Ham}}$ *be as defined in equation* (5.11). *Let* $x_{0k} \in \mathbb{R}^{\mathtt{n}}$ *be as given in the table below, where* $k \in \{0, 1, \cdots, \mathtt{n_f} - 1\}$. *Let* $z_{0k} \in \mathbb{R}^{\mathtt{n}}$, $\bar{x}_{\mathtt{f}k}, \bar{z}_{\mathtt{f}k} \in \mathfrak{C}^{\mathtt{n}}_{\mathtt{imp}}$ *and* $\bar{u}_{\mathtt{f}k} \in \mathfrak{C}_{\mathtt{imp}}$ *as defined as in the table below.*

| $k$ | $x_{0k}$ | $z_{0k}$ | $\bar{x}_{\mathtt{f}k}$ | $\bar{z}_{\mathtt{f}k}$ | $\bar{u}_{\mathtt{f}k}$ |
|---|---|---|---|---|---|
| 0 | $\alpha_0 b$ | $\alpha_0 c^T$ | 0 | 0 | $-\alpha_0 \delta$ |
| 1 | $\alpha_1 A b$ | $-\alpha_1 (cA)^T$ | $-\alpha_1 b \delta$ | $-\alpha_1 c^T \delta$ | $-\alpha_1 \delta^{(1)}$ |
| 2 | $\alpha_2 A^2 b$ | $\alpha_2 (cA^2)^T$ | $-\alpha_2 \left( b\delta^{(1)} + Ab\delta \right)$ | $-\alpha_2 \left( -c^T \delta^{(1)} + (cA)^T \delta \right)$ | $-\alpha_2 \delta^{(2)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathtt{n_f} - 1$ | $\alpha_{\mathtt{n_f}-1} A^{\mathtt{n_f}-1} b$ | $(-1)^{\mathtt{n_f}-1} \alpha_{\mathtt{n_f}-1} (cA^{\mathtt{n_f}-1})^T$ | $-\alpha_{\mathtt{n_f}-1} \sum_{i=0}^{\mathtt{n_f}-2} A^{\mathtt{n_f}-2-i} b \delta^{(i)}$ | $-\alpha_{\mathtt{n_f}-1} \sum_{i=0}^{\mathtt{n_f}-2} (-1)^i (cA^{\mathtt{n_f}-2-i})^T \delta^{(i)}$ | $-\alpha_{\mathtt{n_f}-1} \delta^{(\mathtt{n_f}-1)}$ |

*Let* $x_{0\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} x_{0k}$, $\bar{x}_{\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} \bar{x}_{\mathtt{f}k}$, $\bar{z}_{\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} \bar{z}_{\mathtt{f}k}$, $\bar{u}_{\mathtt{f}} := \sum_{k=0}^{\mathtt{n_f}-1} \bar{u}_{\mathtt{f}k}$ *and* $\bar{y}_{\mathtt{f}} = c\bar{x}_{\mathtt{f}}$. *Then the following statements are true:*

*(1)* $\mathtt{col}\,(\bar{x}_{\mathtt{f}}, \bar{z}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}) \in \Sigma_{\text{Ham}}$ *corresponding to initial condition* $\mathtt{col}\,(x_{0\mathtt{f}}, z_{0\mathtt{f}})$.

*(2)* $\mathtt{col}\,(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}, \bar{y}_{\mathtt{f}}) \in \Sigma$ *corresponding to initial condition* $x_{0\mathtt{f}}$.

*Proof:* On using $\widehat{A}, \widehat{b}$ and $\widehat{c}$ as defined in equation (5.11) and the fact that $y = cx$, the proof of this lemma is exactly similar to the proof of Lemma 3.6. Hence, we do not repeat the proof. ■

Note the primary difference between Lemma 3.6 and Lemma 6.2 is that the costates $\bar{z}$ corresponding to the optimal trajectories in case of singular LQR problems are zero but for singularly passive SISO systems, the costates are non-zero. The reason for the costates in singular LQR problems showing such zero structure is due to Statement *(2)* of Lemma 2.36.

Now we claim that the trajectories characterized in Lemma 6.1 and Lemma 6.2 are indeed lossless trajectories, trajectories for which the rate of change of stored energy is *equal* to the power supplied, of a singularly passive system. However, before that we need to formally define a lossless trajectory.

**Definition 6.3.** *Consider a passive SISO system* $\Sigma$ *with a minimal i/s/o representation as in equation* (5.9). *Let* $\mathtt{col}(x, u, y) \in \mathfrak{C}^{\mathtt{n}+1+1}_{\mathtt{imp}}$ *be a trajectory in* $\Sigma$. *Then,* $\mathtt{col}(x, u, y)$ *is called a* lossless trajectory *if there exists a solution* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *of the corresponding KYP LMI* (5.2) *such that*

$$\frac{d}{dt}\left(x^T K x\right) = 2uy \text{ for all } t \in \mathbb{R}. \tag{6.3}$$

Note that corresponding to an initial condition in $\mathtt{img}\, V_{1\Lambda}$ and with input $\bar{u}_{\mathtt{s}}$ as defined in Lemma 6.1, the unique state and output of the system $\Sigma$ is given by $\bar{x}_{\mathtt{s}}$ and $\bar{y}_{\mathtt{s}}$ defined in Lemma 6.1.

Slow lossless trajectories of a singularly passive system

**Lemma 6.4.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$. Let $\bar{x}_{\mathtt{s}}$, $\bar{u}_{\mathtt{s}}$, and $\bar{y}_{\mathtt{s}}$ be as defined in Lemma 6.1. Then, $\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})$ is a lossless trajectory of $\Sigma$ in the sense of Definition 6.3.*

*Proof:* Let $V := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda})$, where $V_{1\Lambda}, V_{2\Lambda}$ are as defined in Lemma 6.1. As defined in Theorem 5.7, let $W := \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \dots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n_f}}$ and $\begin{bmatrix} V & W \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. Then, by Theorem 5.7, $K := X_2X_1^{-1}$ is a solution of the singular KYP LMI (5.10). Note that

$$\frac{d}{dt}\left(x^TKx\right) = \dot{x}^TKx + x^TK\dot{x} = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^TK+KA & Kb-c^T \\ b^TK-c & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + 2ucx. \tag{6.4}$$

From Statement 3 of Theorem 5.7, we know that $Kb - c^T = 0$. Further, $y = cx$ and therefore,

$$\frac{d}{dt}\left(x^TKx\right) = x^T(A^TK+KA)x + 2uy \tag{6.5}$$

Therefore, from equation (6.5), we have

$$\begin{aligned} \bar{x}_{\mathtt{s}}^T(A^TK+KA)\bar{x}_{\mathtt{s}} + 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}} &= (V_{1\Lambda}e^{\Gamma t}\beta)^T(A^TK+KA)(V_{1\Lambda}e^{\Gamma t}\beta) + 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}} \\ &= \beta^Te^{\Gamma^Tt}V_{1\Lambda}^T(A^TK+KA)V_{1\Lambda}e^{\Gamma t}\beta + 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}}. \end{aligned} \tag{6.6}$$

From equation (5.38), we know that $V_{1\Lambda}^T(A^TK+KA)V_{1\Lambda} = 0$. Using this in equation (6.6) we therefore have

$$\bar{x}_{\mathtt{s}}^T(A^TK+KA)\bar{x}_{\mathtt{s}} + 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}} = 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}}. \tag{6.7}$$

Therefore, equation (6.5) and equation (6.7) gives $\frac{d}{dt}\left(x^TKx\right)|_{\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})} = 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}}$. This proves that $\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})$ is a lossless trajectory. ■

Note that the lossless trajectories obtained in Lemma 6.4 are exponential in nature with exponents given by the spectral zeros of the system. Therefore, we call such trajectories the *slow lossless* trajectories of $\Sigma$. Thus, the slow lossless trajectories of $\Sigma$ are obtained when the initial conditions of the system are from $\mathtt{img}\, V_{1\Lambda}$. This is why we call $\mathtt{img}\, V_{1\Lambda}$ the space of regular initial conditions.

In the next lemma we show that the impulsive trajectories characterized in Lemma 6.2 are indeed lossless trajectories in the sense of Definition 6.3. Note that since the trajectories characterized in Lemma 6.2 are from the space of impulsive-smooth distributions, Definition 6.3 does not preclude the possibility of multiplication of $\delta$ and its derivatives with themselves. Hence we treat equation (6.3) *formally* here, i.e., equation (6.3) is said to hold if the expression $\frac{d}{dt}\left(x^TKx\right) - 2u^Ty$ is zero as a function for $t \in (0,\infty)$ and each of the coefficients of $\delta, \dot{\delta}, \dots, \delta^{(k)}, \dots$ and their powers is zero in the impulsive part of the expression.

Fast lossless trajectories of a singularly passive system

**Lemma 6.5.** *Consider a singularly passive SISO system* $\Sigma$ *of order* $\mathtt{n_s}$. *Let* $\bar{x}_\mathtt{f}$, $\bar{u}_\mathtt{f}$, *and* $\bar{y}_\mathtt{f}$ *be as defined in Lemma 6.2. Then,* $\mathtt{col}(\bar{x}_\mathtt{f}, \bar{u}_\mathtt{f}, \bar{y}_\mathtt{f})$ *is lossless in the sense of Definition 6.3.*

*Proof:* Since $\Sigma_{\mathrm{Ham}}$ is a linear system, from Lemma 6.2, it is clear that the trajectory in $\Sigma_{\mathrm{Ham}}$, corresponding to initial condition $(x_0, z_0) = \sum_{k=0}^{\mathtt{n_f}-1} \widehat{A}^k \widehat{b} \alpha_k$, can be characterized as

$$\begin{bmatrix} \bar{x}_\mathtt{f} \\ \bar{z}_\mathtt{f} \end{bmatrix} = -\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & \delta & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(\mathtt{n_f}-2)} \\ 0 & 0 & \delta & \delta^{(1)} & \cdots & \delta^{(\mathtt{n_f}-3)} \\ 0 & 0 & 0 & \delta & \cdots & \delta^{(\mathtt{n_f}-4)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \delta^{(1)} \\ 0 & 0 & 0 & 0 & \cdots & \delta \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Omega} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{\mathtt{n_f}-1} \end{bmatrix}}_{\alpha} \tag{6.8}$$

$$\bar{u}_\mathtt{f} = -\begin{bmatrix} \delta & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(\mathtt{n_f}-1)} \end{bmatrix} \alpha \tag{6.9}$$

where $W_1 = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix}$ and $W_2 = \begin{bmatrix} c^T & -(cA)^T & \cdots & (-1)^{\mathtt{n_f}-1}(cA^{\mathtt{n_f}-1})^T \end{bmatrix}$. From equation (6.8), we have $\bar{x}_\mathtt{f} = -W_1 \Omega \alpha$.

Now, construct a Lambda-set of $\mathtt{det}(sE - H)$. Let a basis of the $\mathtt{n_s}$-dimensional eigenspace corresponding to $\Lambda$ be the columns of $V_\Lambda := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$, where $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{2\mathtt{n} \times \mathtt{n_s}}$ and $V_{3\Lambda} \in \mathbb{R}^{1 \times \mathtt{n_s}}$. Define $V := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda})$ and $\begin{bmatrix} V & W \end{bmatrix} =: \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix}$. Then, by Theorem 5.7, $K := X_{2\Lambda} X_{1\Lambda}^{-1}$ is a solution of the singular KYP LMI (5.10). Therefore, from equation (6.5), we get

$$\frac{d}{dt}\left(x^T K x\right) = x^T (A^T K + KA) x + 2uy. \tag{6.10}$$

Evaluating the right hand side of equation (6.10) corresponding to the trajectories $\mathtt{col}(\bar{x}_\mathtt{f}, \bar{u}_\mathtt{f}, \bar{y}_\mathtt{f})$, we have

$$\bar{x}_\mathtt{f}^T (A^T K + KA) \bar{x}_\mathtt{f} + 2\bar{u}_\mathtt{f} \bar{y}_\mathtt{f} = \alpha^T \Omega^T W_1^T (A^T K + KA) W_1 \Omega \alpha + 2\bar{u}_\mathtt{f} \bar{y}_\mathtt{f} \tag{6.11}$$

From the proof of Statement 3 of Theorem 5.7, we have

$$W_1^T (A^T K + KA) W_1 = \mathtt{diag}(0_{\mathtt{n_f}-1, \mathtt{n_f}-1}, r), \text{ where } r = (-1)^{\mathtt{n_f}} \widehat{c} \widehat{A}^{2\mathtt{n_f}-1} \widehat{b}. \tag{6.12}$$

Using equation (6.12) and $\Omega$ from equation (6.8) in equation (6.11), we have

$$\bar{x}_{\mathtt{f}}^T(A^TK+KA)\bar{x}_{\mathtt{f}}+2\bar{u}_{\mathtt{f}}\bar{y}_{\mathtt{f}} =$$

$$\alpha^T \left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 0 \\ \delta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{(\mathtt{n_f}-3)} & \delta^{(\mathtt{n_f}-4)} & \cdots & 0 & 0 \\ \hline \delta^{(\mathtt{n_f}-2)} & \delta^{(\mathtt{n_f}-3)} & \cdots & \delta & 0 \end{array}\right] \left[\begin{array}{c|c} 0_{\mathtt{n_f}-1,\mathtt{n_f}-1} & 0 \\ \hline 0 & r \end{array}\right] \left[\begin{array}{cccc|c} 0 & \delta & \cdots & \delta^{(\mathtt{n_f}-3)} & \delta^{(\mathtt{n_f}-2)} \\ 0 & 0 & \cdots & \delta^{(\mathtt{n_f}-4)} & \delta^{(\mathtt{n_f}-3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \delta \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] \alpha + 2\bar{u}_{\mathtt{f}}\bar{y}_{\mathtt{f}}$$

$$= 0 + 2\bar{u}_{\mathtt{f}}\bar{y}_{\mathtt{f}} = 2\bar{u}_{\mathtt{f}}\bar{y}_{\mathtt{f}}. \tag{6.13}$$

Using equation (6.13) in equation (6.11) we have $\frac{d}{dt}\left(x^TKx\right)|_{\mathtt{col}(\bar{x}_{\mathtt{f}},\bar{u}_{\mathtt{f}},\bar{y}_{\mathtt{f}})} = 2\bar{u}_{\mathtt{f}}\bar{y}_{\mathtt{f}}$. Therefore, the trajectory $\mathtt{col}(\bar{x}_{\mathtt{f}},\bar{u}_{\mathtt{f}},\bar{y}_{\mathtt{f}})$ is lossless in the sense of Definition 6.3. ∎

Note that the lossless trajectories obtained in Lemma 6.2 are impulsive in nature. Therefore, we call such trajectories the *fast lossless* trajectories of $\Sigma$. Thus, the fast lossless trajectories of $\Sigma$ are obtained when the initial conditions of the system are from $\mathtt{img}\, W_1$. This is why we call $\mathtt{img}\, W_1$ the space of irregular initial conditions. Interestingly, unlike the slow lossless trajectories of a singularly passive system, the fast lossless trajectories of a singularly passive system do not depend on the spectral zeros of the system.

On combining Lemma 6.4 and Lemma 6.5, we get one of the main results of this chapter.

Lossless trajectories of a singularly passive system

**Theorem 6.6.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with any arbitrary initial condition $x_0$. Define $\bar{x} := \bar{x}_{\mathtt{s}}+\bar{x}_{\mathtt{f}}$, $\bar{u} := \bar{u}_{\mathtt{s}}+\bar{u}_{\mathtt{f}}$ and $\bar{y} := \bar{y}_{\mathtt{s}}+\bar{y}_{\mathtt{f}}$, where $\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})$ and $\mathtt{col}(\bar{x}_{\mathtt{f}},\bar{u}_{\mathtt{f}},\bar{y}_{\mathtt{f}})$ are trajectories as defined in Lemma 6.1 and Lemma 6.5, respectively. Then, the following statements are true:*

*(1) $\mathtt{col}(\bar{x},\bar{u},\bar{y})$ is a unique trajectory of $\Sigma$ corresponding to initial condition $x_0$.*

*(2) $\mathtt{col}(\bar{x},\bar{u},\bar{y})$ is lossless in the sense of Definition 6.3.*

*Proof:* *(1):* From Figure 5.3 we know that any initial condition $x_0$ of the system $\Sigma$ can be uniquely written as $x_0 = x_{0\mathtt{s}}+x_{0\mathtt{f}}$, where $x_{0\mathtt{s}} \in \mathtt{img}\, V_{1\Lambda}$ and $x_{0\mathtt{f}} \in \mathtt{img}\, W_1$. The unique lossless trajectory corresponding to a initial condition $x_{0\mathtt{s}}$ and $x_{0\mathtt{f}}$ of $\Sigma$ is $\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})$ and $\mathtt{col}(\bar{x}_{\mathtt{f}},\bar{u}_{\mathtt{f}},\bar{y}_{\mathtt{f}})$, respectively. Thus, by linearity of the system $\Sigma$, it directly follows that $\mathtt{col}(\bar{x},\bar{u},\bar{y})$ is a unique trajectory in $\Sigma$.

*(2):* In order to prove this statement, we evaluate the right-hand side of the equation (6.5) for the trajectory $\mathtt{col}(\bar{x},\bar{u},\bar{y})$.

$$\frac{d}{dt}(x^TKx)|_{\mathtt{col}(\bar{x},\bar{u},\bar{y})} = \bar{x}^T(A^TK+KA)\bar{x}+2\bar{u}\bar{y} = (\bar{x}_{\mathtt{s}}+\bar{x}_{\mathtt{f}})^T(A^TK+KA)(\bar{x}_{\mathtt{s}}+\bar{x}_{\mathtt{f}})+2\bar{u}\bar{y}. \tag{6.14}$$

From equation (6.7) and equation (6.13), we know that $\bar{x}_{\mathtt{s}}^T(A^TK+KA)\bar{x}_{\mathtt{s}} = 0$ and $\bar{x}_{\mathtt{f}}^T(A^TK+KA)\bar{x}_{\mathtt{f}} = 0$, respectively. Further, using the fact that $\bar{x}_{\mathtt{s}} = V_{1\Lambda}e^{\Gamma t}\beta$ and $\bar{x}_{\mathtt{f}} = W_1\Omega\alpha$ in equation

(6.11), we have

$$\frac{d}{dt}(x^TKx)|_{\texttt{col}(\bar{x},\bar{u},\bar{y})} = \bar{x}_{\mathtt{f}}^T(A^TK+KA)\bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{s}}^T(A^TK+KA)\bar{x}_{\mathtt{f}} + 2\bar{u}\bar{y}$$
$$= \alpha^T\Omega^TW_1^T(A^TK+KA)V_{1\Lambda}e^{\Gamma t}\beta + \beta^Te^{\Gamma^Tt}V_{1\Lambda}^T(A^TK+KA)W_1\Omega\alpha + 2\bar{u}\bar{y}. \quad (6.15)$$

Recall from equation (5.39) that $W_1^T(A^TK+KA)V_{1\Lambda} = 0$. Using this in equation (6.15) , we therefore have $\frac{d}{dt}(x^TKx)|_{\texttt{col}(\bar{x},\bar{u},\bar{y})} = 2\bar{u}\bar{y}$. Thus, $\texttt{col}(\bar{x},\bar{u},\bar{y})$ is lossless. ■

From Theorem 6.6 it is evident that a singularly passive system $\Sigma$ not only admits exponential lossless trajectories but also admits impulsive lossless trajectories.

The lossless trajectories of a singularly passive system corresponding to different initial conditions, characterized in Lemma 6.1 and Lemma 6.2, can therefore be listed in a table as follows:

| $x_0$ | $x(t)$ | $u(t)$ | $y(t)$ |
|---|---|---|---|
| $V_{1\Lambda}\beta$ | $V_{1\Lambda}e^{\Gamma t}\beta$ | $V_{3\Lambda}e^{\Gamma t}\beta$ | $cV_{1\Lambda}e^{\Gamma t}\beta$ |
| $\alpha_0 b$ | $0$ | $-\alpha_0\delta$ | $0$ |
| $\alpha_1 Ab$ | $-\alpha_1 b\delta$ | $-\alpha_1\delta^{(1)}$ | $-\alpha_1 cb\delta$ |
| $\alpha_2 A^2 b$ | $-\alpha_2\left(b\delta^{(1)}+Ab\delta\right)$ | $-\alpha_2\delta^{(2)}$ | $-\alpha_2\left(cb\delta^{(1)}+cAb\delta\right)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\alpha_{\mathtt{n_f}-1}A^{\mathtt{n}-1}b$ | $-\alpha_{\mathtt{n_f}-1}\sum_{i=0}^{\mathtt{n_f}-2}A^{\mathtt{n_f}-2-i}b\delta^{(i)}$ | $-\alpha_{\mathtt{n_f}-1}\delta^{(\mathtt{n_f}-1)}$ | $-\alpha_{\mathtt{n_f}-1}\sum_{i=0}^{\mathtt{n_f}-2}cA^{\mathtt{n_f}-2-i}b\delta^{(i)}$ |

Table 6.1: Table to show the lossless trajectories of a singularly passive system $\Sigma$ corresponding to different initial conditions.

Note that Table 6.1 and Table 3.1 are similar. This hints at an underlying similarity in the theory of singular LQR problems and singularly passive systems. We discuss this in details in Chapter 8.

Recall that for a special class of singularly passive SISO systems called strongly passive systems the Hamiltonian pencil satisfies $\texttt{det}(sE-H) \in \mathbb{R}\setminus\{0\}$ (see Definition 5.17). This implies that such systems do not admit a Lambda-set. Hence, from Table 6.1 it is evident that strongly passive systems do not admit slow lossless trajectories. Such systems admit fast lossless trajectories only. Since these systems do not admit any slow lossless trajectories, loosely speaking, it means that none of the slow trajectories of such a system satisfy the dissipation inequality with an equality. Hence, all the slow trajectories results in dissipation of energy. This is the reason we call such systems strongly passive systems.

Now that we have characterized the lossless trajectories of singularly passive SISO systems, we show that like regularly passive systems, singularly passive systems admit extremal storage functions.

## 6.3 Extremal storage functions

At the very outset of this section, we present the main result of this section that establishes the existence of two special storage functions, denoted by $K_{\mathtt{max}}$ and $K_{\mathtt{min}}$, such that, for every other storage function $K$ we must have $K_{\mathtt{min}} \leqslant K \leqslant K_{\mathtt{max}}$. The matrices $K_{\mathtt{max}}$ and $K_{\mathtt{min}}$ are called the *extremal storage functions* of a passive system. We also show that these extremal storage functions can be constructed using Theorem 5.7 with suitably chosen Lambda-sets.

Extremal solutions of the KYP LMI

**Theorem 6.7.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with a minimal i/s/o representation as in equation* (5.9)*. Let the corresponding Hamiltonian pencil pair $(E,H)$ be as defined in equation* (5.11)*. Let $\Lambda_{\mathtt{min}}$ and $\Lambda_{\mathtt{max}}$ be a pair of Lambda-sets of $\mathtt{det}(sE-H)$ such that $\Lambda_{\mathtt{min}} \subsetneq \mathbb{C}_-$ and $\Lambda_{\mathtt{max}} \subsetneq \mathbb{C}_+$. Suppose $K_{\mathtt{min}}$ and $K_{\mathtt{max}}$ are the storage functions of $\Sigma$ constructed using Theorem 5.7 corresponding to Lambda-sets $\Lambda_{\mathtt{min}}$ and $\Lambda_{\mathtt{max}}$, respectively. Then, for all $K$ that satisfies the singular KYP LMI* (5.10)*, the following inequality holds:*

$$K_{\mathtt{min}} \leqslant K \leqslant K_{\mathtt{max}}.$$

To prove Theorem 6.7, we need a few auxiliary results analogous to Lemma 2.37 for singular LQR problems. The proofs of these auxiliary results follows the same line of reasoning as that of the proof of Lemma 2.37.

Algebraic relations satisfied by the solutions of a KYP LMI

**Lemma 6.8.** *Consider a singularly passive SISO system $\Sigma$ of order $\mathtt{n_s}$ with a minimal i/s/o representation as in equation* (5.9)*. Define $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$. Let $K$ be any solution of the corresponding singular KYP LMI* (5.10)*. Then,*

$$KA^{\alpha}b = (-1)^{\alpha}(cA^{\alpha})^T, \textit{ for any } \alpha \in \{0,1,2,\ldots,\mathtt{n_f}-1\}.$$

*Proof:* We prove Lemma 6.8 using induction and Lemma 5.14.
*Base case:* ($\alpha = 0$) Note that for any solution $K = K^T$ of the LMI (5.10), we have $Kb - c^T = 0$. Thus, $Kb = c^T$ is trivially true.
*Inductive step:* Suppose $\alpha \leqslant \mathtt{n_f} - 1$. Assume $KA^{(\alpha-1)}b = (-1)^{(\alpha-1)}\left(cA^{(\alpha-1)}\right)^T$, we show that $KA^{\alpha}b = (-1)^{\alpha}(cA^{\alpha})^T$ .

Pre- and post-multiplying $A^TK + KA$ by $(A^{(\alpha-1)}b)^T$ and $A^{(\alpha-1)}b$, respectively, we get: $\left(A^{(\alpha-1)}b\right)^T \left(A^TK+KA\right)\left(A^{(\alpha-1)}b\right) \leqslant 0$. Opening the brackets and using the inductive hypothesis, the last inequality takes the form:

$$
\begin{aligned}
(A^{\alpha}b)^T K A^{(\alpha-1)}b + (A^{(\alpha-1)}b)^T K A^{\alpha} b \leqslant 0 \\
\Rightarrow (-1)^{(\alpha-1)} \left((cA^{2\alpha-1}b)^T + (cA^{2\alpha-1}b)\right) \leqslant 0 \\
\Rightarrow (-1)^{(\alpha-1)} \begin{bmatrix} c & -b^T \end{bmatrix} \begin{bmatrix} A^{2\alpha-1} & 0 \\ 0 & (-1)^{2\alpha-1}(cA^{2\alpha-1})^T \end{bmatrix} \begin{bmatrix} b \\ c^T \end{bmatrix} \leqslant 0 \\
\Rightarrow (-1)^{(\alpha-1)} \widehat{c}\widehat{A}^{2\alpha-1}\widehat{b} \leqslant 0. \qquad (6.16)
\end{aligned}
$$

Recall from Lemma 5.14 that $\widehat{c}\widehat{A}^k\widehat{b} = 0$ for $k \in \{1,\ldots,2(\mathtt{n_f}-1)\}$. Since $\alpha \in \{1,2,\cdots,\mathtt{n_f}-1\}$, using Lemma 5.14 in inequality (6.16), we get

$$
\begin{aligned}
(-1)^{(\alpha-1)}\widehat{c}\widehat{A}^{2\alpha-1}\widehat{b} = (-1)^{(\alpha-1)} \left((cA^{2\alpha-1}b)^T + (cA^{2\alpha-1}b)\right) = 0 \\
\Rightarrow \left(A^{(\alpha-1)}b\right)^T \left(A^T K + KA\right) \left(A^{(\alpha-1)}b\right) = 0. \qquad (6.17)
\end{aligned}
$$

Since $A^T K + KA$ is sign-semidefinite, from equation (6.17) we infer that

$$
\left(A^T K + KA\right) A^{(\alpha-1)}b = 0 \Rightarrow A^T K A^{(\alpha-1)} b + KA^{\alpha} b = 0. \qquad (6.18)
$$

Using the inductive hypothesis in equation (6.18), it follows that

$$
A^T (-1)^{\alpha-1} \left(cA^{(\alpha-1)}\right)^T + KA^{\alpha}b = 0 \Rightarrow KA^{\alpha}b = (-1)^{\alpha} \left(cA^{\alpha}\right)^T.
$$

This completes the proof of the lemma. ■

Lemma 6.9 is another auxiliary result that we need for the proof of Theorem 6.7. This lemma shows that, the difference between an arbitrary solution of the KYP LMI (5.2) and a rank-minimizing solution of the KYP LMI (5.2) (ones obtained using Theorem 5.7) satisfies a certain Lyapunov inequality when restricted to a suitable $\mathtt{n_s}$-dimensional subspace.

**Difference between the solutions of a KYP LMI satisfies a Lyapunov equation**

**Lemma 6.9.** *Consider a singularly passive SISO system of order* $\mathtt{n_s}$. *Let the corresponding Hamiltonian pencil pair* $(E,H)$ *be as defined in equation* (5.11). *Let* $K_\Lambda$ *be the rank-minimizing solution of the system computed using Theorem 5.7 corresponding to a Lambda-set* $\Lambda$ *of* $\mathtt{det}(sE-H)$. *Let* $V_{1\Lambda}$ *be as defined in Theorem 5.7. Assume* $K$ *to be any other solution of the KYP LMI* (5.2). *Define* $\Delta_\Lambda := V_{1\Lambda}^T(K-K_\Lambda)V_{1\Lambda} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$. *Then,* $\Delta_\Lambda$ *satisfies the following Lyapunov inequality:*

$$
\Gamma_\Lambda^T \Delta_\Lambda + \Delta_\Lambda \Gamma_\Lambda \leqslant 0.
$$

*Proof:* Recall that the slow lossless trajectories of the system $\Sigma$ are given by $\mathtt{col}(\bar{x}_\mathtt{s},\bar{u}_\mathtt{s},\bar{y}_\mathtt{s})$, where $\bar{x}_\mathtt{s},\bar{u}_\mathtt{s}$ and $\bar{y}_\mathtt{s}$ are as defined in Lemma 6.1. Therefore, we have

$$
\frac{d}{dt}\left(x^T K_\Lambda x\right)|_{\mathtt{col}(\bar{x}_\mathtt{s},\bar{u}_\mathtt{s},\bar{y}_\mathtt{s})} = 2\bar{u}_\mathtt{s}\bar{y}_\mathtt{s} \text{ for all } t \in \mathbb{R}. \qquad (6.19)
$$

From the dissipation inequality of a passive system (inequality (5.5)), it follows that for any solution $K$ of the singular KYP LMI (5.10), the slow lossless trajectories $\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})$ must satisfy the following inequality

$$\frac{d}{dt}\left(x^T K x\right)|_{\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})} \leqslant 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}} \text{ for all } t\in\mathbb{R}. \tag{6.20}$$

Subtracting equation (6.19) from inequality (6.20) gives for all $t\in\mathbb{R}$,

$$\frac{d}{dt}\left(x^T(K-K_\Lambda)x\right)|_{\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})} = \left(\dot{x}^T(K-K_\Lambda)x + x^T(K-K_\Lambda)\dot{x}\right)|_{\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})} \leqslant 0. \tag{6.21}$$

From Table 6.1, we have $\bar{x}_{\mathtt{s}} = V_{1\Lambda}e^{\Gamma_\Lambda t}\beta$, where $x_0 = V_{1\Lambda}\beta$ is the initial condition. Hence, $\dot{x} = V_{1\Lambda}e^{\Gamma_\Lambda t}\Gamma_\Lambda\beta$. Therefore, for all $t\in\mathbb{R}$, we have from inequality (6.21)

$$(V_{1\Lambda}e^{\Gamma_\Lambda t}\Gamma_\Lambda\beta)^T(K-K_\Lambda)(V_{1\Lambda}e^{\Gamma_\Lambda t}\beta) + (V_{1\Lambda}e^{\Gamma_\Lambda t}\beta)^T(K-K_\Lambda)(V_{1\Lambda}e^{\Gamma_\Lambda t}\Gamma_\Lambda\beta) \leqslant 0. \tag{6.22}$$

Since inequality (6.22) is true for all $t$, evaluating it at $t=0$, in particular, we get

$$\beta^T(\Gamma_\Lambda^T V_{1\Lambda}^T(K-K_\Lambda)V_{1\Lambda} + V_{1\Lambda}^T(K-K_\Lambda)V_{1\Lambda}\Gamma_\Lambda)\beta = \beta^T(\Gamma_\Lambda^T\Delta_\Lambda + \Delta_\Lambda\Gamma_\Lambda)\beta \leqslant 0 \tag{6.23}$$

Since inequality (6.23) is true for all $\beta\in\mathbb{R}^{\mathtt{n_s}}$, we infer that $\Gamma_\Lambda^T\Delta_\Lambda + \Delta_\Lambda\Gamma_\Lambda \leqslant 0$. ■

Using Lemma 6.8 and Lemma 6.9 we prove Theorem 6.7 next.

*Proof of Theorem 6.7:* First we prove that $K \geqslant K_{\mathtt{min}}$, i.e., $K - K_{\mathtt{min}} \geqslant 0$. Let the eigenvectors of $(E,H)$ corresponding to the Lambda-set $\Lambda_{\mathtt{min}}$ be given by the columns of $\mathtt{col}(V_{1\mathtt{min}},V_{2\mathtt{min}},V_{3\mathtt{min}})$, where $V_{1\mathtt{min}},V_{2\mathtt{min}}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\mathtt{min}}\in\mathbb{R}^{\mathtt{n_s}}$. Partition $\begin{bmatrix} V_{\Lambda_{\mathtt{min}}} & W\end{bmatrix} =: \begin{bmatrix} X_{1\mathtt{min}} \\ X_{2\mathtt{min}}\end{bmatrix}$, where $W$ is as defined in Theorem 5.7 and $X_{1\mathtt{min}},X_{2\mathtt{min}}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then, $K_{\mathtt{min}} = X_{2\mathtt{min}}X_{1\mathtt{min}}^{-1}$. Further, from equation (5.29) we know that $W$ can be partitioned as

$$W := \begin{bmatrix}\widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b}\end{bmatrix} = \begin{bmatrix} W_1 \\ W_2\end{bmatrix},$$

where $W_1 = \begin{bmatrix} b & Ab & \cdots A^{(\mathtt{n_f}-1)}b\end{bmatrix}$ and $W_2 = \begin{bmatrix} c^T & \cdots & (-1)^{(\mathtt{n_f}-1)}\left(cA^{(\mathtt{n_f}-1)}\right)^T\end{bmatrix}$. Therefore, $X_{1\mathtt{min}} = \begin{bmatrix} V_{1\mathtt{min}} & W_1\end{bmatrix}$.

Note that since $X_{1\mathtt{min}}$ is nonsingular, proving $(K-K_{\mathtt{min}})\geqslant 0$ is equivalent to showing that $X_{1\mathtt{min}}^T(K-K_{\mathtt{min}})X_{1\mathtt{min}}\geqslant 0$. Hence, we evaluate $X_{1\mathtt{min}}^T(K-K_{\mathtt{min}})X_{1\mathtt{min}}$ next.

$$X_{1\mathtt{min}}^T(K-K_{\mathtt{min}})X_{1\mathtt{min}} = \begin{bmatrix} V_{1\mathtt{min}}^T(K-K_{\mathtt{min}})V_{1\mathtt{min}} & V_{1\mathtt{min}}^T(K-K_{\mathtt{min}})W_1 \\ W_1^T(K-K_{\mathtt{min}})V_{1\mathtt{min}} & W_1^T(K-K_{\mathtt{min}})W_1\end{bmatrix} \tag{6.24}$$

From Lemma 6.8, we know that

$$K\begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b\end{bmatrix} = \begin{bmatrix} c^T & -(cA)^T & \cdots & (-1)^{\mathtt{n_f}-1}(cA^{\mathtt{n_f}-1})^T\end{bmatrix} \Rightarrow KW_1 = W_2.$$

Since $K_{\mathtt{min}}$ is a storage function of $\Sigma$, $K_{\mathtt{min}}W_1 = W_2$. Therefore, $(K - K_{\mathtt{min}})W_1 = 0$. Using this in equation (6.24) we have

$$X_{1\mathtt{min}}^T (K - K_{\mathtt{min}}) X_{1\mathtt{min}} = \begin{bmatrix} V_{1\mathtt{min}}^T (K - K_{\mathtt{min}}) V_{1\mathtt{min}} & 0 \\ 0 & 0_{\mathtt{n_f},\mathtt{n_f}} \end{bmatrix} = \begin{bmatrix} \Delta_{\Lambda\mathtt{min}} & 0 \\ 0 & 0_{\mathtt{n_f},\mathtt{n_f}} \end{bmatrix}, \qquad (6.25)$$

where $\Delta_{\Lambda\mathtt{min}} := V_{1\mathtt{min}}^T (K - K_{\mathtt{min}}) V_{1\mathtt{min}}$. Further, from Lemma 6.9, we know that $\Delta_{\Lambda\mathtt{min}}$ satisfies the Lyapunov inequality: $(\Gamma_{\mathtt{min}}^T \Delta_{\Lambda\mathtt{min}} + \Delta_{\Lambda\mathtt{min}} \Gamma_{\mathtt{min}}) \leqslant 0$. Since $\sigma(\Gamma_{\mathtt{min}}) \subsetneq \mathbb{C}_-$, by property of Lyapunov operator, $\Delta_{\Lambda\mathtt{min}} \geqslant 0$. Using this positive-semidefiniteness property of $\Delta_{\Lambda\mathtt{min}}$ in equation (6.25), we infer $X_{1\mathtt{min}}^T (K - K_{\mathtt{min}}) X_{1\mathtt{min}} \geqslant 0 \Rightarrow K - K_{\mathtt{min}} \geqslant 0$.

Now we prove that $K \leqslant K_{\mathtt{max}}$, i.e., $K - K_{\mathtt{max}} \leqslant 0$. Let an $\mathtt{n_s}$-dimensional eigenspace basis corresponding to Lambda-set $\Lambda_{\mathtt{max}}$ be given by the columns of $\mathtt{col}(V_{1\mathtt{max}}, V_{2\mathtt{max}}, V_{3\mathtt{max}})$, where $V_{1\mathtt{max}}, V_{2\mathtt{max}} \in \mathbb{R}^{\mathtt{n} \times \mathtt{n_s}}$ and $V_{3\mathtt{max}} \in \mathbb{R}^{\mathtt{n_s}}$. Partition $\begin{bmatrix} V_{1\mathtt{max}} & W \end{bmatrix} =: \begin{bmatrix} X_{1\mathtt{max}} \\ X_{2\mathtt{max}} \end{bmatrix}$, where $W$ is as defined in Theorem 5.7 and $X_{1\mathtt{max}}, X_{2\mathtt{max}} \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$. Then, $K_{\mathtt{max}} = X_{2\mathtt{max}} X_{1\mathtt{max}}^{-1}$. Thus, $X_{1\mathtt{max}} = \begin{bmatrix} V_{1\mathtt{max}} & W_1 \end{bmatrix}$.

Using the same line of reasoning as given for the proof $K - K_{\mathtt{min}} \geqslant 0$, proving $K - K_{\mathtt{max}} \leqslant 0$ is equivalent to proving that $X_{1\mathtt{max}}^T (K - K_{\mathtt{max}}) X_{1\mathtt{max}} \leqslant 0$. Since $K_{\mathtt{max}} W_1 = W_2$, we have $(K - K_{\mathtt{max}}) W_1 = 0$. Therefore,

$$X_{1\mathtt{max}}^T (K - K_{\mathtt{max}}) X_{1\mathtt{max}} = \begin{bmatrix} V_{1\mathtt{max}}^T (K - K_{\mathtt{max}}) V_{1\mathtt{max}} & 0 \\ 0 & 0_{\mathtt{n_f},\mathtt{n_f}} \end{bmatrix} = \begin{bmatrix} \Delta_{\Lambda\mathtt{max}} & 0 \\ 0 & 0_{\mathtt{n_f},\mathtt{n_f}} \end{bmatrix}, \qquad (6.26)$$

where $\Delta_{\Lambda\mathtt{max}} := V_{1\mathtt{max}}^T (K - K_{\mathtt{max}}) V_{1\mathtt{max}}$. Further, from Lemma 6.9 we know that $\Gamma_{\mathtt{max}}^T \Delta_{\Lambda\mathtt{max}} + \Delta_{\Lambda\mathtt{max}} \Gamma_{\mathtt{max}} \leqslant 0$. Since $\sigma(\Gamma_{\mathtt{max}}) \subsetneq \mathbb{C}_+$, we infer that $\Delta_{\Lambda\mathtt{max}} \leqslant 0 \Rightarrow X_{1\mathtt{max}}^T (K - K_{\mathtt{max}}) X_{1\mathtt{max}} \leqslant 0$. Thus, $K - K_{\mathtt{max}} \leqslant 0$. ■

From the proof of Theorem 6.7, it follows that the rank-minimizing solutions of the KYP LMI computed using the basis of the eigenspace corresponding to Lambda-sets that are either subsets of $\mathbb{C}_-$ or $\mathbb{C}_+$ induces the *extremal* storage functions of a singularly passive SISO system. The storage function corresponding to the Lambda-set that is a subset of $\mathbb{C}_-$ gives the minimal storage function and the one corresponding to a subset of $\mathbb{C}_+$ gives the maximal storage function.

**Example 6.10.** *Recall for the system in Example 5.16, the minimal storage function corresponds to the one computed using Lambda-set $\Lambda_1$ and the maximal storage function corresponds to that of $\Lambda_4$. Therefore, for the system in Example 5.16, we have*

$$K_{\mathtt{min}} = \begin{bmatrix} 40.75 & 44.20 & 11 \\ 44.20 & 48.10 & 12 \\ 11 & 12 & 3 \end{bmatrix}, \qquad K_{\mathtt{max}} = \begin{bmatrix} 345.25 & 44.20 & 11 \\ 44.20 & 139.90 & 12 \\ 11 & 12 & 3 \end{bmatrix}.$$

The initial part of the proof of Theorem 6.7, especially the application of Lemma 6.8, reveals an interesting fact about the set of all solutions of the singular KYP LMI. This part of the proof shows that *every* solution $K$ of the KYP LMI satisfies $KW_1 = W_2$. Let us denote by $M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ the matrix obtained by appending column vectors to the left of $W_1$ to complete it into a nonsingular matrix. That is, $M := \begin{bmatrix} Y & W_1 \end{bmatrix}$, where $Y \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ such that $M$ is nonsingular. Clearly, the columns of $M$ form a basis of the state-space $\mathbb{R}^{\mathtt{n}}$. Note that, under this basis, the solutions of the KYP LMI with transformed system matrices would be related with the old solutions through a congruence transformation: indeed, every solution of the transformed LMI would be of the form $M^TKM$, where $K$ is a solution of the original LMI. However, writing out this congruence transformation explicitly and using the fact that $KW_1 = W_2$ we get

$$M^TKM = \begin{bmatrix} Y^TKY & Y^TKW_1 \\ W_1^TKY & W_1^TKW_1 \end{bmatrix} = \begin{bmatrix} Y^TKY & Y^TW_2 \\ W_2^TY & W_1^TW_2 \end{bmatrix}.$$

From Lemma 6.8 then it follows that the last $\mathtt{n_f}$ rows and the last $\mathtt{n_f}$ columns of $M^TKM$, i.e., the submatrices $W_1^TW_2$, $W_2^TY$ and $Y^TW_2$ of $M^TKM$, remains the same for *all* solutions $K = K^T$ of the KYP LMI. Therefore, under this basis, the difference between any pair of solutions $K_1$ and $K_2$ of the singular KYP LMI would be of the form $\begin{bmatrix} Y^T(K_1-K_2)Y & 0 \\ 0 & 0_{\mathtt{n_f},\mathtt{n_f}} \end{bmatrix}$.

**Example 6.11.** *Note that in Example 5.16, the difference of the rank-minimizing solutions of the KYP LMI are as follows:*

$$K_{\Lambda_2} - K_{\Lambda_1} = \begin{bmatrix} 198.78 & 79.6 & 0 \\ 79.6 & 139.88 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_{\Lambda_3} - K_{\Lambda_2} = \begin{bmatrix} -93.06 & 0 & 0 \\ 0 & 28.05 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*Similarly, the difference between the other rank-minimizing solutions obtained in Example 5.16 will also have a similar structure as the one shown above. Now, we demonstrate that this is true for any arbitrary solution of the KYP LMI* (5.2).

*Let $K$ be a solution of the KYP LMI* (5.2) *corresponding to the system in this example. Define* $K := \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}$. *Since $K$ is solution of the KYP LMI* (5.2), *it has to satisfy $Kb - c^T = 0$. Then, we must have*

$$Kb - c^T = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 11 \\ 12 \\ 3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} k_{13} \\ k_{23} \\ k_{33} \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 3 \end{bmatrix}.$$

*Thus, all solutions of the KYP LMI* (5.2) *must be of the form* $K = \begin{bmatrix} k_{11} & k_{12} & 11 \\ k_{12} & k_{22} & 12 \\ 11 & 12 & 3 \end{bmatrix}$. *Therefore,* *it is evident that the last row and last column of the difference between any two solutions of the KYP LMI* (5.2)*, corresponding to the system in this example, must be zero.*

Next we compute explicitly the lossless trajectories of the circuit that admits the transfer function given in Example 5.16.

**Example 6.12.** *An RC circuit corresponding to the impedance function $G(s)$ as given in Example 5.16 is as follows:*

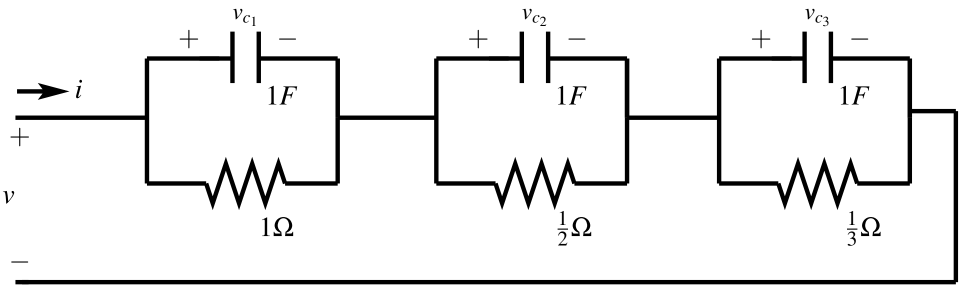



Figure 6.1: An RC network with an impedance function $Z(s) = G(s) = \dfrac{3s^2+12s+11}{s^3+6s^2+11s+6}$.

Slow lossless trajectories: *Corresponding to the spectral zeros of the system in $\mathbb{C}_-$, i.e., $-\sqrt{4+\sqrt{5}}$ and $-\sqrt{4-\sqrt{5}}$, the eigenvectors of $(E,H)$ are given by the columns of the matrix*

$$\begin{bmatrix} -0.34 & 0.85 & -2.12 & 0.35 & 0.36 & 0.09 & -0.13 \\ -0.55 & 0.73 & -0.97 & -0.80 & -0.82 & -0.19 & 0.20 \end{bmatrix}^T.$$

*Therefore, the slow lossless trajectory $\texttt{col}(\bar{x}_{\texttt{s}},\bar{u}_{\texttt{s}},\bar{y}_{\texttt{s}})$ of the RC circuit corresponding to an initial condition* $x_{0\texttt{s}} = \begin{bmatrix} -0.34 & -0.55 \\ 0.85 & 0.73 \\ -2.12 & -0.97 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$*, where $\beta_1, \beta_2 \in \mathbb{R}$ is*

$$\bar{x}_{\texttt{s}} = \begin{bmatrix} -0.34 & -0.55 \\ 0.85 & 0.73 \\ -2.12 & -0.97 \end{bmatrix} \begin{bmatrix} e^{-\sqrt{4+\sqrt{5}}t} & 0 \\ 0 & e^{-\sqrt{4-\sqrt{5}}t} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -0.34e^{-\sqrt{4+\sqrt{5}}t}\beta_1 - 0.55e^{-\sqrt{4-\sqrt{5}}t}\beta_2 \\ 0.85e^{-\sqrt{4+\sqrt{5}}t}\beta_1 + 0.73e^{-\sqrt{4-\sqrt{5}}t}\beta_2 \\ -2.12e^{-\sqrt{4+\sqrt{5}}t}\beta_1 - 0.97e^{-\sqrt{4-\sqrt{5}}t}\beta_2 \end{bmatrix},$$

$$\bar{u}_{\texttt{s}} = \begin{bmatrix} -0.13 & 0.20 \end{bmatrix} \begin{bmatrix} e^{-\sqrt{4+\sqrt{5}}t} & 0 \\ 0 & e^{-\sqrt{4-\sqrt{5}}t} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = -0.31e^{-\sqrt{4+\sqrt{5}}t}\beta_1 + 0.21e^{-\sqrt{4-\sqrt{5}}t}\beta_2, \text{ and}$$

$$\bar{y}_{\texttt{s}} = c\bar{x}_{\texttt{s}} = 0.09e^{-\sqrt{4+\sqrt{5}}t}\beta_1 - 0.19e^{-\sqrt{4-\sqrt{5}}t}\beta_2.$$

*These slow lossless trajectories corresponds to a Lambda-set $\Lambda$ of $\texttt{det}(sE-H)$ such that $\Lambda \subsetneq \mathbb{C}_-$. Recall that the initial condition $x_{0\texttt{s}}$ here is from the space of regular initial condition and hence, the lossless trajectories are exponential in nature. Next we look at lossless trajectories when the initial condition of the system is from the space of irregular initial conditions.*

Fast lossless trajectories: *Let the initial condition of the system be* $x_{0\mathtt{f}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \alpha$, *where* $\alpha \in \mathbb{R}$. *Then, using Table 6.1 the fast lossless trajectory* $(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}, \bar{y}_{\mathtt{f}})$ *of the RC circuit is*

$$\bar{x}_{\mathtt{f}} = 0_{\mathtt{n},1}, \ \bar{u}_{\mathtt{f}} = -\alpha\delta, \ \text{and } \bar{y}_{\mathtt{f}} = 0.$$

*Note that in this case* $\frac{d}{dt}(x^T Kx)|_{\mathtt{col}(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}, \bar{y}_{\mathtt{f}})} = 0$ *and* $2\bar{u}_{\mathtt{f}}\bar{y}_{\mathtt{f}} = 0$. *Hence, the rate of change of stored energy is equal to the power supplied; confirms that* $(\bar{x}_{\mathtt{f}}, \bar{u}_{\mathtt{f}}, \bar{y}_{\mathtt{f}})$ *is a lossless trajectory.*

*Thus, corresponding to an initial condition* $x_0 = x_{0\mathtt{s}} + x_{0\mathtt{f}} = \begin{bmatrix} -0.34 & -0.55 & 0 \\ 0.85 & 0.73 & 0 \\ -2.12 & -0.97 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \alpha \end{bmatrix}$, *the lossless trajectory of the system is given by*

$$\bar{x} = \bar{x}_{\mathtt{s}} + \bar{x}_{\mathtt{f}} = \begin{bmatrix} -0.34e^{-\sqrt{4+\sqrt{5}}t}\beta_1 - 0.55e^{-\sqrt{4-\sqrt{5}}t}\beta_2 \\ 0.85e^{-\sqrt{4+\sqrt{5}}t}\beta_1 + 0.73e^{-\sqrt{4-\sqrt{5}}t}\beta_2 \\ -2.12e^{-\sqrt{4+\sqrt{5}}t}\beta_1 - 0.97e^{-\sqrt{4-\sqrt{5}}t}\beta_2 \end{bmatrix},$$

$$\bar{u} = \bar{u}_{\mathtt{s}} + \bar{u}_{\mathtt{f}} = -0.31e^{-\sqrt{4+\sqrt{5}}t}\beta_1 + 0.21e^{-\sqrt{4-\sqrt{5}}t}\beta_2 - \alpha\delta,$$

$$\bar{y} = \bar{y}_{\mathtt{s}} + \bar{y}_{\mathtt{f}} = 0.09e^{-\sqrt{4+\sqrt{5}}t}\beta_1 - 0.19e^{-\sqrt{4-\sqrt{5}}t}\beta_2.$$

*The presence of* $\delta$ *in* $\bar{u}$ *indicates that with the capacitors initially charged to* $x_0$, *if one discharges the capacitors very fast, i.e. in the limit of a sequence of exponential decays: instantaneously*[1], *then it is possible to extract the capacitors' entire stored energy through the port. On the other hand, if* $\alpha \neq 0$ *then nonzero dissipation at the resistor R is inevitable if one does not discharge instantaneously.*

Note that although Example 6.12 do not present a scenario where we encounter products of $\delta$ and its derivatives while evaluating stored energy $x^T Kx$ or power supply $2uy$, there can be scenarios when we encounter such products. The next example demonstrates this.

**Example 6.13.** *Consider a singularly passive SISO system with a transfer function* $G(s) = \dfrac{s^2+qs+1}{s^3+qs^2+ds+1}$ *such that* $q,d \in \mathbb{R}_+ \setminus 0$ *and* $d = 1 + \dfrac{1}{q}$. *For example, let* $q = 2$ *and* $d = \dfrac{3}{2}$. *Then, an i/s/o representation of the system is given by*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1.5 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

*An RLC circuit corresponding to* $G(s)$ *as an impedance function is given in Figure 6.2.*

[1] Instantaneous discharge of capacitor $C$ (by a controller at the port, which is, in this case 'short') can be viewed as a limit of a sequence of exponentially decaying extractions, with increasing magnitudes of decay rates.

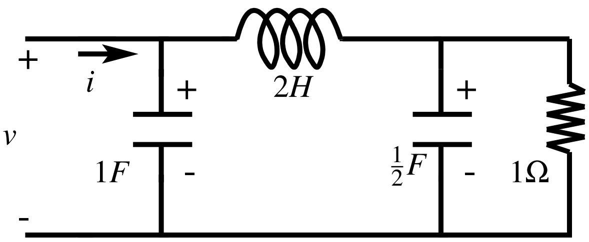



Figure 6.2: A RLC network with impedance transfer function $G(s) = \dfrac{s^2+2s+1}{s^3+2s^2+1.5s+1}$

*Note that* $\mathtt{n_f} = 3$ *here. Hence, using Theorem 5.7,* $W_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 2.5 \end{bmatrix}$. *Since* $\mathtt{n} = \mathtt{n_f}$, *the RLC circuit in Figure 6.2 do not admit any slow lossless trajectories. Therefore, corresponding to an initial condition* $x_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 2.5 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_0 b + \alpha_1 Ab + \alpha_2 A^2 b$, *the lossless trajectory of this system from Table 6.1 is given by*

$$\bar{x} = 0 - \alpha_1 b\delta - \alpha_2 \left(b\dot{\delta} + Ab\delta\right) = -\alpha_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta - \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\delta} - \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \delta = - \begin{bmatrix} 0 \\ \alpha_2\delta \\ (\alpha_1 - 2\alpha_2)\delta + \alpha_2\dot{\delta} \end{bmatrix},$$

$\bar{u} = -\alpha_0\delta - \alpha_1\delta^{(1)} - \alpha_2\delta^{(2)}$, *and* $\bar{y} = c\bar{x} = \alpha_1\delta + \alpha_2\dot{\delta}$.

*It is evident here that power supplied* $2\bar{u}\bar{y}$ *involves product of* $\delta$ *and its derivatives.*

Thus, it is clear that there are many scenarios when we encounter products of $\delta$ and its derivatives while evaluating the power supply $2uy$ or the stored energy $x^TKx$. Multiplication of $\delta$ and its derivatives has been defined in [Fuc84], [Tre09]. Such multiplications are defined in the literature using *Fuchssteiner multiplication*. However, physical interpretation of the products of $\delta$ and its derivatives is an open question to the best of our knowledge. We do not dwell into the physical interpretation of such products here. This is a matter of future research.

In the next section we show that similar to a singular LQR problem, there exist state-feedback controllers for a singularly passive SISO system, as well, that confine the set of trajectories of such a system to the lossless ones.

## 6.4 Controllers to confine the set of system trajectories to its lossless trajectories

In this section we design state-feedback controllers that confine the set of trajectories of a singularly passive SISO system to its lossless trajectories. Due to the similarity in the structure of the lossless trajectories of a singularly passive SISO system (see Table 6.1) and the optimal trajectories of a singular LQR problem (see Lemma 3.6 and Lemma 3.7) the design of the controllers is the same as described in Section 3.4. Therefore, we claim that a PD state-feedback control law of the form $x = F_{\mathtt{p}}x + F_{\mathtt{d}}\frac{d}{dt}x$ confines the set of trajectories of a singularly passive SISO system to its lossless trajectories.

Using the fact that $X_{1\Lambda}$ is nonsingular (Theorem 5.7), we define $F_{\mathtt{p}}$, $F_{\mathtt{d}} \in \mathbb{R}^{1\times\mathtt{n}}$ as follows:

$$F_{\mathtt{p}} := \begin{bmatrix} V_{3\Lambda} & f_0 & f_1 & \cdots & f_{\mathtt{n_f}-1} \end{bmatrix} X_{1\Lambda}^{-1}, \tag{6.27}$$
$$F_{\mathtt{d}} := \begin{bmatrix} 0_{1,\mathtt{n_s}} & 1 & -f_0 & \cdots & -f_{\mathtt{n_f}-2} \end{bmatrix} X_{1\Lambda}^{-1}, \tag{6.28}$$

where $V_{3\Lambda}$ is as defined in Theorem 5.7 and $f_i \in \mathbb{R}$ for $i \in \{0, 1, \ldots, \mathtt{n_f}-1\}$. The closed loop system obtained on application of $u = F_{\mathtt{p}}x + F_{\mathtt{d}}\frac{d}{dt}x$ to $\Sigma$ is as follows:

$$E_{\mathtt{c}}\frac{d}{dt}x = A_{\mathtt{c}}x \quad \text{and } y = cx, \quad \text{where } (I_{\mathtt{n}} - bF_{\mathtt{d}}) =: E_{\mathtt{c}}, (A + bF_{\mathtt{p}}) =: A_{\mathtt{c}}. \tag{6.29}$$

We use the symbol $\Sigma_{\mathtt{lossless}}$ to represent the closed loop system in equation (6.29). Following the same line of reasoning as in Section 3.4 of Chapter 3, we therefore have the following results.

Existence of $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ such that the matrix pencil $(sE_{\mathtt{c}} - A_{\mathtt{c}})$ is regular

**Lemma 6.14.** *Let $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ be as defined in equation* (6.27) *and equation* (6.28)*, respectively. Then, there exist $f_0, \ldots, f_{\mathtt{n_f}-1} \in \mathbb{R}$ such that* $\det(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$*, where $E_{\mathtt{c}}$, $A_{\mathtt{c}}$ are as defined in equation*(6.29).

*Proof:* The proof is exactly the same as that of Lemma 3.10. ■

Trajectories of the closed loop system $\Sigma_{\mathtt{lossless}}$

**Theorem 6.15.** *Let $\Sigma_{\mathtt{lossless}}$ be the system defined in equation* (6.29)*, where $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ are as defined in equations* (6.27) *and* (6.28)*, respectively, with* $\det(sE_{\mathtt{c}} - A_{\mathtt{c}}) \neq 0$. *Consider an arbitrary initial condition of the system $\Sigma_{\mathtt{lossless}}$ given as $x_0 =: V_{1\Lambda}\beta + W_1\alpha$, $\beta \in \mathbb{R}^{\mathtt{n_s}}$, $\alpha \in \mathbb{R}^{\mathtt{n_f}}$, where $V_{1\Lambda}$ and $W_1$ are as defined in Theorem 5.7 and equation* (5.29)*, respectively. Let $\bar{x}$ be as defined in Theorem 6.6. Then, the unique trajectory in $\Sigma_{\mathtt{lossless}}$ corresponding to $x_0$ is $\bar{x}$.*

*Proof:* The proof is exactly the same as that of Theorem 3.11. ■

Trajectories of the closed loop system $\Sigma_{\texttt{lossless}}$ are the lossless trajectories

**Theorem 6.16.** *Consider a singularly passive system $\Sigma$ of order $\mathtt{n_s}$. Assume $F_{\mathtt{p}} \in \mathbb{R}^{1\times \mathtt{n}}$ and $F_{\mathtt{d}} \in \mathbb{R}^{1\times \mathtt{n}}$ to be as defined in equation* (6.27) *and equation* (6.28)*, respectively with* $\det\left(s(I_{\mathtt{n}} - bF_{\mathtt{d}}) - (A + bF_{\mathtt{p}})\right) \neq 0$*. Let the closed loop system obtained on application of the PD state-feedback law $u = F_{\mathtt{p}}x + F_{\mathtt{d}}\frac{d}{dt}x$ to $\Sigma$ be as defined in equation* (6.29)*. Then, for an arbitrary initial condition $x_0$, the corresponding trajectory of the closed loop system $\Sigma_{\texttt{lossless}}$ is a lossless trajectory in the sense of Definition 6.3.*

*Proof:* The proof of this theorem directly follows from Theorem 6.6 and Theorem 6.15. ■

Slow and fast subspaces of the closed-loop system $\Sigma_{\texttt{lossless}}$

**Corollary 6.17.** *Consider the system $\Sigma_{\texttt{lossless}}$ with the state-space equation of the form given in equation* (6.29)*, where $F_{\mathtt{p}}$ and $F_{\mathtt{d}}$ are as defined in Theorem 6.16. Define $\mathscr{V} :=$ $\mathtt{img}\, V_{1\Lambda}$ and $\mathscr{W} := \mathtt{img}\, W_1$ with $V_{1\Lambda}$ and $W_1$ as defined in Theorem 5.7. Then, $\mathscr{V}$ and $\mathscr{W}$ are the slow subspace and fast subspace of the system $\Sigma_{\texttt{lossless}}$, respectively.*

*Proof:* The proof is exactly the same as that of Corollary 3.13. ■

From Corollary 6.17 it is evident that the slow and fast subspace of the system $\Sigma_{\texttt{lossless}}$ is the same as the space of regular and irregular initial condition of a singularly passive system, respectively. We illustrate the design of such state-feedback matrices for the RC circuit in Figure 6.1 next.

**Example 6.18.** *Corresponding to the Lambda-set $\Lambda_{\mathtt{min}} = \left\{-\sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}\right\}$, recall that $V_{3\Lambda} = \begin{bmatrix} -0.13 & 0.20 \end{bmatrix}$ and $X_{1\Lambda} = \begin{bmatrix} -0.34 & -0.55 & 0 \\ 0.85 & 0.73 & 0 \\ -2.12 & -0.97 & 1 \end{bmatrix}$. Therefore, choosing $f_0 = 0$ in equation* (6.27) *and equation* (6.28)*, we have*

$$F_{\mathtt{p}} = \begin{bmatrix} -0.13 & 0.21 & 0 \end{bmatrix} \begin{bmatrix} -0.34 & -0.55 & 0 \\ 0.85 & 0.73 & 0 \\ -2.12 & -0.97 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1.21 & -0.64 & 0 \end{bmatrix},$$

$$F_{\mathtt{d}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.34 & -0.55 & 0 \\ 0.85 & 0.73 & 0 \\ -2.12 & -0.97 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3.32 & 3.83 & 1 \end{bmatrix}.$$

*The closed-loop system obtained on application of* $u = F_{\mathtt{p}}x + \frac{d}{dt}F_{\mathtt{d}}$ *to* $\Sigma$ *is given by*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3.32 & -3.83 & 0 \end{bmatrix} \frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7.21 & -11.64 & -6 \end{bmatrix} x \tag{6.30}$$

*Evidently, the closed-loop system in equation* (6.30) *is a singular descriptor system. Further, the trajectories of this system are the lossless trajectories of* $\Sigma$*. Note that for the closed-loop system in equation* (6.30)*, we have* $\mathtt{det}(sE_{\mathtt{c}} - A_{\mathtt{c}}) = 7.21 + 8.32s + 2.17s^2$ *and as expected* $\mathtt{roots}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) = \left\{-\sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}\right\} = \Lambda_{\mathtt{min}}$.

*Similarly, corresponding to the Lambda-set* $\Lambda_{\mathtt{max}} = \left\{\sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\}$*, we have* $V_{3\Lambda} = \begin{bmatrix} 40.04 & -17.97 \end{bmatrix}$ *and* $X_{1\Lambda} = \begin{bmatrix} 0.46 & -0.54 & 0 \\ 1.16 & -0.71 & 0 \\ 2.89 & -0.95 & 1 \end{bmatrix}$*. On choosing* $f_0 = 0$ *in equation* (6.27) *and equation* (6.28)*, we have the following matrices:*

$$F_{\mathtt{p}} = \begin{bmatrix} -26.59 & 45.27 & 0 \end{bmatrix}, \qquad F_{\mathtt{d}} = \begin{bmatrix} 3.32 & -3.83 & 1 \end{bmatrix}.$$

*The corresponding closed-loop therefore is*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3.32 & 3.83 & 0 \end{bmatrix} \frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -32.59 & 34.27 & -6 \end{bmatrix} x \tag{6.31}$$

*The trajectories of corresponding to this system are lossless, as well. Note that for the closed-loop system in equation* (6.31)*, we have* $\mathtt{det}(sE_{\mathtt{c}} - A_{\mathtt{c}}) = 32.59 - 37.59s + 9.83s^2$ *and as expected* $\mathtt{roots}\,(sE_{\mathtt{c}} - A_{\mathtt{c}}) = \left\{\sqrt{4+\sqrt{5}}, \sqrt{4-\sqrt{5}}\right\} = \Lambda_{\mathtt{max}}$.

Although the set of trajectories of a singularly passive SISO system can always be confined to its lossless trajectories using the PD controllers proposed in Theorem 6.16, not all lossless trajectories can be designated as optimal-charging and optimal-discharging trajectories. Recall that in Example 6.13 the lossless trajectories obtained involved multiplication of $\delta$ and its derivatives. Hence, in such a case the integrals in equation (6.1) and equation (6.2) are no longer well-defined. Since such a scenario involving products of $\delta$ and its derivatives arises when the initial conditions are *not* from the space of regular initial conditions, we need to consider initial conditions only from the space of regular initial conditions when we deal with optimal-charging and optimal-discharging of a singularly passive SISO system. Hence, if we are interested in designing controllers to confine the set of trajectories of a singularly passive SISO system to its optimal-charging and optimal-discharging trajectories, then we can do so using a static state-feedback control law. We present this as a result next.

Controllers for optimal-charging/optimal-discharging

**Theorem 6.19.** *Consider a singularly passive system $\Sigma$ of order $\mathtt{n_s}$ with the corresponding Hamiltonian matrix pair $(E,H)$ as defined in equation (5.11). Assume $F_{\mathrm{p}} \in \mathbb{R}^{1\times\mathtt{n}}$ to be as defined in equation (6.27), where $V_{3\Lambda}$ corresponds to a Lambda-set $\Lambda$ of $\mathtt{det}(sE-H)$. Suppose the closed-loop system obtained on application of the static state-feedback law $u = F_{\mathrm{p}}x$ be $\Sigma_{\mathtt{opt}}$. Let the initial/final condition of the system be from the subspace $x_0 \in \mathtt{img}\, V_{1\Lambda}$. Then, the following statements are true*

*(1) If $\Lambda \subsetneq \mathbb{C}_-$, then trajectories of $\Sigma_{\mathtt{opt}}$ are trajectories of optimal-discharging when the system is brought to rest from $x_0$.*

*(2) If $\Lambda \subsetneq \mathbb{C}_+$, then trajectories of $\Sigma_{\mathtt{opt}}$ are trajectories of optimal-charging when the system's states changes from zero to $x_0$.*

*Proof:* Application of the state-feedback $u = F_{\mathrm{p}}x$ to the system $\Sigma$ results in the closed-loop system $\frac{d}{dt}x = (A+bF_{\mathrm{p}})x$. The state-trajectories of the system therefore are $x(t) = e^{(A+bF_{\mathrm{p}})t}x_0$, where $x_0 := V_{1\Lambda}\beta$ for any $\beta \in \mathbb{R}^{\mathtt{n_s}\times 1}$. From equation (5.12) we know that $AV_{1\Lambda} + bV_{3\Lambda} = V_{1\Lambda}\Gamma$, where $\sigma(\Gamma) = \Lambda$. Using this and the fact that $F_{\mathrm{p}}V_{1\Lambda} = V_{3\Lambda}$, the input, output and state-trajectory of the closed-loop system corresponding to an initial/final condition $x_0 = V_{1\Lambda}\beta$ is as follows

$$\begin{aligned} x(t) = e^{(A+bF_{\mathrm{p}})t}V_{1\Lambda}\beta &= \left(I_{\mathtt{n}} + t(A+bF_{\mathrm{p}}) + \frac{t^2}{2!}(A+bF_{\mathrm{p}})^2 + \cdots\right)V_{1\Lambda}\beta \\ &= \left(I_{\mathtt{n}} + t(A+bF_{\mathrm{p}})V_{1\Lambda} + \frac{t^2}{2!}(A+bF_{\mathrm{p}})^2V_{1\Lambda} + \cdots\right)\beta \\ &= \left(I_{\mathtt{n}} + tV_{1\Lambda}\Gamma + \frac{t^2}{2!}V_{1\Lambda}\Gamma^2 + \cdots\right)\beta = V_{1\Lambda}e^{\Gamma t}\beta, \end{aligned} \tag{6.32}$$

$$u(t) = F_{\mathrm{p}}x(t) = F_{\mathrm{p}}V_{1\Lambda}e^{\Gamma t}\beta = V_{3\Lambda}e^{\Gamma t}\beta, \tag{6.33}$$

$$y(t) = cx(t) = cV_{1\Lambda}e^{\Gamma t}\beta. \tag{6.34}$$

Comparing equations (6.32), (6.33) and (6.34) with Table 6.1 it is evident that the trajectories of $\Sigma_{\mathtt{opt}}$ corresponding to initial/final condition $x_0 \in \mathtt{img}\, V_{1\Lambda}$ are lossless, i.e, $x(t) = \bar{x}_{\mathtt{s}}, u(t) = \bar{u}_{\mathtt{s}}$, and $y(t) = \bar{y}_{\mathtt{s}}$. Now we prove each statement of this theorem one-by-one.

*(1):* Let $K_{\mathtt{min}}$ be the minimal rank-minimizing solution of the KYP LMI (5.2). Corresponding to a Lambda-set $\Lambda \subsetneq \mathbb{C}_-$, the lossless trajectories $\mathtt{col}(\bar{x}_{\mathtt{s}}, \bar{u}_{\mathtt{s}}, \bar{y}_{\mathtt{s}})$ must therefore satisfy $\frac{d}{dt}(\bar{x}_{\mathtt{s}}^T K_{\mathtt{min}}\bar{x}_{\mathtt{s}}) = 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}}$ by Definition 6.3. Integrating both sides of this equation, we have

$$\int_0^\infty \frac{d}{dt}(\bar{x}_{\mathtt{s}}^T K_{\mathtt{min}}\bar{x}_{\mathtt{s}})dt = \int_0^\infty (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt \Rightarrow (\bar{x}_{\mathtt{s}}^T K_{\mathtt{min}}\bar{x}_{\mathtt{s}})|_{t=\infty} - (\bar{x}_{\mathtt{s}}^T K_{\mathtt{min}}\bar{x}_{\mathtt{s}})|_{t=0} = \int_0^\infty (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt \tag{6.35}$$

Since $\sigma(\Gamma) = \Lambda \subsetneq \mathbb{C}_-$, we must have

$$(\bar{x}_{\mathtt{s}}^T K_{\mathtt{min}}\bar{x}_{\mathtt{s}})|_{t=\infty} = (\beta^T e^{\Gamma^T t}{V_{1\Lambda}}^T K_{\mathtt{min}}V_{1\Lambda}e^{\Gamma t}\beta)|_{t=\infty} = 0. \tag{6.36}$$

Further, we also have

$$(\bar{x}_{\mathtt{s}}^T K_{\mathtt{min}} \bar{x}_{\mathtt{s}})|_{t=0} = (\beta^T e^{\Gamma^T t} V_{1\Lambda}{}^T K_{\mathtt{min}} V_{1\Lambda} e^{\Gamma t} \beta)|_{t=0} = \beta^T V_{1\Lambda}{}^T K_{\mathtt{min}} V_{1\Lambda} \beta = x_0^T K_{\mathtt{min}} x_0. \quad (6.37)$$

Therefore, using equation (6.36) and equation (6.37) in equation (6.35), we get

$$\int_0^{\infty} (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt = -(x_0^T K_{\mathtt{min}} x_0) \Rightarrow x_0^T K_{\mathtt{min}} x_0 = -\int_0^{\infty} (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt. \quad (6.38)$$

Comparing equation (6.38) with equation (6.2) it is evident that $\mathtt{col}(\bar{u},\bar{y})$ are the trajectories of optimal-discharging.

*(2):* Let $K_{\mathtt{max}}$ be the maximal rank-minimizing solution of the KYP LMI (5.2). Corresponding to a Lambda-set $\Lambda \subsetneq \mathbb{C}_+$, the lossless trajectories $\mathtt{col}(\bar{x}_{\mathtt{s}},\bar{u}_{\mathtt{s}},\bar{y}_{\mathtt{s}})$ must therefore satisfy $\frac{d}{dt}(\bar{x}_{\mathtt{s}}^T K_{\mathtt{max}} \bar{x}_{\mathtt{s}}) = 2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}}$ by Definition 6.3. Similar to the proof of Statement *(1)* of this theorem, we therefore have the following equation

$$\int_{-\infty}^{0} \frac{d}{dt}(\bar{x}_{\mathtt{s}}^T K_{\mathtt{max}} \bar{x}_{\mathtt{s}})dt = \int_{-\infty}^{0} (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt \Rightarrow (\bar{x}_{\mathtt{s}}^T K_{\mathtt{max}} \bar{x}_{\mathtt{s}})|_{t=0} - (\bar{x}_{\mathtt{s}}^T K_{\mathtt{max}} \bar{x}_{\mathtt{s}})|_{t=-\infty} = \int_{-\infty}^{0} (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt \quad (6.39)$$

Since $\sigma(\Gamma) = \Lambda \subsetneq \mathbb{C}_+$, we must have

$$(\bar{x}_{\mathtt{s}}^T K_{\mathtt{max}} \bar{x}_{\mathtt{s}})|_{t=-\infty} = (\beta^T e^{\Gamma^T t} V_{1\Lambda}{}^T K_{\mathtt{max}} V_{1\Lambda} e^{\Gamma t} \beta)|_{t=-\infty} = 0. \quad (6.40)$$

Further, we also have

$$(\bar{x}_{\mathtt{s}}^T K_{\mathtt{max}} \bar{x}_{\mathtt{s}})|_{t=0} = (\beta^T e^{\Gamma^T t} V_{1\Lambda}{}^T K_{\mathtt{max}} V_{1\Lambda} e^{\Gamma t} \beta)|_{t=0} = \beta^T V_{1\Lambda}{}^T K_{\mathtt{max}} V_{1\Lambda} \beta = x_0^T K_{\mathtt{max}} x_0. \quad (6.41)$$

Thus, using equation (6.40) and equation (6.41) in equation (6.39), we have

$$x_0^T K_{\mathtt{max}} x_0 = \int_0^{\infty} (2\bar{u}_{\mathtt{s}}\bar{y}_{\mathtt{s}})dt. \quad (6.42)$$

Comparing equation (6.42) with equation (6.1) it is evident that $\mathtt{col}(\bar{u},\bar{y})$ are the trajectories of optimal-charging. ■

From Theorem 6.19 it is clear that the closed-loop system obtained on application of the static state-feedback $u = F_{\mathrm{p}}x$ have trajectories of optimal-charging/optimal-discharging when the initial/final condition of a singularly passive SISO system is from the space of regular initial conditions. The space of regular initial condition is of dimension $\mathtt{n_s}$. Therefore, for singularly passive systems there always exist a $\mathtt{n} - \mathtt{n_s} = \mathtt{n_f}$ dimensional subspace complementary to the space of regular initial conditions where the system do not admit trajectories of optimal-charging/optimal-discharging. We call such complementary subspaces the *space of inadmissible initial conditions*. The subspace $\mathtt{img}\, W_1 \subsetneq \mathbb{R}^{\mathtt{n}}$ is one such subspace.

Further, if we compute $V_{3\Lambda}$ and $X_{1\Lambda}$ corresponding to a Lambda-set $\Lambda_{\mathtt{min}}$ of $\mathtt{det}(sE-H)$ such that $\Lambda_{\mathtt{min}} \subsetneq \mathbb{C}_-$ and design $F_{\mathrm{p}}$ using equation (6.27), then the closed-loop system $\Sigma_{\mathtt{opt}}$ obtained on application of the static state-feedback $u = F_{\mathrm{p}}x$ to a singularly passive SISO system $\Sigma$ admits the trajectories of optimal-discharging. Similarly, for the case when the Lambda-set $\Lambda_{\mathtt{max}}$ is such that $\Lambda \subsetneq \mathbb{C}_+$, the closed-loop system $\Sigma_{\mathtt{opt}}$ admits the trajectories of optimal-charging. We illustrate this with the system in Example 6.12 next.

**Example 6.20.** Optimal-discharging: *The state-feedback matrix, corresponding to Lambda-set $\Lambda_{\mathtt{min}}$, with $f_0 = 0$ is given by $F_{\mathrm{p}} = \begin{bmatrix} -1.21 & -0.64 & 0 \end{bmatrix}$ (see Example 6.18). The closed-loop system obtained on application of $u = F_{\mathrm{p}}x$ is given by*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7.21 & -11.64 & -6 \end{bmatrix} x. \tag{6.43}$$

*If the initial condition of the system is $x_{0\mathtt{s}} = \begin{bmatrix} -0.34 & -0.55 \\ 0.85 & 0.73 \\ -2.12 & -0.97 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \mathtt{img}\, V_{\Lambda_{\mathtt{min}}}$, where $\beta_1, \beta_2 \in \mathbb{R}$. Then, the state-trajectory of the system in equation* (6.43) *is*

$$x(t) = \begin{bmatrix} 1 & 1 & 1 \\ -1.33 & -2.17 & -2.50 \\ 1.76 & 4.73 & 6.24 \end{bmatrix} \begin{bmatrix} e^{-\sqrt{4-\sqrt{5}t}} & & \\ & e^{-2.17t} & \\ & & e^{-\sqrt{4+\sqrt{5}t}} \end{bmatrix} \begin{bmatrix} 0 & -0.55 \\ 0 & 0 \\ -0.34 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$
$$= \begin{bmatrix} -0.34e^{-\sqrt{4+\sqrt{5}t}} - 0.55e^{-\sqrt{4-\sqrt{5}t}} \\ 0.85e^{-\sqrt{4+\sqrt{5}t}} + 0.73e^{-\sqrt{4-\sqrt{5}t}} \\ -2.12e^{-\sqrt{4+\sqrt{5}t}} - 0.97e^{-\sqrt{4-\sqrt{5}t}} \end{bmatrix}.$$ *(This is a slow lossless trajectory: see Example 6.12)*

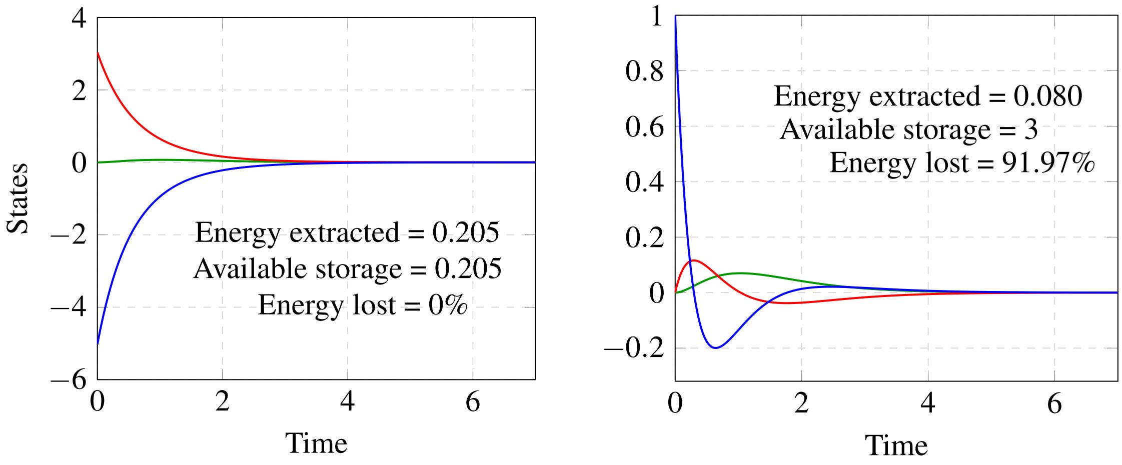
*This is the optimal-discharging trajectory when the system $\Sigma$ is discharged from initial condition $x_{0\mathtt{s}}$ to zero. Figure 6.3 show the state-trajectories and the percentage of energy extraction corresponding to the controller $F_{\mathrm{p}}$ when the initial conditions of the closed loop system in equation* (6.43) *are from the space of admissible and inadmissible initial conditions.*

Optimal-charging: *The state-feedback matrix, corresponding to Lambda-set $\Lambda_{\mathtt{max}}$, with $f_0 = 0$ is given by $F_{\mathrm{p}} = \begin{bmatrix} -26.59 & 45.27 & 0 \end{bmatrix}$ (see Example 6.18). The closed-loop system obtained on application of $u = F_{\mathrm{p}}x$ is given by*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -32.59 & 34.27 & -6 \end{bmatrix} x. \tag{6.44}$$

*Corresponding to a final condition $x_{0\mathtt{s}} = \begin{bmatrix} 0.46 & -0.54 \\ 1.16 & -0.71 \\ 2.89 & -0.95 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \mathtt{img}\, V_{\Lambda_{\mathtt{max}}}$, the trajectory of the system in equation* (6.44) *is given by*

$$\bar{x} = \begin{bmatrix} 0.46e^{\sqrt{4+\sqrt{5}t}}\beta_1 - 0.54e^{\sqrt{4-\sqrt{5}t}}\beta_2 \\ 1.16e^{\sqrt{4+\sqrt{5}t}}\beta_1 - 0.71e^{\sqrt{4-\sqrt{5}t}}\beta_2 \\ 2.89e^{\sqrt{4+\sqrt{5}t}}\beta_1 - 0.95e^{\sqrt{4-\sqrt{5}t}}\beta_2 \end{bmatrix}.$$




(a) $x_0 \in \mathtt{img}\, V_{1\Lambda}$: $\beta_1 = 1, \beta_2 = 3$ (b) $x_0 = \mathtt{col}(0,0,1) \notin \mathtt{img}\, V_{1\Lambda}$

Figure 6.3: Optimal discharging trajectories of the system with $F_{\mathrm{p}}$ corresponding to (a) an admissible initial condition and (b) an inadmissible initial condition.

*This is the optimal-charging state-trajectory when the system $\Sigma$ goes from rest to the final state.*

*Figure 6.3(a) shows the optimal discharging state-trajectories of the system corresponding to an admissible initial condition ($\beta_1 = 1, \beta_2 = 3$). Note that in such a case the entire available stored energy can be extracted out. On the other hand, when initial conditions are inadmissible, e.g. in case of Figure 6.3(b),* 91.97% *of the stored energy is lost.*

## 6.5 Summary

In this chapter, we showed that the storage functions of a singularly passive SISO system admit extremal storage functions (Theorem 6.7) and these extremal storage functions can be computed using a suitable choice of Lambda-set in Algorithm 5.16. To prove the existence of such extremal solutions we characterized the lossless trajectories of a singularly passive SISO system (Table 6.1). We showed that if the initial conditions of the system are from the space of regular initial conditions, then the lossless trajectories are exponential in nature (Lemma 6.1). On the other hand, the lossless trajectories are impulsive if the initial conditions are from the space of irregular initial condition (Lemma 6.2). Further, we also presented a method, similar to that in Chapter 3, to design a PD state-feedback control law that confines the set of trajectories of a singularly passive SISO system to its lossless trajectories (Theorem 6.16). Thus, in this chapter we showed that in case of a regularly or singularly passive system the rank-minimizing solutions of the corresponding KYP LMI can be used to confine the set of system trajectories to its lossless trajectories.

In the next section, we look into a special but familiar class of passive systems for which the rate of change of stored energy is always equal to the power supplied to the system. These are systems that do not admit any dissipation in energy and hence are aptly called *lossless* *systems*. In other words, all the trajectories of these systems are lossless. Hence, in terms of Definition 6.3, there must exist a solution of the KYP LMI for a lossless system such that equation (6.3) is satisfied for all the trajectories the system. However such a solution of the KYP LMI cannot be computed using Theorem 5.7, since a lossless system admits spectral zeros on the imaginary axis. Therefore, in the next section, we present different methods to compute the storage function, i.e. solution of the KYP LMI corresponding to a lossless system. These methods reveal interesting properties about lossless systems and their bounded-real counterparts allpass systems.

# Chapter 7

# Storage functions of lossless systems

## 7.1 Introduction

Lossless systems are a class of passive systems that is well-studied in the literature [VD89], [PW02], [RR08]. Traditionally, lossless systems have played a crucial role in classical electrical network theory. Examples of lossless electrical networks include networks composed of inductors, capacitors, transformers, and gyrators but no resistive elements. Such systems also find applications in digital signal processing [VD89], induction heating [RLC17], communication systems, etc. A typical example of a lossless system is the resonant circuit shown in Figure 7.1. The LC circuit in Figure 7.1 oscillates at its natural resonant frequency and stores energy. The stored energy in such a circuit oscillates back and forth between the capacitor and the inductor. Since such systems do not dissipate this energy in ideal conditions, such systems are called lossless. Hence, lossless systems are those passive systems for which the energy extracted from the system equals the energy supplied to the system. In other words, these are systems that do not dissipate energy. Therefore, lossless systems satisfy the dissipation inequality (5.5) with equality. Thus, a passive system with a minimal i/s/o representation

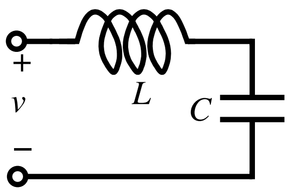



Figure 7.1: A resonant ciruit

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \tag{7.1}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$, is lossless if and only if there exists a matrix $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that for every $\mathtt{col}(x,u,y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}+2\mathtt{p}})$ that satisfies equation (7.1), we must have (see [Wil72])

$$\frac{d}{dt}\left(x^T K x\right) = 2u^T y \text{ for all } t \in \mathbb{R}. \tag{7.2}$$

Similar to regularly and singularly passive systems, it can be shown that $K$ must satisfy the KYP LMI, albeit with equality. Recall that $x^T K x$ (or $K$) is called the storage function of the system. Our primary objective in this chapter is to compute storage functions of lossless systems.

From studies in classical network analysis it is well-known that for a lossless system with a transfer function $G(s) \in \mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$, the corresponding Popov function is a zero function, i.e., $G(s)+G(-s)^T = 0$ (see [AV06, Theorem 2.7.4]). It therefore follows that for a lossless system the feed-through term $D$ in equation (7.1) must satisfy $D+D^T = 0$. Recall that $D+D^T$ is the feed-through regularity condition for the KYP LMI. This indicates that lossless systems do not admit an ARE. Thus, a natural question arises: can the method to compute storage function, proposed in Chapter 5, for singularly passive systems applied to lossless systems? Since $G(s)+G(-s)^T = 0$ for lossless systems, the entire $\mathbb{C}$-plane are the spectral zeros for such systems. This implies that lossless systems admit spectral zeros on the imaginary axis, as well. Therefore, at first glance it seems that the method to compute storage functions for singularly passive SISO systems (Theorem 5.7 and Algorithm 5.16) cannot be used to compute the storage functions of lossless systems. However, in Section 7.3 we show that Algorithm 5.16 can indeed be used to compute the storage functions of lossless systems, as well. Apart from Algorithm 5.16, we present a few other methods to compute the storage functions of lossless systems. These methods are developed using different salient features of a lossless system.

It is well-known that synthesis of lossless transfer functions result in LC networks. Traditionally, LC realizations of lossless transfer functions are non-unique; Foster 1 & 2, Cauer 1 & 2 and their combinations, for example. The values of the capacitances and inductances would be highly varied across these realizations, due to which, for a given amount of stored energy, the capacitor-voltages and inductor-currents would be different across the realizations. Further, for a given lossless transfer function there are many state-space realizations that need not correspond to an LC realization, this also adds to the non-uniqueness in the values of states for a given stored energy. In spite of this non-uniqueness, it is known that the energy stored, when expressed in terms of the external variables (port-variables) and their derivatives, is exactly unique and is independent of both the LC realization and the state-space realization. In other words, lossless systems admit unique storage functions ($x^T Kx$). For the design of an algorithm to compute storage function, this property can be exploited in the sense that the LC realization or state-space realization can be chosen in a form so that new methods (possibly with better numerical/flop-count properties) to compute the stored energy are revealed by the chosen realization. Apart from a method based on Algorithm 5.16, this chapter proposes four different approaches to characterize the stored energy; each approach unfolds new results and algorithms to compute the storage function. The four concepts on which the main results of this chapter are based on are as follows:

1. States and costates of a lossless system. (Section 7.4)

2. Partial fraction method: Foster/Cauer and their combinations, (Section 7.5 and Section 7.6)

3. Bezoutian of two polynomials using Euclidean long division, Pseudo-inverse/Left-inverse, and Two dimensional discrete Fourier transform. (Section 7.7 and Section 7.8)

4. Controllability/Observability Gramians. (Section 7.9)

In order to develop the results in this chapter we need a few preliminaries that we present next.

## 7.2 Preliminaries

Note that in terms of the solution of the KYP LMI, it is evident that a passive system with a minimal i/s/o representation as in equation (7.1) is lossless if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that the following linear matrix equality (LME) is satisfied:

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} A^T K + KA = 0 \\ KB - C^T = 0. \end{cases} \tag{7.3}$$

Hence, our primary objective is to find algorithms to compute $K$ that satisfies equation (7.3).

### 7.2.1 Minimal Polynomial Basis

The *degree* of a polynomial vector $r(s) \in \mathbb{R}[s]^{\mathtt{n}}$ is defined as the maximum degree amongst the $\mathtt{n}$ components of the vector. Degree of the zero polynomial and the zero vector $\mathbb{R}[s]^{\mathtt{n}}$ is defined as $-\infty$.

Consider the polynomial matrix $R(s) \in \mathbb{R}[s]^{\mathtt{n}\times\mathtt{m}}$ of normal rank $\mathtt{n}$. Let the polynomial matrix $P(s) \in \mathbb{R}[s]^{\mathtt{m}\times(\mathtt{n}-\mathtt{m})}$ be such that $R(s)P(s) = 0$ and $\mathtt{nrank}(P(s)) = \mathtt{n}-\mathtt{m}$. Then, the columns of $P(s)$ are said to form a basis of the nullspace of $R(s)$. Suppose the columns of $P(s)$ are $\{p_1(s), p_2(s), \ldots, p_{\mathtt{m}-\mathtt{n}}(s)\}$ ordered with degrees $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{\mathtt{m}-\mathtt{n}}$. The set $\{p_1(s), p_2(s), \ldots, p_{\mathtt{m}-\mathtt{n}}(s)\}$ is said to be a *minimal polynomial basis* of $R(s)$ if every other nullspace basis $\{q_1(s), q_2(s), \ldots, q_{\mathtt{m}-\mathtt{n}}(s)\}$ with degree $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_{\mathtt{m}-\mathtt{n}}$ is such that $d_i \leqslant c_i$, for $i = 1, 2, \ldots, \mathtt{m}-\mathtt{n}$. The degrees of the vectors of minimal polynomial basis of $R(s)$ are called the *Forney invariant minimal indices* or *Kronecker indices* (more details in [GF75], [Kai80, Section 6.5.4]).

### 7.2.2 Hamiltonian systems corresponding to MIMO lossless systems

Corresponding to a MIMO lossless system with a minimal i/s/o representation of the form given in equation (7.1), the Hamiltonian system is given by:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{p}} \end{bmatrix}}_{E} \frac{d}{dt} \begin{bmatrix} x \\ z \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & 0 \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u \end{bmatrix}. \tag{7.4}$$

Recall that the components of $x$ and $z$ are the states and costates of the system, respectively. Analogous to Chapter 5, we call the matrix pair $(E, H)$, the Hamiltonian matrix pair and the system in equation (7.4) is called a Hamiltonian system. The matrix pencil $(sE - H)$ is called the Hamiltonian pencil. We use the symbol $\Sigma_{\text{Ham}}$ to represent such a system. The output-nulling representation of the system $\Sigma_{\text{Ham}}$ is

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \widehat{A}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{B}u, \quad 0 = \widehat{C}\begin{bmatrix} x \\ z \end{bmatrix}, \tag{7.5}$$

where $\widehat{A} := \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$, $\widehat{B} := \begin{bmatrix} B \\ C^T \end{bmatrix}$ and $\widehat{C} := \begin{bmatrix} C & -B^T \end{bmatrix}$. It is important to note that for lossless systems the Hamiltonian pencil $(sE - H)$ is singular. We prove this in the next lemma.

Lossless systems admit a singular Hamiltonian pencil

**Lemma 7.1.** *Consider a lossless system $\Sigma$ with a minimal i/s/o representation as given in equation* (7.1)*. Let the corresponding Hamiltonian matrix pair be $(E, H)$ as defined in equation* (7.4)*. Then,* $\texttt{det}(sE - H) = 0$.

*Proof:* On evaluating $\texttt{det}(sE - H)$ using Schur-complement and using the fact that for lossless systems $G(s) + G(-s)^T = 0$, it follows that

$$\begin{aligned}
\texttt{det}(sE - H) &= \texttt{det}\left(\begin{array}{cc|c} sI_{\mathtt{n}} - A & 0 & -B \\ 0 & sI_{\mathtt{n}} + A^T & -C^T \\ \hline -C & B^T & 0 \end{array}\right) \\
&= \texttt{det}\left(\begin{bmatrix} -C & B^T \end{bmatrix}\begin{bmatrix} sI_{\mathtt{n}} - A & 0 \\ 0 & sI_{\mathtt{n}} + A^T \end{bmatrix}^{-1}\begin{bmatrix} -B \\ -C^T \end{bmatrix}\right) \times \texttt{det}(sI_{\mathtt{n}} - A) \times \texttt{det}(sI_{\mathtt{n}} + A^T) \\
&= \texttt{det}\left\{-C(sI_{\mathtt{n}} - A)^{-1}B + B^T(sI_{\mathtt{n}} + A^T)^{-1}C^T\right\} \times \texttt{det}(sI_{\mathtt{n}} - A) \times \texttt{det}(sI_{\mathtt{n}} + A^T) \\
&= \texttt{det}\left\{-G(s) - G(-s)^T\right\} \times \texttt{det}(sI_{\mathtt{n}} - A) \times \texttt{det}(sI_{\mathtt{n}} + A^T) = 0.
\end{aligned}$$

This completes the proof of the lemma. ■

### 7.2.3 Quotient ring and Gröbner basis

Consider a commutative ring $\mathfrak{R}$ (with multiplicative identity 1) and an ideal $\mathbb{I} \subseteq \mathfrak{R}$. We define an equivalence relation over $\mathfrak{R}$ such that two elements $p_1, p_2 \in \mathfrak{R}$ are related if $p_1 - p_2 \in \mathbb{I}$. The set of all equivalence classes originating from this equivalence relation is denoted by $\mathfrak{R}/\mathbb{I}$. It is well-known that the set $\mathfrak{R}/\mathbb{I}$ has a ring structure with addition and multiplication operations inherited from $\mathfrak{R}$. This ring is known in the literature as the *quotient ring*. The equivalence class of an element $p \in \mathfrak{R}$ is represented as $[p]$. In this chapter, we deal with the two-variable polynomial ring $\mathbb{C}[x_1, x_2]$. Let $\mathbb{I} \subseteq \mathbb{C}[x_1, x_2]$ be the ideal generated by a given set of polynomials $\{f_1, f_2, \ldots, f_{\mathtt{t}}\} \in \mathbb{C}[x_1, x_2]$. Given a polynomial $p \in \mathbb{C}[x_1, x_2]$ we want to uniquely represent it in the quotient ring $\mathbb{C}[x_1, x_2]/\mathbb{I}$. This is a standard problem in commutative algebra and Gröbner basis helps here: see [CLO92, Chapter 2].

Given an ideal $\mathbb{I} \subseteq \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$, a Gröbner basis is a special set of polynomials, which generate $\mathbb{I}$ and possesses useful properties for computational analysis. The first step in finding a Gröbner basis of an ideal is to fix the ordering of the monomials in the polynomial ring. Such an ordering is called *term ordering*[1]. In this chapter, we use lexicographic term ordering on the set of monomials of $\mathbb{C}[x,y]$. Note that, although Gröbner basis is not always unique, however, for this chapter and the application dealt with here the Gröbner basis of the ideal is "reduced" and hence unique. For example, with respect to lexicographic ordering, the reduced and unique Gröbner basis for the ideal $\langle x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1 \rangle \subsetneq \mathbb{C}[x,y]$ is given by $\{x^{\mathtt{N}} - 1, y^{\mathtt{N}} - 1\}$: for details on reduced Gröbner basis refer to [CLO92, Chapter 2]. For the rest of this chapter, by "the Gröbner basis" we mean the reduced and unique Gröbner basis. Next we state a property of Gröbner basis that we use in this chapter.

**Proposition 7.2.** [CLO92, Section 2.6] *Let $G = \{g_1, g_2, \ldots, g_{\mathtt{t}}\}$ be the Gröbner basis for an ideal $\mathbb{I} \subseteq \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ with respect to a term ordering and let $p \in \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$. Then there exist polynomials $q_1, q_2, \ldots, q_{\mathtt{t}}, r \in \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ such that $p = q_1 g_1 + q_2 g_2 + \cdots + q_{\mathtt{t}} g_{\mathtt{t}} + r$ where $r$ is the remainder with respect to $G$ and leading*[2] *monomial (LM) of $r \prec LM(g_i)$ for every $i = 1, 2, \ldots, \mathtt{t}$. Moreover, $\underline{r \text{ is unique}}$ and independent of the order of division.*

Thus, given an ideal $\mathbb{I} \in \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ and given the Gröbner basis, $G = \{g_1, g_2, \ldots, g_{\mathtt{t}}\}$, the map $\Pi : \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}] \rightarrow \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]/\mathbb{I}$ maps any element $p \in \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ to its unique remainder $r$ obtained by multivariate division of $p$ by $G$ (irrespective of the order of division). The equivalence class $[p]$ can therefore be represented by the remainder $r \in \mathbb{C}[x_1, x_2, \cdots, x_{\mathtt{n}}]$. It is important to note that $\Pi$ is a ring homomorphism.

A well-known result that is used in Section 7.7.3 is the Hilbert's Nullstellensatz. We present this result as a proposition for ease of reference. Before we present the proposition we need to define the radical of an ideal first.

**Definition 7.3.** [CLO92, Definition 2] *The radical of $\mathbb{J}$, denoted by $\sqrt{\mathbb{J}}$, is the set $\{g \in \mathbb{F}[x_1, \ldots, x_2] : g^m \in \mathbb{J}$ for some $m \geqslant 1\}$. Further, an ideal $\mathbb{J}$ is said to be a* radical ideal *if $\sqrt{\mathbb{J}} = \mathbb{J}$.*

Next we present Hilbert's Nullstellensatz as a proposition.

**Proposition 7.4.** [CLO92, Theorem 2] *If $\mathbb{F}$ is an algebraically closed field and $\mathbb{J}$ is an ideal in $\mathbb{F}[x_1, x_2, \ldots, x_{\mathtt{n}}]$, then $\mathscr{I}(\mathbb{V}(\mathbb{J})) = \sqrt{\mathbb{J}}$.*

An ideal $\mathbb{J} \subsetneq \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ is a *zero-dimensional* ideal if $\mathbb{V}(\mathbb{J}) \subsetneq \mathbb{C}^{\mathtt{n}}$ is a finite set: see [CLO92, Chapter 2, Finiteness Theorem] for details.

[1] A *term ordering* on $\mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ is any relation on the set of monomials $x^\alpha, \alpha \in \mathscr{Z}^{\mathtt{n}}_{\geqslant 0}$ satisfying:
(i) $\succ$ is a total ordering on $\mathscr{Z}^{\mathtt{n}}_{\geqslant 0}$. (ii) If $\alpha \succ \beta$ and $\gamma \in \mathscr{Z}^{\mathtt{n}}_{\geqslant 0}$, then $\alpha + \gamma \succ \beta + \gamma$. (iii) $\succ$ is a well-ordering on $\mathscr{Z}^{\mathtt{n}}_{\geqslant 0}$ i.e. every nonempty subset of $\mathscr{Z}^{\mathtt{n}}_{\geqslant 0}$ has a smallest element under $\succ$.

There are different types of term ordering viz. lexicographic, graded lexicographic, graded reverse lexicographic, etc. : see [CLO92, Chapter 2] for a detailed exposition.

[2] Let $f = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x_1, x_2, \ldots, x_{\mathtt{n}}]$ and let $\succ$ be a term ordering. Then the leading monomial of $f$ is $LM(f) := x^{d(f)}$ (with coefficient 1), where the multidegree $d(f)$ of $f$ is defined as $d(f) := \max(\alpha \in \mathscr{Z}^{\mathtt{n}}_{\succeq 0} : a_\alpha \neq 0)$: see [CLO92, Chapter 2].

### 7.2.4 Two dimensional-discrete Fourier transform (2D-DFT)

This section contains a quick review of 2D-DFT (see [Ciz86] for more). The 2D-DFT of a matrix $U = [u_{pq}]_{p,q=1,2,\dots,\mathtt{N}-1} \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is represented as $\mathscr{F}(U) = [f_{mn}]_{m,n=0,1,\dots,\mathtt{N}-1} \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ and $f_{mn}$ is defined by

$$f_{mn} := \sum_{p=0}^{\mathtt{N}-1} \left( \sum_{q=0}^{\mathtt{N}-1} u_{pq}\omega^{qm} \right) \omega^{pn}, \quad \text{where } \omega = e^{-j\frac{2\pi}{\mathtt{N}}}. \tag{7.6}$$

A matrix representation of $\mathscr{F}(U)$ is given by

$$\mathscr{F}(U) = \Omega_{\mathtt{N}}^T U \Omega_{\mathtt{N}}, \text{ where } \Omega_{\mathtt{N}} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{\mathtt{N}-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(\mathtt{N}-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{\mathtt{N}-1} & \omega^{2(\mathtt{N}-1)} & \cdots & \omega^{(\mathtt{N}-1)^2} \end{bmatrix} \tag{7.7}$$

The elements of the matrix $\mathscr{F}(U)$ in equation (7.7) can also be computed using polynomials. Such a polynomial interpretation of 2D-DFT is essential for the results in this section. In order to compute the 2D-DFT of $U$, we need to first construct a two-variable polynomial $g(x,y) := \mathbf{X}^T U \mathbf{Y}$, where $\mathbf{X} := \mathtt{col}\left(1, x, x^2, \dots, x^{\mathtt{N}-1}\right)$ and $\mathbf{Y} := \mathtt{col}\left(1, y, y^2, \dots, y^{\mathtt{N}-1}\right)$. Then, the elements of the matrix $\mathscr{F}(U)$ are given by $f_{mn} = g(\omega^m, \omega^n)$, where $m,n = 0,1,\dots,\mathtt{N}-1$. Note that the inverse 2D-DFT is given by

$$u_{pq} = \frac{1}{\mathtt{N}^2} \left\{ \sum_{m=0}^{\mathtt{N}-1} \left( \sum_{n=0}^{\mathtt{N}-1} f_{mn}\omega^{-mq} \right) \omega^{-np} \right\}, \quad \text{where } \omega = e^{-j\frac{2\pi}{\mathtt{N}}}. \tag{7.8}$$

### 7.2.5 Bounded-real and allpass systems

Analogous to passive systems, a system $\Sigma$ with a minimal i/s/o representation as in equation (7.1) is *bounded-real* if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $K$ satisfies the following LMI:

$$\begin{bmatrix} A^T K + KA + C^T C & KB + C^T D \\ B^T K + D^T C & -(I - D^T D) \end{bmatrix} \leqslant 0. \tag{7.9}$$

We call LMI (7.9) the *bounded-real LMI*. The bounded-real LMI originates from a more fundamental law known as the dissipation inequality: a system with minimal i/s/o representation (7.1) is bounded-real if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that for every $\mathtt{col}(x,u,y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}+2\mathtt{p}})$ that satisfies equation (7.1), we must have

$$\frac{d}{dt}\left(x^T K x\right) \leqslant u^T u - y^T y \text{ for all } t \in \mathbb{R}. \tag{7.10}$$

It can be shown that $K$ satisfies the dissipation inequality (7.10) if and only if $K$ is a solution of the bounded-real LMI (7.9). Analogous to passive systems, we call $K$ to be a storage function of the bounded-real system.

It is well-known in the literature that a system $\Sigma_{\mathtt{br}}$ with input-output variables $(u,y)$ is bounded-real if and only if a system $\Sigma_{\mathtt{pas}}$ with input-output variables $\left(\frac{u+y}{\sqrt{2}},\frac{u-y}{\sqrt{2}}\right)$ is passive: see [HC08, Chapter 5]. Therefore, in this chapter, we call $\Sigma_{\mathtt{pas}}$ to be the *passive counterpart* of $\Sigma_{\mathtt{br}}$ and $\Sigma_{\mathtt{br}}$ to be the *bounded real counterpart* of $\Sigma_{\mathtt{pas}}$. Further, if an i/s/o representation of $\Sigma_{\mathtt{br}}$ is

$$\frac{d}{dt}x = Ax+Bu, y = Cx+Du, \text{ where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}, \text{ and } D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}, \tag{7.11}$$

then an i/s/o representation of the corresponding passive system $\Sigma_{\mathtt{pas}}$ is

$$\begin{aligned} \frac{d}{dt}x &= \left(A - B(I+D)^{-1}C\right)x + \frac{1}{\sqrt{2}}\left(B + B(I+D)^{-1}(I-D)\right)v, \\ r &= -\sqrt{2}(I+D)^{-1}Cx + (I+D)^{-1}(I-D)v, \text{ where } v := \frac{u+y}{\sqrt{2}}, r := \frac{u-y}{\sqrt{2}}. \end{aligned} \tag{7.12}$$

Similarly, if equation (7.11) is a minimal i/s/o representation of a passive system, equation (7.12) is the minimal i/s/o representation of its bounded-real counterpart.

It is known that bounded-real systems admit a special class of systems called the allpass systems. A bounded-real system is called allpass if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$\frac{d}{dt}\left(x^T K x\right) = u^T u - y^T y, \text{ for all } \mathtt{col}(x,u,y) \text{ that satisfy the i/s/o equations of the system.}$$

Hence, an allpass system satisfies the bounded-real LMI (7.9) with equality, i.e., for an allpass system there always exists a unique $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$\begin{bmatrix} A^T K + KA + C^T C & KB + C^T \\ B^T K + C & 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} A^T K + KA + C^T C = 0, \\ KB + C^T = 0. \end{cases} \tag{7.13}$$

We review a property of allpass systems next that we crucially used in Section 7.9. Since the property directly follows from a result presented in [Glo84, Theorem 5.1], we present it as a proposition next.

**Proposition 7.5.** [Glo84, Theorem 5.1] *Consider an allpass system with a minimal i/s/o representation as in equation* (7.1). *Let $P$ and $Q$ be its controllability and observability Gramian, respectively. Then, $PQ = I_{\mathtt{n}}$.*

### 7.2.6 Gramian and balancing

Consider a stable, controllable, and observable system with a minimal i/s/o representation as in equation (7.1). Then the Lyapunov equations $AP + PA^T + BB^T = 0$ and $A^T Q + QA + C^T C = 0$ have unique solutions $P > 0$ and $Q > 0$, respectively: see [Ant05, Section 4.3]. $P$ and $Q$ are called the (infinite) controllability and observability Gramian matrices, respectively.

For one of the main results of this chapter we also need the concept of balancing and we review this next. A system is said to be represented in a *balanced state-space basis* if the controllability Gramian $P$ and observability Gramian $Q$ are equal. The proposition next gives a procedure to compute the balancing transformation of a system.

**Proposition 7.6.** [Ant05, Lemma 7.3] *Consider a controllable, observable and stable system with a minimal i/s/o representation as in equation* (7.1). *Let the corresponding controllability Gramian and observability Gramian be $P$ and $Q$, respectively. Assume $P := UU^*$ and $U^*QU = KS^2K^*$ then a balancing transformation is given by $T = \sqrt{S}\ K^*U^{-1}$ (see footnote 3 for definition*[3] *of $\sqrt{S}$).*

Now that we have reviewed the preliminaries required for the main results in this chapter, we proceed to state the main results in the next section.

## 7.3 Controllability matrix method

At the very outset, we present the method to compute the storage function of a lossless system using Algorithm 5.16. Note that this method is already known in the literature [AV06, Section 6.5]. However we present this method here to demonstrate that as a special case of Algorithm 5.16, we can retrieve a classic and well-known algorithm for lossless systems. In order to get to the main result of this section, we need to first present a property of the Markov parameters of the Hamiltonian system corresponding to a lossless system. This is an adaptation of Lemma 5.14 in Chapter 5 to lossless systems.

Markov parameters of $\Sigma_{\text{Ham}}$ corresponding to a lossless system are all zero

**Lemma 7.7.** *Consider a lossless SISO system $\Sigma$ with a minimal i/s/o representation as in equation* (5.9). *Let the corresponding Hamiltonian system be as defined in equation* (5.11). *Then, $\widehat{c}\widehat{A}^k\widehat{b} = 0$ for all $k \in \mathbb{N}$.*

*Proof:* Recall from Lemma 5.13 that $G(s) + G(-s)^T = \widehat{c}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{b}$. We know that for a lossless system $G(s)+G(-s)^T = 0$, therefore we must have

$$\widehat{c}(sI_{2\mathrm{n}} - \widehat{A})^{-1}\widehat{b} = 0 \Rightarrow \widehat{c}\widehat{A}^k\widehat{b} = 0 \text{ for all } k \in \mathbb{N}.$$

This completes the proof of the lemma. ■

Now we present the main result of this section.

---

[3] A matrix $R = R^T \geqslant 0$ is said to be the square root of another matrix $S = S^T \geqslant 0$ if $R^2 = S$. We denote such a matrix as $\sqrt{S} := R$.

Storage function computation using controllability matrix

**Theorem 7.8.** *Consider a lossless SISO system $\Sigma$ with a minimal i/s/o representation as given in equation* (5.9). *Let $\widehat{A},\widehat{b},\widehat{c}$ be as defined in equation* (5.11). *Define $W := \begin{bmatrix}\widehat{b} & \widehat{A}\widehat{b} & \cdots & \widehat{A}^{\mathrm{n}-1}\widehat{b}\end{bmatrix} \in \mathbb{R}^{2\mathrm{n}\times\mathrm{n}}$. Partition $W = \begin{bmatrix}W_1\\ W_2\end{bmatrix}$, where $W_1, W_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Then, the following statements are true:*

*(1) $W_1$ is invertible.*

*(2) $K = W_2W_1^{-1}$ is symmetric.*

*(3) $K$ is the unique solution to the corresponding KYP LME* (7.3).

*Proof: (1):* Note that on simple multiplication it can be verified that

$$W_1 = \begin{bmatrix}b & Ab & \cdots & A^{\mathrm{n}-1}b\end{bmatrix}, \text{ and } W_2 = \begin{bmatrix}c^T & -(cA)^T & \cdots & (-1)^{\mathrm{n}-1}(cA^{\mathrm{n}-1})^T\end{bmatrix}.$$

Since the system $\Sigma_{\mathrm{Ham}}$ is minimal, i.e., controllable, it is evident that $W_1$ is invertible.
*(2):* This directly follows from the proof of Statement *(2)* of Theorem 5.7.
*(3):* Note that $W_2W_1^{-1}b = \begin{bmatrix}c^T & -(cA)^T & \cdots & (-1)^{\mathrm{n}-1}(cA^{\mathrm{n}-1})^T\end{bmatrix} e_1 = c^T$. Thus, $Kb - c^T = 0$. Now we show that $A^TK + KA \leqslant 0$. Similar to the proof of Statement *(3)* of Theorem 5.7 instead of proving $A^TK + KA \leqslant 0$, we prove $W_1^T(A^TK + KA)W_1 \leqslant 0$. From equation (5.41), we know that $W_1^T(A^TK + KA)W_1 =: [\ell_{ki}]_{k,i\in\{1,2,\ldots,\mathrm{n_f}\}}$, where $\ell_{ki} = (-1)^{k-1}\widehat{c}\widehat{A}^{k+i-1}\widehat{b}$. Using Lemma 7.7 we therefore have $W_1^T(A^TK + KA)W_1 = 0 \Rightarrow (A^TK + KA) = 0$.

Now we prove the uniqueness of $K$. Let us assume that there exists another solution $K_1$ of the KYP LME (7.3). Then, we have $K_1b = c^T$ and $Kb = c^T$. Subtracting these two equations, we have $(K_1 - K)b = 0_{\mathrm{n},1}$. Further, $A^TK_1 + K_1A = 0$ and $A^TK + KA = 0$. Subtracting these two equations, we have

$$A^T(K_1 - K) + (K_1 - K)A = 0_{\mathrm{n},\mathrm{n}} \tag{7.14}$$

Post-multiplying equation (7.14) with $b$ and using the fact that $(K_1 - K)b = 0_{\mathrm{n},1}$, we have $(K_1 - K)Ab = 0_{\mathrm{n},1}$. Proceeding in a similar way, we can show that

$$(K_1 - K)\begin{bmatrix}b & Ab & \cdots & A^{\mathrm{n}-1}b\end{bmatrix} = 0_{\mathrm{n},\mathrm{n}} \Rightarrow (K_1 - K)W_1 = 0_{\mathrm{n},\mathrm{n}}. \tag{7.15}$$

Since $W_1$ is invertible, it is evident from equation (7.15) that $K_1 - K = 0_{\mathrm{n},\mathrm{n}} \Rightarrow K_1 = K$. ■

From Theorem 7.8 we infer that Algorithm 5.16 can be used to compute the storage function of a lossless system. Note that the matrix $W_1$ here is the controllability matrix of the lossless system. Since this method uses the controllability matrix of the system to compute the storage function of the system, we call this method the controllability matrix method. Interestingly, this method is already known in the literature and an alternate proof to the same can be found in [AV06, Section 6.5].

**Example 7.9.** *Consider a lossless system with a transfer function* $G(s) = \dfrac{8s^2+1}{6s^3+s}$ *and an i/s/o representation:*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{6} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \frac{1}{6}\begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}^T u \tag{7.16}$$

*Here* $W_1 = \dfrac{1}{6}\begin{bmatrix} 0 & 0 & 6 \\ 0 & 6 & 0 \\ 6 & 0 & -1 \end{bmatrix}$ *and* $W_2 = \dfrac{1}{18}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 24 & 0 & -1 \end{bmatrix}$. *Then,* $K = W_2W_1^{-1} = \frac{1}{36}\begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & 0 \\ 6 & 0 & 48 \end{bmatrix}$.
*It can be easily verified that* $A^TK + KA = 0$ *and* $Kb = c^T$.

We adapt Algorithm 5.16 for lossless systems and present it next.

**Algorithm 7.10** Controllability matrix method to compute storage function of a lossless system.

**Input:** $(A,b,c)$ matrices corresponding to a lossless SISO system $\Sigma_{\mathtt{loss}}$.
**Output:** $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

1: Construct $\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$, $\widehat{b} = \begin{bmatrix} b \\ c^T \end{bmatrix}$ and $\widehat{c} = \begin{bmatrix} c & -b^T \end{bmatrix}$.
2: Construct $W := \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \widehat{A}^2\widehat{b} & \cdots & \widehat{A}^{\mathtt{n}-1}\widehat{b} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$.
3: Partition $X$ as $X =: \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix}$ where $X_{1\Lambda}, X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.
4: Compute the storage function: $K = X_{2\Lambda}X_{1\Lambda}^{-1} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

In what follows we provide four alternate methods to compute the storage function of a lossless system. The first among these is a method based on the algebraic relations between the states and costates of a lossless system.

## 7.4 Minimal polynomial basis (MPB) method

In this section we present a method to compute the storage function of a lossless system using the notion of minimal polynomial basis. This method is developed using certain algebraic relations between the states and costates of a lossless system. In what follows, we show that these algebraic relations provide a method to compute the unique storage function (unique solution of the KYP LME (7.18)) of a lossless system.

Storage function computation using algebraic-relations between states and costates

**Theorem 7.11.** *Consider a lossless system $\Sigma$ with a minimal i/s/o representation $\frac{d}{dt}x = Ax+Bu$, $y = Cx$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$. Let the corresponding Hamiltonian system $\Sigma_{\text{Ham}}$ be as defined in equation* (7.4). *Then, the following statements are true:*

*(1)* $\Sigma_{\text{Ham}}$ *is not autonomous, i.e.,* $\mathtt{det}(sE-H) = 0$.

*(2) there exists a unique* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *such that*

$$\frac{d}{dt}\left(x^T Kx\right) = 2u^T y \text{ for all } \mathtt{col}(x,u,Cx) \in \Sigma. \tag{7.17}$$

*(3) there exists a unique* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *such that*

$$\mathtt{nrank}\begin{bmatrix} sI_{\mathtt{n}}-A & 0 & -B \\ 0 & sI_{\mathtt{n}}+A^T & -C^T \\ -C & B^T & 0 \end{bmatrix} = \mathtt{nrank}\begin{bmatrix} sI_{\mathtt{n}}-A & 0 & -B \\ 0 & sI_{\mathtt{n}}+A^T & -C^T \\ -C & B^T & 0 \\ -K & I_{\mathtt{n}} & 0 \end{bmatrix}. \tag{7.18}$$

*Further, for* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$,

$$K \text{ satisfies equation (7.17)} \quad \text{if and only if} \quad K \text{ satisfies equation (7.18)}. \tag{7.19}$$

In order to prove Theorem 7.11, we need a result that we present next. This result states that the difference dynamics $x(t)-Kz(t)$ of a lossless system is orthogonal to the subspace $\mathtt{img}\, B$ for all $t>0$ if and only if it is orthogonal to the controllable subspace spanned by columns of $\begin{bmatrix} B & AB & \cdots & A^{\mathtt{n}-1}B \end{bmatrix}$ for all $t>0$. We use the symbol $\perp$ to denote orthogonality between subspaces.

Orthogonality between controllable subspace and difference dynamics $x(t)-Kz(t)$

**Lemma 7.12.** *Consider a controllable, lossless system $\Sigma$. An i/s/o representation of $\Sigma$ is $\frac{d}{dt}x = Ax+Bu$, and $y = Cx$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$. Let the corresponding Hamiltonian system be as defined in equation* (7.4). *Let $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be a solution of the LME* (7.3). *Then the difference dynamics $z(t)-Kx(t)$ satisfies the following*

$$\Big(z(t)-Kx(t)\Big) \perp \mathtt{img}\, B \Longleftrightarrow \Big(z(t)-Kx(t)\Big) \perp \mathtt{img}\begin{bmatrix} B & AB & \cdots & A^{\mathtt{n}-1}B \end{bmatrix} \text{ for all } t>0.$$

*Proof:* Let the controllability matrix be $\mathscr{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{\mathtt{n}-1}B \end{bmatrix}$. All the arguments here are true for each $t>0$.
($\Leftarrow$) Given $\mathscr{C}^T(z-Kx) = 0$, i.e., $B^T(z-Kx) = 0$. Hence, $z(t)-Kx(t) \perp \mathtt{img}\, B$ for all $t>0$.
($\Rightarrow$) We use the principle of mathematical induction to prove that $(A^kB)^T(z-Kx) = 0$ for $k \in \{0,1,\ldots,\mathtt{n}-1\}$.

*Base step:* (k = 0) Since $K$ is a solution of the LME (7.3), we have $KB = C^T$. Therefore, using equation (7.4), $B^T(z - Kx) = B^T z - Cx = 0$.
*Induction step:* Assume $(A^k B)^T(z - Kx) = 0$ for $k < \mathtt{n} - 1$. Then, we prove that $(A^{k+1}B)^T(z - Kx) = 0$. Note that

$$B^T(A^T)^{k+1}(z - Kx) = B^T(A^T)^k(A^T z - A^T Kx) \tag{7.20}$$

From equation (7.4), we know that $\dot{z} = -A^T z + C^T u \Rightarrow A^T z = -\dot{z} + C^T u$ and $\dot{x} = Ax + Bu \Rightarrow Ax = \dot{x} - Bu$. Using these in equation (7.20) and the fact that $A^T K + KA = 0$, we have

$$B^T(A^T)^{k+1}(z - Kx) = B^T(A^T)^k(A^T z + KAx) = B^T(A^T)^k\left(-\dot{z} + C^T u + K(\dot{x} - Bu)\right) \tag{7.21}$$

Since $KB - C^T = 0$, we therefore infer from equation (7.21) that

$$B^T(A^T)^{k+1}(z - Kx) = B^T(A^T)^k(K\dot{x} - \dot{z}) = \frac{d}{dt}(A^k B)^T(Kx - z). \tag{7.22}$$

Using the induction hypothesis, we infer from equation (7.22) that $B^T(A^T)^{k+1}(z - Kx) = 0$. ■

Using Lemma 7.12, we prove Theorem 7.11 next.
*Proof of Theorem 7.11: (1):* From Lemma 7.1, we have $\mathtt{det}(sE - H) = 0$. Using [PW98, Section 3.2], we know that a system is autonomous if and only $\mathtt{det}(sE - H) \neq 0$. Therefore, $\Sigma_{\text{Ham}}$ is non-autonomous.
*(2):* Since $\Sigma$ is a lossless system, it is passive. Hence, there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ that satisfies equation (7.17). We prove uniqueness of $K$ next. Assume $K_1 = K_1^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $K_2 = K_2^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ induces the storage function of a lossless system. Since the storage function of a lossless system is unique ([WT98, Remark 5.13]), $x^T(t)K_1x(t) = x^T(t)K_2x(t)$ for all $t \in \mathbb{R}$. This is true if and only if $K_1 = K_2$. This proves the uniqueness of $K$.
*(3):* Using equation (7.4), we have

$$\frac{d}{dt}(x^T z) = (Ax + Bu)^T z + x^T(-A^T z + C^T u) = u^T B^T z + u^T Cx = 2u^T Cx = 2u^T y \tag{7.23}$$

Using equation (7.17) in equation (7.23) and expanding, we get

$$\frac{d}{dt}(x^T z) = \frac{d}{dt}(x^T Kx) \Rightarrow \dot{x}^T z + x^T \dot{z} - \dot{x}^T Kx - x^T K\dot{x} = 0. \tag{7.24}$$

Using equation (7.4) and the LME (7.3) in equation (7.24), we have

$$\begin{aligned}(Ax + Bu)^T z + x^T(-A^T z + C^T u) - (Ax + Bu)^T Kx - x^T K(Ax + Bu) = 0\\ \Rightarrow u^T B^T z - x^T(A^T K + KA)x - x^T(KB - C^T)u - x^T KBu = 0\\ \Rightarrow u^T B^T z - x^T KBu = 0 \Rightarrow u^T B^T(z - Kx) = 0.\end{aligned} \tag{7.25}$$

Since equation (7.25) is true for all system trajectories $\mathtt{col}(x, u, y) \in \Sigma$, $B^T(z - Kx) = 0$. Using Lemma 7.12, we have $z - Kx \in \mathtt{ker}\,\mathscr{C}^T$. However $(A, B)$ is a controllable system with minimal state representation, hence $z - Kx = 0$ is true for *all* trajectories in $\Sigma$. Thus,

$$z - Kx = 0 \Rightarrow \begin{bmatrix} -K & I_{\mathtt{n}} & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix} = 0$$

Therefore equation (7.18) follows. Next we prove equation (7.19).
*Only if:* This follows from the proof of Statement *(3)* of this theorem.
*If:* Note that equation (7.18) means the Hamiltonian system $\Sigma_{\text{Ham}}$ has trajectories $\texttt{col}(x,z,u)$ that satisfy $z = Kx$ for all $t > 0$. Further, from [WT98, Section 10], it is clear that the states and costates of a system satisfy $\frac{d}{dt}\left(x^T z\right) = 2u^T y$. In this equation, replacing $z$ by $Kx$, we therefore have $\frac{d}{dt}\left(x^T Kx\right) = 2u^T y$ for all $t > 0$. ■

Equation (7.19) implies that for a lossless system with storage function $K$ the Hamiltonian pencil satisfies the following rank condition:

$$\texttt{rank}\begin{bmatrix} \lambda I_{\text{n}} - A & 0 & -B \\ 0 & \lambda I_{\text{n}} + A^T & -C^T \\ -C & B^T & 0 \end{bmatrix} = \texttt{rank}\begin{bmatrix} \lambda I_{\text{n}} - A & 0 & -B \\ 0 & \lambda I_{\text{n}} + A^T & -C^T \\ -C & B^T & 0 \\ -K & I_{\text{n}} & 0 \end{bmatrix}, \text{ for all } \lambda \in \mathbb{C}. \tag{7.26}$$

However, consider a passive system $\Sigma$ that is *not* lossless. Let $K$ be a rank-minimizing solution of the KYP LMI corresponding to a Lambda-set $\Lambda$ of the Hamiltonian pencil. Then, we have

$$\texttt{rank}\begin{bmatrix} \lambda I_{\text{n}} - A & 0 & -B \\ 0 & \lambda I_{\text{n}} + A^T & -C^T \\ -C & B^T & 0 \end{bmatrix} \leqslant \texttt{rank}\begin{bmatrix} \lambda I_{\text{n}} - A & 0 & -B \\ 0 & \lambda I_{\text{n}} + A^T & -C^T \\ -C & B^T & 0 \\ -K & I_{\text{n}} & 0 \end{bmatrix}, \text{ for } \lambda \in \mathbb{C}. \tag{7.27}$$

The equality in equation (7.27) is achieved only at those complex numbers that are elements of $\Lambda$. We illustrate this with an example next.

**Example 7.13.** *Consider a passive system $\Sigma$ (not lossless) with transfer function $G(s) = \dfrac{s+2}{s+1}$. A minimal i/s/o representation of the system is $\dot{x} = -x + u$ and $y = x + u$. Note that the feedthrough regularity condition is satisfied here and hence, Proposition 5.4 can be used to compute the storage functions of such a system. The Hamiltonian pencil for this system is*

$$R(s) := (sE - H) = \begin{bmatrix} s+1 & 0 & -1 \\ 0 & s-1 & -1 \\ -1 & 1 & -2 \end{bmatrix}$$

*Here $\texttt{det}(R(s)) = 4 - 2s^2 \neq 0$ and therefore, the Lambda-sets possible are $\Lambda_1 = \{-\sqrt{2}\}$ and $\Lambda_2 = \{\sqrt{2}\}$. For $\Lambda_1$, the storage function is $K_{\Lambda_1} = 3 - 2\sqrt{2}$. Note that*

$$\texttt{rank}\begin{bmatrix} -\sqrt{2}+1 & 0 & -1 \\ 0 & -\sqrt{2}-1 & -1 \\ -1 & 1 & -2 \end{bmatrix} = 2 \quad \textit{and} \quad \texttt{rank}\begin{bmatrix} -\sqrt{2}+1 & 0 & -1 \\ 0 & -\sqrt{2}-1 & -1 \\ -1 & 1 & -2 \\ -(3-2\sqrt{2}) & 1 & 0 \end{bmatrix} = 2.$$

*Consider any other arbitrary value of $K$ which is not a solution to the ARE corresponding to the system $\Sigma$. Say $K = 1$ then*

$$\texttt{rank} \begin{bmatrix} -\sqrt{2}+1 & 0 & -1 \\ 0 & -\sqrt{2}-1 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & 0 \end{bmatrix} = 3.$$

*Hence for any other arbitrary value of $K$,* $\texttt{rank} \begin{bmatrix} & \lambda E - H & \\ -K & I & 0 \end{bmatrix} \neq \texttt{rank}(\lambda E - H)$.

We use Theorem 7.11 to propose a theorem next that leads to the algorithm to compute the storage function of a lossless system.

MPB based method to compute storage function of a lossless system

**Theorem 7.14.** *Consider the Hamiltonian matrix pair $(E, H)$ as defined in equation* (7.4). *Let $M(s) \in \mathbb{R}[s]^{(2\mathtt{n}+\mathtt{p})\times \mathtt{p}}$ be any minimal polynomial nullspace basis (MPB) for the Hamiltonian pencil $(sE - H)$. Partition $M = \begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}$ with $M_1 \in \mathbb{R}[s]^{2\mathtt{n}\times \mathtt{p}}$. Let $N(s)$ be any MPB for $M_1(s)^T$. Then, the following statements are true.*

1. *Each of the first* $\mathtt{n}$ *Forney invariant minimal indices of $N(s)$ are* 0*, i.e., first* $\mathtt{n}$ *columns of $N(s)$ are* constant *vectors.*

2. *Partition $N(s)$ into $\begin{bmatrix} N_1 & N_2(s) \end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathtt{n}\times \mathtt{n}}$ and further partition $N_1 = \begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix}$ with $N_{12} \in \mathbb{R}^{\mathtt{n}\times \mathtt{n}}$. Then $N_{12}$ is invertible and $K := -N_{11}N_{12}^{-1}$ is the storage function for $\Sigma$.*

*Proof: (1):* We first prove that the first $\mathtt{n}$ minimal indices of the Hamiltonian pencil $sE - H$ are 0. Since $\texttt{nrank}(sE - H) = 2\mathtt{n}$ there exists $M(s) \in \mathbb{R}[s]^{(2\mathtt{n}+\mathtt{p})\times \mathtt{p}}$ with $\texttt{nrank}(M(s)) = \mathtt{p}$ such that $R(s)M(s) = 0$. From Theorem 7.11, we know that $\begin{bmatrix} -K & I_\mathtt{n} & 0 \end{bmatrix}$ is in the row span of $sE - H$. Therefore,

$$\begin{bmatrix} -K & I_\mathtt{n} & 0 \end{bmatrix} M(s) = 0 \Rightarrow \begin{bmatrix} -K & I_\mathtt{n} & 0 \end{bmatrix} \begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix} = 0 \Rightarrow M_1(s)^T \begin{bmatrix} -K \\ I_\mathtt{n} \end{bmatrix} = 0. \qquad (7.28)$$

The nullspace of $M_1(s)^T$ must have $\mathtt{n}$ constant polynomial vectors. Hence the first $\mathtt{n}$ (Forney invariant) minimal indices are 0.

*(2):* A minimal polynomial nullspace basis of $M_1(s)^T$ is the columns of $N(s) \in \mathbb{R}[s]^{2\mathtt{n}\times(2\mathtt{n}-\mathtt{p})}$. Partition $N$ into $\begin{bmatrix} N_1 & N_2(s) \end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathtt{n}\times \mathtt{n}}$ and further partition $N_1 = \begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix}$ with $N_{12} \in$

$\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. From equation (7.28), we infer that

$$\mathtt{img}\begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix} = \mathtt{img}\begin{bmatrix} -K \\ I \end{bmatrix} \Rightarrow K = -N_{11}N_{12}^{-1}.$$

The construction of $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ in the proof is done such a way that $\begin{bmatrix} -K & I & 0 \end{bmatrix}$ is in the row span of $sE - H$. Therefore, from Theorem 7.11, $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ satisfies equation (7.17) and hence $K$ induces the storage function of $\Sigma$. ■

Using the system in Example 7.9 we illustrate the method to compute the storage function of a lossless system using Theorem 7.14.

**Example 7.15.** *The MPB corresponding to the Hamiltonian pencil $(sE - H)$ of the system given in Example 7.9 is*

$$[\underbrace{36 \quad 36s \quad 36s^2 \quad 1+6s^2 \quad 2s \quad 6+48s^2}_{M_1(s)^T} \quad 6s+36s^3]^T$$

*By Theorem 7.14 the first $\mathtt{n} = 3$ columns of the minimal polynomial basis of the $M_1(s)^T$ have Forney indices 0. The first 3 columns of the minimal polynomial basis of $M_1(s)^T$ are*

$$\left[\begin{array}{rrr} -0.0189 & 0.0025 & -0.0987 \\ -0.0002 & -0.0554 & -0.0013 \\ -0.0960 & 0.0195 & -0.7921 \\ \hline 0.9938 & -0.0017 & -0.0470 \\ 0.0028 & 0.9981 & 0.0243 \\ -0.0522 & -0.0144 & 0.6000 \end{array}\right] =: \left[\begin{array}{c} N_{11} \\ \hline N_{12} \end{array}\right]$$

*Therefore,* $K = -N_{11}N_{12}^{-1} = \frac{1}{36}\begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & 0 \\ 6 & 0 & 48 \end{bmatrix}$ *is the storage function of the system.*

Next we present the algorithm to compute the storage function of a lossless system using the theory developed in Theorem 7.11 and Theorem 7.14.

**Algorithm 7.16** Minimal polynomial basis algorithm.

**Input:** $R(s) := sE - H \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}$.

**Output:** $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with $x^T Kx$ the storage function.

1: Compute a minimal polynomial nullspace basis of $R(s)$. Result: A full column rank polynomial matrix $M(s) \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times\mathtt{p}}$.

2: Partition $M(s)$ as $\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}$ where $M_1(s) \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$.

3: Compute a minimal polynomial nullspace basis of $M_1(s)^T$. Result: A full column rank polynomial matrix $N(s) \in \mathbb{R}[\xi]^{2\mathtt{n}\times(2\mathtt{n}-\mathtt{p})}$.

4: Partition $N(s) = \begin{bmatrix} N_{11} & N_{12}(s) \\ N_{21} & N_{22}(s) \end{bmatrix}$ with $N_{11}, N_{21} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. (See Theorem 7.14)

5: Define $K := -N_{11}N_{21}^{-1} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Algorithm 7.16 is based on computation of nullspace basis of polynomial matrices. Efficient and stable computation of nullspace basis of a polynomial matrix can be done by block Toeplitz matrix algorithm: more details can be found in [KPB10].

Both the controllability matrix method and the MPB method described above do not depend on the basis in which the system matrices are represented. However, in what follows, we present methods to compute the storage function of a lossless system using special basis to represent the system matrices.

## 7.5 Partial fraction method: SISO case

In this section we use partial fraction/continued fraction expansion of the transfer function of a lossless system to compute the storage function of the system. The capacitor voltages and inductor currents in the electrical network corresponding to the system's transfer function are taken as the states while computing the storage function in this section. In other words, this method is based on viewing the lossless transfer function $G(s)$ as the driving point impedance or driving point admittance of an *LC* network. Since the system is lossless, the poles and zeros of the system are all on the imaginary axis. Expansion of the proper transfer function $G(s)$ into its partial fractions using the Foster form gives

$$G(s) = \frac{r_0}{s} + \sum_{q=1}^{\mathtt{m}} \frac{r_q s}{s^2+\omega_q^2} \tag{7.29}$$

where $r_0 \geqslant 0, r_1, r_2, \ldots, r_{\mathtt{m}} > 0$ and each $\omega_q > 0$. For a system with proper transfer function $G(s)$ as in equation (7.29), a minimal i/s/o representation

$$\frac{d}{dt}x_f = A_f x_f + B_f u_f \qquad \text{and} \qquad y_f = C_f x_f \tag{7.30}$$

is given by

$$A_f := \mathtt{diag}(A_0, A_1, \cdots, A_{\mathtt{m}}) \text{ where } A_0 := 0, A_q := \begin{bmatrix} 0 & -r_q \\ \dfrac{\omega_q^2}{r_q} & 0 \end{bmatrix} \text{ with } q \in \{1,2,\cdots,\mathtt{m}\}, \text{ and}$$

$$B_f := \mathtt{col}\,(r_0, r_1, 0, r_2, 0, \ldots, r_{\mathtt{m}}, 0) \in \mathbb{R}^{2\mathtt{m}+1}, C_f := \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\mathtt{m}+1}.$$

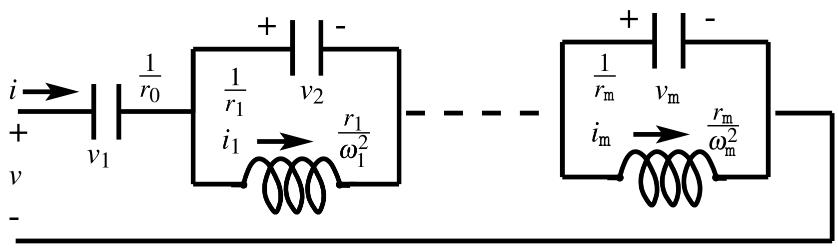



Figure 7.2: LC realization based on partial fractions: Foster-I form

On the other hand, expansion of a proper transfer function $G(s) = g_q(s)$ in continued fraction using Cauer-II methods involves the following iteration:

$$g_q(s) = \frac{h_q}{s} + \frac{1}{g_{q+1}(s)}, \quad g_{\mathtt{n}}(s) := \frac{h_{\mathtt{n}}}{s} \tag{7.31}$$

where $q = 1, 2, \cdots, \mathtt{n}$ and $\mathtt{n}$ is the McMillan degree[4] of the system. For the sake of simplicity, we assume that the McMillan degree $\mathtt{n}$ of the system is odd. Consider the vectors $V := \begin{bmatrix} v_1 & v_2 & \cdots & v_{\mathtt{m}} \end{bmatrix}^T$, $I := \begin{bmatrix} i_1 & i_2 & \cdots & i_{\mathtt{m}-1} \end{bmatrix}^T$ and $B_2 := \begin{bmatrix} h_1 & h_3 & \cdots & h_{\mathtt{n}} \end{bmatrix}^T \in \mathbb{R}^{\mathtt{m}}$ where $\mathtt{m} := \frac{\mathtt{n}+1}{2}$. For $p = 1, 2, \cdots, \mathtt{m}-1$, define $H^u, H^l \in \mathbb{R}^{(\mathtt{m}-1)\times(\mathtt{m}-1)}$ such that

$$H^u_{pj} := \begin{cases} 0 & \text{for } p > j \\ h_{2p} & \text{for } p \leqslant j \end{cases} \quad H^\ell_{pj} := \begin{cases} h_{2p+1} & \text{for } p \geqslant j \\ 0 & \text{for } p < j \end{cases}$$

A minimal representation of the system $G(s)$,

$$\dot{x}_c = A_c x_c + B_c u_c \qquad \text{and} \qquad y_c = C_c x_c \tag{7.32}$$

is given by the following matrices:

$$A_c := \begin{bmatrix} 0 & & H^u \\ & 0 & \\ -H^\ell & & \end{bmatrix}, \; B := \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \text{ and } C := \begin{bmatrix} 0 \\ 1_{\mathtt{m}} \end{bmatrix}^T \text{ where } x_c := \begin{bmatrix} I \\ V \end{bmatrix}.$$

The physical realization of transfer function in equation (7.29) in an LC network can be done using the Foster method (as shown in Figure 7.2) and the Cauer method (as in Figure 7.3). Using equation (7.30) and equation (7.32) we present a method to compute the storage function of a lossless system.

[4]The McMillan degree of a linear time-invariant system is the order of any minimal state-space realization of the system.

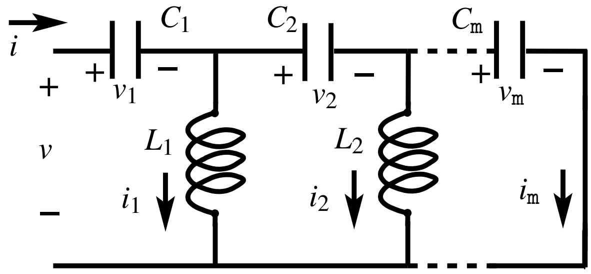



Figure 7.3: LC realization based on continued fractions: Cauer-II form

Foster & Cauer method to compute the storage function of a lossless system

**Theorem 7.17.** *Consider a controllable, lossless system with a strictly proper transfer function $G(s)$ of the form given in equation (7.29) and equation (7.31). Assume the McMillan degree of $G(s)$ is odd. For each of the two cases ((a) & (b) below corresponding to Foster and Cauer realizations), consider the state-space realizations in which the states are the capacitor voltages and inductor currents. Then, the stored energy is*

$$x^T K x = \sum_{L_j} L_j i_j^2 + \sum_{C_q} C_q v_q^2.$$

*More precisely,*

*(a) For systems with proper impedance function as in equation (7.29) and a minimal i/s/o representation as in (7.30), the unique storage function is $x_f^T K_f x_f$, with the diagonal matrix $K_f \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ defined by*

$$K_f := \mathtt{diag}\left(\frac{1}{r_0}, K_1, K_2, \ldots, K_{\mathtt{m}}\right), \text{ where } K_q := \begin{bmatrix} r_q^{-1} & 0 \\ 0 & r_i\omega_i^{-2} \end{bmatrix} \text{ with } q \in \{1, 2, \cdots, \mathtt{m}\}.$$

*(b) For systems with proper admittance function as in equation (7.31) and a minimal i/s/o representation as in equation (7.32), the unique storage function is $x_c^T K_c x_c$, with the diagonal matrix $K_c \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ defined by*

$$K_c := \mathtt{diag}\left(\frac{1}{h_2}, \frac{1}{h_4}, \cdots, \frac{1}{h_{\mathtt{n}-1}}, \frac{1}{h_1}, \frac{1}{h_3}, \cdots, \frac{1}{h_{\mathtt{n}}}\right).$$

*Proof:* *(a)* Note that $\omega_q \neq \omega_j$ for $q \neq j$ guarantees controllability and observability[5] of the system. We prove the theorem for $\mathtt{n} = 5$, the general case follows from it. The transfer function in partial fraction form is

$$G(s) = \frac{r_0}{s} + \frac{r_1 s}{s^2 + \omega_1^2} + \frac{r_2 s}{s^2 + \omega_2^2}.$$

[5]It can be verified that the controllability matrix $[B_f \; A_f B_f \; \cdots \; A_f^{n-1} B_f]$ and observability matrix $[C_f^T \; A_f^T C_f^T \; \cdots \; A_f^{n-1^T} C_f^T]$ has full rank if and only if $\omega_q \neq \omega_j$ for $q \neq j$.

Hence, by simple inspection we can infer that

$$A = \mathtt{diag}\left(0, \begin{bmatrix} 0 & -r_1 \\ \frac{r_1}{\omega_1^2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -r_2 \\ \frac{r_2}{\omega_2^2} & 0 \end{bmatrix}\right), B = \begin{bmatrix} r_0 & r_1 & 0 & r_2 & 0 \end{bmatrix}^T, C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Consider $K_f = K_f^T = [k_{xy}]_{x,y=0,1,\cdots,4} \in \mathbb{R}^{5\times 5}$. Since $\omega_1 \neq \omega_2$ and $K_f$ satisfies $A_f^T K_f + K_f A_f = 0$, we have $K_f = \mathtt{diag}(k_{00}, k_{11}, \cdots, k_{44})$, $\frac{k_{22}}{k_{11}} = \frac{r_1}{r_2}$, and $\frac{k_{44}}{k_{33}} = \frac{r_3}{r_4}$. Further, on use of the equation $B_f^T K_f - C_f = 0$ we get a unique $K_f$ of the form $K_f = \mathtt{diag}\left(\frac{1}{r_0}, \frac{1}{r_1}, \frac{r_1}{\omega_1^2}, \frac{1}{r_2}, \frac{r_2}{\omega_2^2}\right)$. This completes the proof Statement *(a)* of Theorem 7.17.
*(b)* We give a brief outline of the proof here. We show it for a system with McMillan degree $\mathtt{n} = 5$. The proof for the general case follows from it. Using equation (7.32), we have

$$A_c = \begin{bmatrix} & & & h_2 & h_2 \\ & & & 0 & h_4 \\ & & 0 & & \\ -h_3 & 0 & & & \\ -h_5 & -h_5 & & & \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ h_1 \\ h_3 \\ h_5 \end{bmatrix}, C_c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T$$

Solving the matrix equations $A_c^T K_c + K_c A_c = 0$ and $B_c^T K_c - C_c = 0$, we get the unique diagonal matrix $K_c := \mathtt{diag}\left(\frac{1}{h_2}, \frac{1}{h_4}, \frac{1}{h_1}, \frac{1}{h_3}, \frac{1}{h_5}\right)$. Hence $K_c$ induces the storage function $x_c^T K_c x_c$ of the system. This completes the proof Statement b) of Theorem 7.17. ∎

Note that for systems with even McMillan degree the term $\frac{r_0}{s}$ and $\frac{h_1}{s}$ in equation (7.30) and equation (7.32) will not be present. Therefore, in Theorem 7.17 while computing the storage function we have to drop the terms $\frac{1}{r_0}$ and $\frac{1}{h_1}$ from the expressions for $K_f$ and $K_c$, respectively. We revisit the example in Example 7.9 to demonstrate Theorem 7.17 next.

**Example 7.18.** *For the lossless system $\Sigma$ with transfer function $G(s) = \dfrac{8s^2+1}{6s^3+s}$ in Example 7.9, by partial fraction expansion we get*

$$G(s) = \frac{1}{s} + \frac{s/3}{s^2+1/6} \Rightarrow r_0 = 1, r_1 = \frac{1}{3}, \textit{ and } \omega_1^2 = \frac{1}{6}.$$

*The LC circuit corresponding to the Foster realization therefore is as given in Figure 7.4.*

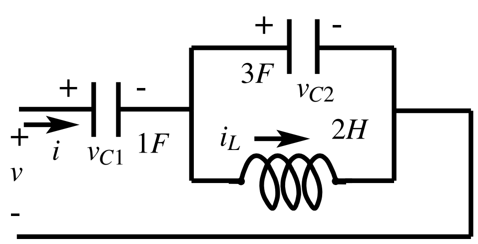



Figure 7.4: LC realization of transfer function $G(s)$ in Example 7.18

*Foster realization (Theorem 7.17 (a)): the state-vector is* $x = \mathtt{col}(v_{C1}, v_{C2}, i_L)$. *Here*

$$A_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, B_f = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{3} \end{bmatrix}, C_f = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T, K_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

*By continued fraction expansion, we have the following*

$$G(s) = \frac{1}{s} + \frac{1}{(1/2s) + \frac{1}{1/3s}} \Rightarrow h_1 = 1, h_2 = \frac{1}{2}, \text{ and } h_3 = \frac{1}{3}.$$

*The LC circuit corresponding to the Cauer realization is also as shown in Figure 7.4.*
*Cauer realization (Theorem 7.17 (b)): the state-vector is* $x = \mathtt{col}(i_L, v_{C1}, v_{C2})$. *Here*

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}, C_c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^T, K_c = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Based on Theorem 7.17, we present an algorithm to compute the storage function of a lossless system referred to as the 'Partial fraction' algorithm next. We do not explicitly write down the algorithm for the Cauer form, since it is almost similar to the one presented below.

**Algorithm 7.19** Partial fraction based algorithm - SISO.

**Input:** Strictly proper transfer function of the lossless system $G(s)$.
**Output:** $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with $x^T Kx$ the storage function.

1: Calculate the partial fraction expansion:

$$G(s) = \frac{r_0}{s} + \sum_{i=1}^{\mathtt{m}} G_i(s), \text{ where } G_i(s) = \frac{r_i s}{s^2 + \omega_i^2}, \text{ and } \omega_i > 0.$$

2: For each $G_i(s)$, obtain $(A_i, B_i, C_i)$ triple, where $A_i \in \mathbb{R}^{2\times 2}$, $B_i \in R^2$ and $C_i \in R^{1\times 2}$ using equation (7.30).
3: Obtain $K_i$ from each triple $(A_i, B_i, C_i)$ using Theorem 7.17.
4: The storage function is given by

$$K := \mathtt{diag}\left(\frac{1}{r_0}, K_1, K_2, \ldots, K_{\mathtt{m}}\right) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

## 7.6 Partial fraction method: MIMO case

In this section we generalize the SISO result based on Foster LC realization to MIMO lossless systems. Gilbert's realization is a well-known method to find the i/s/o representation of MIMO systems [Kai80, Section 6.1]. However, such a method does not guarantee an i/s/o representation with the inductor currents and capacitor voltages as the states in an LC realization. We need such a form of $A$ since the proposed method uses energy stored in inductor and capacitor as the storage function. In this section we present a method to find the i/s/o representation of a lossless system such that the inductor current and capacitor voltage are the states of the system. We then proceed to compute the storage function matrix $K$ with respect to these states.

### 7.6.1 Gilbert's realization adapted to lossless systems

Before we present the main results of the section, we revisit Gilbert's theorem as given in [Gil63, Theorem 7]. This proposition gives a method to compute the McMillan degree of a linear time-invariant MIMO system.

**Proposition 7.20.** [Gil63, Theorem 7] *Consider a proper rational transfer-function matrix $G(s)$ whose elements have semi-simple poles at $s=\lambda_i$, $i=1,2,\ldots,\mathtt{q}$. Let the partial fraction expansion of $G(s)$ be*

$$G(s)=\sum_{i=1}^{\mathtt{q}}\frac{R_i}{s-\lambda_i}+D,\ \textit{where}\ R_i=\lim_{s\to\lambda_i}(s-\lambda_i)G(s)\ \textit{and}\ D=\lim_{s\to\infty}G(s).$$

*Corresponding to each $\lambda_i$, let the rank of the $R_i$ matrix be $\mathtt{r}_i$. Then McMillan degree corresponding to the system is given by $\mathtt{n}=\sum_{i=1}^{\mathtt{q}}\mathtt{r}_i$.*

For systems with imaginary axis poles, Proposition 7.20 is adapted and presented as Lemma 7.22 next. However, before that we review necessary and sufficient conditions for a real rational matrix to be positive real and/or lossless.

**Proposition 7.21.** [AV06, Section 2.7] *A transfer matrix $G(s)\in\mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$ is positive real if and only if*

1. *Each element of $G(s)$ is analytic in the open right half s-plane.*
2. *$G(j\omega)+G(-j\omega)^T\geqslant 0$ for all $\omega\in\mathbb{R}$ such that $j\omega$ is not a pole of any element of $G(s)$.*

*Further, a positive real rational transfer matrix $G(s)$ is lossless if and only if $G(s)+G(-s)^T=0$.*

Let $G(s)=[g_{ij}]$ and poles of $g_{ij}$ be represented as $\mathscr{P}(g_{ij})$. Using Proposition 7.21, we conclude that a necessary condition for a positive real transfer matrix to be lossless is $\mathscr{P}(g_{ij})_{i\neq j}\subseteq\mathscr{P}(g_{ij})_{i=j}$ i.e. the poles of the off-diagonal entries of $G(s)$ are also poles of the diagonal entries. This means that if the poles corresponding to the diagonal entries of the transfer function $G(s)$ are not poles of the off-diagonal entries of $G(s)$ then the residue matrix corresponding to such poles will be diagonal. Therefore, a more generalized case would be when the poles of the off-diagonal and diagonal entries of $G(s)$ are the same: thus ensuring non-diagonal residue matrix. Hence, we deal with such systems only.

Next we adapt Proposition 7.20 for the case of imaginary axis poles and present it as a lemma. We will use the lemma to construct the minimal i/s/o representation of a MIMO lossless system.

McMillan degree of a lossless system

**Lemma 7.22.** *Consider a rational transfer matrix $G(s)$ whose elements have simple poles at $s = 0$ and/or $s = \pm j\omega_i$, $i = 1, 2, \cdots \mathtt{q}$. Let the partial expansion of $G(s)$ be*

$$\frac{R_0}{s} + \sum_{\ell=1}^{\mathtt{q}} \frac{R_\ell(s)}{s^2 + \omega_\ell^2} + D = \frac{R_0}{s} + \sum_{\ell=1}^{\mathtt{q}} \left( \frac{Z_\ell}{s + j\omega_\ell} + \frac{Z_\ell^*}{s - j\omega_\ell} \right) + D,$$

*where $R_0 = \lim_{s\to 0} sG(s) \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}, D = \lim_{s\to\infty} G(s) \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$ are the residual matrices and $R_\ell(s)$ is the residue matrix corresponding to the conjugate pair of poles on the imaginary axis. Let $\mathtt{r}_0 = \mathtt{rank}(R_0)$ and $\mathtt{r}_\ell = \mathtt{rank}(Z_\ell)$. Then, the minimal order of the i/s/o representation of the system is*

$$\mathtt{n} = \mathtt{r}_0 + \sum_{\ell=1}^{\mathtt{q}} 2 \times \mathtt{r}_\ell.$$

*Proof:* Note that rank$(Z_\ell)$ = rank$(Z_\ell^*)$ = $\mathtt{r}_\ell$. Hence using Proposition 7.20, minimum number of states for the system is

$$\mathtt{n} = \mathtt{r}_0 + \sum_{\ell=1}^{\mathtt{q}} \operatorname{rank}(Z_\ell) + \sum_{\ell=1}^{\mathtt{q}} \operatorname{rank}(Z_\ell^*) = \mathtt{r}_0 + \sum_{\ell=1}^{\mathtt{q}} 2 \times \mathtt{r}_\ell.$$

This completes the proof of the lemma. ■

From Proposition 7.21, we know that for a lossless system $G(s) = -G(-s)^T$. Hence the partial fractions corresponding to each of the poles $\omega_\ell$ have a skew symmetric structure. Consider $G_\ell = [g_{iv}^\ell]$. The general structure[6] of $G_\ell(s)$ is given by

$$g_{iv}^\ell(s) = \frac{\alpha_{iv}^\ell s - \beta_{iv}^\ell}{s^2 + \omega_i^2}, \text{ where } \beta_{iv}^\ell = -\beta_{vi}^\ell. \tag{7.33}$$

We state and prove a theorem next which gives a procedure for construction of the $(A, B, C)$ matrices for lossless systems. For simplicity, we consider that the transfer matrix has only one pair of conjugate poles on the imaginary axis, i.e., we consider $\mathtt{q} = 1$ in $G(s) = \sum_{\ell=1}^{\mathtt{q}} G_\ell(s)$. In the next theorem we present a method to compute the storage function of a MIMO lossless system for $\mathtt{q} = 1$ case. For the general case, i.e., $\mathtt{q} > 1$ we just have to apply the theorem described below on each partial fraction individually.

[6]In general the elements of the transfer matrix $G(s)$ may not be proper. However, there always exists an input-output partition such that the transfer matrix is proper [WT98, Section 2]. Hence without loss of generality, we assume the transfer matrix to be proper.

Partial fraction method to compute the storage function of a lossless system

**Theorem 7.23.** *Consider a lossless transfer matrix* $G(s) = \frac{R(s)}{s^2+\omega^2} = \frac{Z}{s+j\omega} + \frac{Z^*}{s-j\omega}$*, where* $R(s) = sX - Y \in \mathbb{R}[s]^{\mathtt{p}\times\mathtt{p}}$ *and* $Z, X,$ *and* $Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$*. Elements of* $G(s)$ *are of the form given in equation* (7.33)*. Assume* $\mathtt{rank}(Z) = \mathtt{r}$*. The i-th rows of* $R(s)$*,* $X$ *and* $Y$ *are represented as* $z_i$*,* $x_i$ *and* $y_i$ *respectively. Then, the following state-space realization is minimal.*

1. $A := \mathtt{diag}(J_\omega, J_\omega, \ldots, J_\omega) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$*, where* $\mathtt{n} := 2\times\mathtt{r}$ *and* $J_\omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$.
2. *Construct* $B_i := \mathtt{col}(x_i, \frac{y_i}{\omega})$ *for each* $i = 1, 2, \ldots, \mathtt{p}$ *such that* $B_i \neq 0$ *and define* $B :=$ col $(B_1, B_2, \cdots, B_{\mathtt{r}}) \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$.
3. *There exist* $c_1, c_2, \cdots, c_i \in \mathbb{R}^{1\times\mathtt{n}}$ *such that* $C :=$ col $(e_1, e_3, \cdots, e_{2\mathtt{r}-1}, c_1, c_2, \cdots c_i) \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$ *where* $c_i \in$ *span of* $\{e_1, e_2, \cdots, e_{2\mathtt{r}-1}\}$ *and* $e_i$ *is the i-th row of* $I_{\mathtt{n}}$.

*Proof:* Proof for the general case follows from the proofs of the following two special cases: 1. $G(s)$ is nonsingular and $\mathtt{p} = 2$. 2. $G(s)$ is singular and $\mathtt{p} = 3$.

*$G(s)$ is nonsingular and* $\mathtt{p} = 2$*:* Using equation (7.33), we have

$$R(s) = s\begin{bmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{12} & \alpha_2 \end{bmatrix} - \begin{bmatrix} 0 & \beta_{12} \\ -\beta_{12} & 0 \end{bmatrix} = sX - Y. = \frac{R(s)}{s^2+\omega^2}.$$

Since $G(s)$ is nonsingular, $\mathtt{r} = 2$ and $\mathtt{n} = 4$ (By Lemma 7.22). Here

$$A = \mathtt{diag}\left(\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\right), B^T = \begin{bmatrix} \alpha_1 & 0 & \alpha_{12} & -\frac{\beta_{12}}{\omega} \\ \alpha_{12} & \frac{\beta_{12}}{\omega} & \alpha_2 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Using $(A, B, C)$, it is easy to verify that $C(sI-A)^{-1}B = G(s)$. Applying the same construction procedure for the matrices $A, B, C$, the theorem is proved for any nonsingular $G(s)$ of arbitrary order $\mathtt{n}$.

*$G(s)$ is singular and* $\mathtt{p} = 3$*:* Using equation (7.33), we have

$$R(s) = s\begin{bmatrix} \alpha_1 & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_2 & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_3 \end{bmatrix} - \begin{bmatrix} 0 & \beta_{12} & \beta_{13} \\ -\beta_{12} & 0 & \beta_{23} \\ -\beta_{13} & -\beta_{23} & 0 \end{bmatrix} = sX - Y.$$

Since $G(s)$ is singular, consider the case when the rows of $R(s)$ are related by the following relation $a \times z_1 + b \times z_2 =: z_3$. Here $\mathtt{r} = 2$ and hence $\mathtt{n} = 4$ (By Lemma 7.22). Therefore, consider

$$A = \mathtt{diag}\left(\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\right), B^T = \begin{bmatrix} \alpha_1 & 0 & \alpha_{12} & -\frac{\beta_{12}}{\omega} \\ \alpha_{12} & \frac{\beta_{12}}{\omega} & \alpha_2 & 0 \\ \alpha_{13} & \frac{\beta_{13}}{\omega} & \alpha_{23} & \frac{\beta_{23}}{\omega} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & b & 0 \end{bmatrix}.$$

With the constructed triplet $(A, B, C)$, we have $C(sI-A)^{-1}B = G(s)$. ■

### 7.6.2 Storage function using adapted Gilbert's realization

Given the minimal state-space realization of $G(s)$ as in Theorem 7.23, we compute the storage function associated with the system next. The storage function of a lossless system with transfer function $G(s)$ must satisfy the matrix equations (7.3) where the storage function is induced by the symmetric matrix $K$. Let $K = [k_{iv}]$ and $K = K^T$. Since $K$ satisfies the first matrix equation in equation (7.3), $K$ has to have the form

$$K = \begin{bmatrix} k_{11} & 0 & k_{13} & k_{14} & \cdots & k_{1(\mathtt{n}-1)} & k_{1\mathtt{n}} \\ 0 & k_{11} & -k_{14} & k_{13} & \cdots & -k_{1\mathtt{n}} & k_{1(\mathtt{n}-1)} \\ k_{13} & -k_{14} & k_{33} & 0 & \cdots & k_{3(\mathtt{n}-1)} & k_{3\mathtt{n}} \\ k_{14} & k_{13} & 0 & k_{33} & \cdots & -k_{3\mathtt{n}} & k_{3(\mathtt{n}-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{1(\mathtt{n}-1)} & -k_{1\mathtt{n}} & k_{3(\mathtt{n}-1)} & -k_{3\mathtt{n}} & \cdots & k_{\mathtt{nn}} & 0 \\ k_{1\mathtt{n}} & k_{1(\mathtt{n}-1)} & k_{3\mathtt{n}} & k_{3(\mathtt{n}-1)} & \cdots & 0 & k_{\mathtt{nn}} \end{bmatrix}.$$

Writing the $\frac{\mathtt{n}^2}{4}$ unknown elements in $K$ as a vector $y_k$, we define

$$y_k^T := \begin{bmatrix} k_{11} & k_{13} & \cdots & k_{1\mathtt{n}} & k_{33} & k_{35} & \cdots & k_{3\mathtt{n}} & \cdots & k_{\mathtt{nn}} \end{bmatrix} \in \mathbb{R}^{\frac{\mathtt{n}^2}{4}}. \tag{7.34}$$

The matrix $K$ has to further satisfy the second matrix equation in (7.3). Hence we have $\mathtt{p} \times \mathtt{n}$ linear equations of the form $Py_k = q$ where $q \in \mathbb{R}^{\mathtt{pn}}$. Solution to these set of linear equations gives us the desired storage function $K$. Note that for a *controllable* lossless system there exists a unique symmetric matrix $K$ that induces the storage function $x^T Kx$ (as seen in Theorem 7.8). Hence for such a system the vector $y_k$ defined in equation (7.34) is unique as well. Further, the facts that every conservative system admits a storage function $K$ and the unique $K$ satisfies matrix equations (7.41) together guarantee that $q \in \mathtt{img}\, P$ in equation $Py_k = q$.

Note that Theorem 7.23 gives a minimal realization of $G(s)$. With the same $A$ obtained as in Statement *(1)* of Theorem 7.23, we can have different $(B,C)$ pairs. Depending on the specific $(B,C)$, we get different sets of linear equations. We compute the storage function of the system using the triplet $(A,B,C)$ obtained in Theorem 7.23. The unknown elements of $K$ are hence given by $y_k = P^\dagger q$ where $P^\dagger$ is the pseudo-inverse of $P$.

The special structure of the triplet $(A,B,C)$ is used to create $P$ and $q$ in the set of linear equations $Py_k = q$. For simplicity, we show the construction of $P$ and $q$ for a lossless system with a nonsingular transfer function $G(s) \in \mathbb{R}^{2\times 2}(s)$. Since $G(s)$ is nonsingular, the minimum number of states of $G(s)$ is 4.

*Construction of $P \in \mathbb{R}^{8\times 4}$ and $q \in \mathbb{R}^8$:* Let

$$B^T := \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \in \mathbb{R}^{2\times 4}, \text{ and } C := \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \in \mathbb{R}^{2\times 4}.$$

1. Construction of matrix $P$: Partition $P^T = \begin{bmatrix} P_1^T & P_2^T & \cdots & P_{\frac{n}{2}}^T \end{bmatrix}$ where $P_i \in \mathbb{R}^{2\mathtt{p}\times\mathtt{p}^2}$. Further, each $P_i$ is partitioned as $\begin{bmatrix} P_{i1} & P_{i2} & \cdots & P_{i\mathtt{p}} \end{bmatrix}$ where $P_{iv} \in \mathbb{R}^{2\mathtt{p}\times(2\mathtt{p}-2v+1)}$. Therefore

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} b_1 & b_3 & b_4 & 0 \\ b_2 & b_4 & -b_3 & 0 \\ \hline 0 & b_1 & -b_2 & b_3 \\ 0 & b_2 & b_1 & b_4 \end{array}\right].$$

2. Construction of vector $q$: $q = \text{col}\begin{pmatrix} c_1, & c_2, & c_3, & c_4 \end{pmatrix}$.

This construction is to be used for any lossless system with nonsingular $G(s)$. For a singular $G(s)$, a slightly modified construction procedure is to be used after appropriate zero padding in $B$ and $C$: this is skipped since the procedure is straightforward.

Next we present the algorithms to compute storage functions of lossless system based on the notion of partial fraction described above.

**Algorithm 7.24** Partial fraction based algorithm - MIMO.

**Input:** Strictly proper transfer function matrix of the lossless system $G(s)$.
**Output:** $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with $x^T K x$ the storage function.

```
Find the minimal state-space realization (A,B,C) of G(s) using Theorem 7.23.
Define a matrix P such that it is partitioned into row blocks P_i ∈ R^{2p×p^2}.
Partition each P_i in column blocks P_iv ∈ R^{2p×(2p−2v+1)}. P_iv is the i-th row block and v-th
column block of P.
if i = v then
    P̂_ii = [ b_{2i−1}  b_{2i}      b_{2i+2}  b_{2i+3}   ···  b_{2p}
             b_{2i}    −b_{2i−1}   b_{2i+3}  −b_{2i+2}  ···  −b_{2p−1} ]
    Delete second column of P̂_ii. Result: P_ii ∈ R^{2p×(2p−2i+1)}.
else
    if i < v then
        P_iv = 0 ∈ R^{2p×(2p−2i+1)}
    else   (i.e. if i > v)
        Construct
        L_v := [ b_{2v−1}  −b_{2v} ; b_{2v}  b_{2v−1} ],      L̂_v = [0  L_v ⊗ I_{p−v}]
        col (P_{(v+1)v}, P_{(v+2)v}, ··· , P_{pv}) := L̂_v
                                 where v = 1,2,··· ,p−1.
    end if
end if
q = col (c_1, c_2, ··· , c_{2p})
Compute y_k = P^† q where y is as defined in equation (7.34).
```

We illustrate the method in Algorithm 7.24 with the help of an example next.

**Example 7.25.** *Consider a lossless system with a transfer function*

$$G(s) = \begin{bmatrix} \frac{s}{s^2+16} & \frac{2s-5}{s^2+16} \\ \frac{2s+5}{s^2+16} & \frac{2s}{s^2+16} \end{bmatrix} = \frac{\begin{bmatrix} s & 2s-5 \\ 2s+5 & 2s \end{bmatrix}}{s^2+16} = \frac{s\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}}{s^2+16}$$

*The minimal state-space realization of $G(s)$ using Theorem 7.23 is:*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & -4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 4 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & \frac{5}{4} \\ 2 & 2 \\ -\frac{5}{4} & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x.$$

*Using Algorithm 7.24, the linear equation we obtain is:*

$$\left[\begin{array}{ccc|c} 1 & 2 & -1.25 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & -1.25 & -2 & 0 \\ 1.25 & 0 & -2 & 0 \\ \hline 0 & 1 & 0 & 2 \\ 0 & 2 & -1.25 & 2 \\ 0 & 0 & 1 & -1.25 \\ 0 & 1.25 & 2 & 0 \end{array}\right] \begin{bmatrix} k_{11} \\ k_{13} \\ k_{14} \\ k_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} k_{11} \\ k_{13} \\ k_{14} \\ k_{33} \end{bmatrix} = \begin{bmatrix} -0.56 \\ 0.56 \\ -0.35 \\ -0.28 \end{bmatrix}.$$

*Therefore, the unique storage function of the lossless system is:*

$$K = \begin{bmatrix} -0.56 & 0 & 0.56 & -0.35 \\ 0 & -0.56 & 0.35 & 0.56 \\ 0.56 & 0.35 & -0.28 & 0 \\ -0.35 & 0.56 & 0 & -0.28 \end{bmatrix}.$$

## 7.7 Bezoutian method: SISO case

In this section we compute the storage function of a lossless SISO system using the notion of Bezoutian. Consider a lossless SISO system with transfer function $G(s) = \frac{n(s)}{d(s)}$, where $n(s)$ and $d(s)$ are coprime and $d(s)$ monic. It is known in the literature [WT98] that the storage function of such a system can be computed using a two-variable polynomial expression of the form:

$$\Psi(\zeta,\eta) := \frac{d(\zeta)n(\eta) + d(\eta)n(\zeta)}{\zeta+\eta} \tag{7.35}$$

Note that by definition $\Psi(\zeta,\eta)$ is a symmetric polynomial, i.e., $\Psi(\zeta,\eta)=\Psi(\eta,\zeta)^T$. It is easy to verify that with each two variable polynomial in $R[\zeta,\eta]$ a coefficient matrix with elements from $\mathbb{R}$ is uniquely associated. Similarly, we associate the matrix $\tilde{\Psi}$ and $\tilde{\Phi}$ with $\Psi(\zeta,\eta)$ and $d(\zeta)n(\eta)+d(\eta)n(\zeta)$, respectively. Let $\tilde{\Psi}_{i,k}$ and $\tilde{\Phi}_{i,k}$ be the $i$-th row and $k$-th column element of $\tilde{\Psi}$ and $\tilde{\Phi}$, respectively. Therefore, equation (7.35) can be rewritten as:

$$\Psi(\zeta,\eta):=\frac{d(\zeta)n(\eta)+d(\eta)n(\zeta)}{\zeta+\eta}\Rightarrow\sum_{i,k}\tilde{\Psi}_{i,k}\zeta^i\eta^k=\frac{\sum_{i,k}\tilde{\Phi}_{i,k}\zeta^i\eta^k}{\zeta+\eta}. \tag{7.36}$$

Here $\tilde{\Psi}$ is the matrix that induces the storage function of the lossless system with transfer function $G(s)$. The storage function can be calculated by what may be called, "polynomial long division technique" which is based on Euclidean long division of polynomials. We state this as a result below.

**Bezoutian method to compute the storage function of a lossless system**

**Theorem 7.26.** *Consider a lossless SISO system $\Sigma$ with transfer function $G(s)=\frac{n(s)}{d(s)}$, where $n(s)$, $d(s)$ are coprime and $d(s)$ is a monic polynomial. Let the controller canonical i/s/o representation of the system be:*

$$\frac{d}{dt}x=Ax+Bu \qquad \textit{and} \qquad y=Cx. \tag{7.37}$$

*Construct the two variable polynomial $z_b(\zeta,\eta)$*

$$\Psi(\zeta,\eta):=\frac{n(\zeta)d(\eta)+n(\eta)d(\zeta)}{\zeta+\eta}=\begin{bmatrix}1\\ \zeta\\ \vdots\\ \zeta^{\mathrm{n}-1}\end{bmatrix}^T\tilde{\Psi}\begin{bmatrix}1\\ \eta\\ \vdots\\ \eta^{\mathrm{n}-1}\end{bmatrix}, \tag{7.38}$$

*where $\Psi\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is the corresponding coefficient matrix. Then, $x^T\tilde{\Psi}x$ is the unique storage function for the lossless system, i.e., $\frac{d}{dt}\left(x^T\tilde{\Psi}x\right)=2uy$ with the states in the controller canonical representation.*

*Proof:* This is a direct consequence of the result in [WT98, Theorem 5.7, Remark 5.13]. Hence, we do not explicitly prove this theorem here. ∎

The conventional Bezoutian of two polynomials $p(x)$ and $q(x)$ is defined as

$$B_z(x,y):=\frac{p(x)q(y)-p(y)q(x)}{x-y}.$$

Note the change in sign between this conventional Bezoutian definition and the one defined in equation (7.38): in spite of the sign change, we call $\Psi(\zeta,\eta)$ the Bezoutian due to the following reasons. In any lossless system with transfer function $\frac{n(s)}{d(s)}$, when the order of the system is even then $n(s)$ is an odd polynomial, i.e., $n(-s)=-n(s)$ and $d(s)$ is even polynomial i.e. $d(-s)=$

$d(s)$. Thus, our definition is same as the conventional one if we substitute $x = -\zeta$; $y = \eta$ when the order of the system is even and $x = \zeta$; $y = -\eta$ when the order of the system is odd. Hence for lossless case $B_z(x,y) = \Psi(\zeta,\eta)$ where $x = -\zeta$, $y = \eta$ for even order systems and $x = \zeta$, $y = -\eta$ for odd order systems. In fact, $\tilde{\Psi}$ of equation (7.38) is nonsingular if and only if $n(s)$ and $d(s)$ are coprime. This justifies the use of the term 'Bezoutian' for $\Psi(\zeta,\eta)$ defined in equation (7.38).

*Methods to compute the Bezoutian:* There are various methods of finding the Bezoutian $\Psi(\zeta,\eta)$ of two polynomials. In this chapter we propose three different methods to compute $\Psi(\zeta,\eta)$:

(a) Euclidean long division method,

(b) Pseudo-inverse method, and

(c) 2 dimensional discrete Fourier transform method.

### 7.7.1 Euclidean long division method

Though Theorem 7.26 involves *bivariate* polynomial manipulation, Algorithm 7.31 below uses only *univariate* polynomial operations. The algorithm is similar to Euclidean long division. As in [BT02], write $\Phi(\zeta,\eta) = \phi_0(\eta) + \zeta\phi_1(\eta) + \cdots + \zeta^{\mathrm{n}}\phi_{\mathrm{n}}(\eta)$. Then the storage function $\Psi(\zeta,\eta) = \psi_0(\eta) + \zeta\psi_1(\eta) + \cdots + \zeta^{\mathrm{n}-1}\psi_{\mathrm{n}-1}(\eta)$ can be computed by the following recursion (see [BT02, Section 6.5]) with $k = 1,\ldots,\mathrm{n}-1$:

$$\psi_0(\xi) := \frac{\phi_0(\xi)}{\xi}, \qquad \psi_k(\xi) := \frac{\phi_k(\xi) - \psi_{k-1}(\xi)}{\xi}. \tag{7.39}$$

In the sequel, we present an algorithm to implement this method using simple matrix operations in Algorithm 7.31.

### 7.7.2 Pseudo-inverse method

Recall that the coefficient matrix corresponding to $\Psi(\zeta,\eta)$ and $\Phi(\zeta,\eta)$ are $\tilde{\Psi} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and $\tilde{\Phi} \in \mathbb{R}^{(\mathrm{n}+1)\times(\mathrm{n}+1)}$, respectively. From equation (7.36), we have

$$(\zeta+\eta)\Psi(\zeta,\eta) = M(\zeta)^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M(\eta) = \Phi(\zeta,\eta), \text{ where } M(\xi) := \begin{bmatrix} d(\xi) \\ n(\xi) \end{bmatrix} \tag{7.40}$$

Using equation (7.40), we have

$$\sigma_R(\tilde{\Psi}) + \sigma_D(\tilde{\Psi}) = \tilde{\Phi}, \tag{7.41}$$

where $\sigma_R(\tilde{\Psi}) := \begin{bmatrix} 0_{\mathrm{n},1} & \tilde{\Psi} \\ 0 & 0_{1,\mathrm{n}} \end{bmatrix}$, and $\sigma_D(\tilde{\Psi}) := \begin{bmatrix} 0_{1,\mathrm{n}} & 0 \\ \tilde{\Psi} & 0_{\mathrm{n},1} \end{bmatrix}$: see also [TR99]. Due to the symmetry of $\Psi(\zeta,\eta)$ and $\Phi(\zeta,\eta)$ the number of unknowns in $\tilde{\Psi}$ is $\frac{\mathrm{n}(\mathrm{n}+1)}{2}$ and the number of distinct elements in $\tilde{\Phi}$ is $\frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2}$. Hence the matrix equation (7.41) can be rewritten as linear equations of the form

$$P_b y = q_b, \text{ where } P_b \in \mathbb{R}^{\frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2} \times \frac{\mathtt{n}(\mathtt{n}+1)}{2}},\ q_b \in \mathbb{R}^{\frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2}} \\ y := \mathtt{col}\,(\Psi_{11}, \Psi_{12}, \cdots, \Psi_{1\mathtt{n}}, \Psi_{22}, \cdots, \Psi_{2\mathtt{n}}, \cdots, \Psi_{\mathtt{nn}})\,. \tag{7.42}$$

For a passive system, a storage function exists and this guarantees $q_b \in \operatorname{im} P_b$. We compute the pseudo-inverse of the matrix $P_b$, i.e., $P_b^\dagger$ and obtain $y := P_b^\dagger q_b$. This gives the storage function.

Computation of the storage function of a lossless system using linear equations

**Corollary 7.27.** *Consider a minimal, passive system $\Sigma$ with transfer function $G(s) = \frac{n(s)}{d(s)}$ where $d(s)$ is a monic polynomial and construct $\tilde{\Phi} = [\tilde{\Phi}_{i,j}]$ as in equation* (7.36). *Let the matrix equation* (7.41) *be written in the linear equation form $P_b y = q_b$. Then the following are equivalent*

1. *$\Sigma$ is lossless.*
2. *There exists a unique symmetric $K$ such that $x^T K x = 2u^T y$.*
3. *$P_b$ is full column rank and $q_b \in \mathtt{img}\, P_b$.*

*Proof:* The proof directly follows from the discussion before the corollary. ■

### 7.7.3 2D-DFT method

In this section we propose a method to compute the coefficient matrix $\tilde{\Psi}$ of the Bezoutian using two-dimensional discrete Fourier transform (2D-DFT). In order to develop this method we present a theorem next establishing a link between 2D-DFT and two-variable polynomial multiplication first. We use the symbols $\odot$ and $\oslash$ to denote elementwise multiplication and elementwise division, respectively.

Two-variable polynomial multiplication and 2D-DFT

**Theorem 7.28.** *Consider the polynomial ring $\mathbb{C}[\zeta,\eta]$ and the ideal $\mathbb{A} := \langle \zeta^{\mathtt{N}}-1, \eta^{\mathtt{N}}-1\rangle$, where $\mathtt{N} \in \mathbb{N}$. Define the map $\Pi : \mathbb{C}[\zeta,\eta] \longrightarrow \mathbb{C}[\zeta,\eta]/\mathbb{A}$. Let $P,Q,R \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ be such that $p(\zeta,\eta) := \mathbf{X}^T P \mathbf{Y}$, $q(\zeta,\eta) := \mathbf{X}^T Q \mathbf{Y}$ and $r(\zeta,\eta) := \mathbf{X}^T R \mathbf{Y}$, where $\mathbf{X} := \mathtt{col}(1,\zeta,\ldots,\zeta^{\mathtt{N}-1})$ and $\mathbf{Y} := \mathtt{col}(1,\eta,\ldots,\eta^{\mathtt{N}-1})$. Let $\mathscr{F}(P), \mathscr{F}(Q)$ and $\mathscr{F}(R)$ be the 2D-DFT matrices of $P,Q$ and $R$, respectively. Then,*

$$\Pi\Big(p(\zeta,\eta)q(\zeta,\eta)\Big) = r(\zeta,\eta) \text{ if and only if } \mathscr{F}(P)\odot\mathscr{F}(Q) = \mathscr{F}(R).$$

*Proof: Only if:* Since $\Pi\Big(p(\zeta,\eta)q(\zeta,\eta)\Big) = r(\zeta,\eta)$ and the set $\{x^{\mathtt{N}}-1, y^{\mathtt{N}}-1\}$ is the Gröbner basis of $\mathbb{A}$, there exists unique $a(\zeta,\eta), b(\zeta,\eta) \in \mathbb{C}[\zeta,\eta]$ such that

$$p(\zeta,y)q(\zeta,\eta) = a(\zeta,\eta)(x^{\mathtt{N}}-1) + b(\zeta,\eta)(y^{\mathtt{N}}-1) + r(\zeta,\eta). \tag{7.43}$$

On evaluating the right-hand side of the equation (7.43) at $\zeta = \omega^m$ and $\eta = \omega^n$ for every $m,n = 0,1,2,\dots,\mathtt{N}-1$, where $\omega = e^{-j\frac{2\pi}{\mathtt{N}}}$ and using the fact that $\omega^{m\mathtt{N}} = \omega^{n\mathtt{N}} = 1$, we have

$$a(\omega^m,\omega^n)(\omega^{m\mathtt{N}}-1) + b(\omega^m,\omega^n)(\omega^{n\mathtt{N}}-1) + r(\omega^m,\omega^n) = r(\omega^m,\omega^n). \tag{7.44}$$

Therefore, for every $m,n = 0,1,2,\dots,\mathtt{N}-1$, the left-hand side of equation (7.43) becomes

$$p(\omega^m,\omega^n)q(\omega^m,\omega^n) = r(\omega^m,\omega^n) \tag{7.45}$$

Let $\mathscr{F}(P) =: [e_{mn}]_{m,n=0,1,\dots,\mathtt{N}-1}$, $\mathscr{F}(Q) =: [f_{mn}]_{m,n=0,1,\dots,\mathtt{N}-1}$ and $\mathscr{F}(R) =: [g_{mn}]_{m,n=0,1,\dots,\mathtt{N}-1}$. As discussed in Section 7.2.4, computation of 2D-DFT of a matrix $U \in \mathbb{C}^{\mathtt{N}\times\mathtt{N}}$ is essentially the computation of $\mathbf{X}^T U \mathbf{Y}$ at $\zeta = \omega^m$ and $\eta = \omega^n$ for every $m,n = 0,1,2,\dots,\mathtt{N}-1$. Hence, it is clear that $e_{mn} = p(\omega^m,\omega^n)$, $f_{mn} = q(\omega^m,\omega^n)$ and $g_{mn} = r(\omega^m,\omega^n)$. Rewriting equation (7.45) in matrix form, we have $\mathscr{F}(P) \odot \mathscr{F}(Q) = \mathscr{F}(R)$.
*If:* Since $\mathscr{F}(P) \odot \mathscr{F}(Q) = \mathscr{F}(R)$, for every $m,n = 0,1,\dots,\mathtt{N}-1$ and $\omega := e^{-j\frac{2\pi}{\mathtt{N}}}$, we get

$$p(\omega^m,\omega^n)q(\omega^m,\omega^n) = r(\omega^m,\omega^n). \tag{7.46}$$

Note that the variety of $\mathbb{A}$, denoted by $\mathbb{V}(\mathbb{A})$, is the set $\left\{\mathtt{col}(\alpha,\beta) \in \mathbb{C}^2 | \alpha^{\mathtt{N}} = 1 \text{ and } \beta^{\mathtt{N}} = 1\right\}$. Let $\mathscr{I}\Big(\mathbb{V}(\mathbb{A})\Big)$ be the ideal consisting of all polynomials $g(\zeta,\eta) \in \mathbb{C}[\zeta,\eta]$ such that $g(\alpha,\beta) = 0$ for all $\mathtt{col}(\alpha,\beta) \in \mathbb{V}(\mathbb{A})$. Since equation (7.46) is true for every $m,n = 0,1,\dots,\mathtt{N}-1$, we have $p(\alpha,\beta)q(\alpha,\beta) - r(\alpha,\beta) = 0$ for all $(\alpha,\beta) \in \mathbb{V}(\mathbb{A})$. Therefore, $p(\zeta,y)q(\zeta,\eta) - r(\zeta,\eta) \in \mathscr{I}\Big(\mathbb{V}(\mathbb{A})\Big)$.
By Hilbert's Nullstellensatz in Proposition 7.4, we infer that the two-variable polynomial

$$p(\zeta,\eta)q(\zeta,\eta) - r(\zeta,\eta) \in \mathscr{I}\Big(\mathbb{V}(\mathbb{A})\Big) = \sqrt{\mathbb{A}}.$$

Note that $\mathbb{V}(\mathbb{A}) \subsetneq \mathbb{C}^2$ is a finite set and therefore $\mathbb{A}$ is a zero-dimensional ideal. Hence using [CLO92, Proposition 2.7], we can verify that $\mathbb{A}$ is a radical ideal i.e. $\sqrt{\mathbb{A}} = \mathbb{A}$. Therefore,

$$\begin{aligned} p(\zeta,\eta)q(\zeta,\eta) - r(\zeta,\eta) \in \sqrt{\mathbb{A}} &\Rightarrow p(\zeta,y)q(\zeta,\eta) - r(\zeta,\eta) \in \mathbb{A} \\ &\Rightarrow \Pi\Big(p(\zeta,\eta)q(\zeta,\eta) - r(\zeta,\eta)\Big) = 0. \end{aligned} \tag{7.47}$$

Since $LM\Big(r(\zeta,\eta)\Big) \preceq (\zeta\eta)^{\mathtt{N}-1}$, we have $\Pi\Big(r(\zeta,\eta)\Big) = r(\zeta,\eta)$. It therefore follows from equation (7.47) that

$$\Pi\Big(p(\zeta,\eta)q(\zeta,\eta)\Big) = \Pi\Big(r(\zeta,\eta)\Big) \Rightarrow \Pi\Big(p(\zeta,\eta)q(\zeta,\eta)\Big) = r(\zeta,\eta).$$

This completes the proof of the theorem. ∎

Note that for a system with McMillan degree $\mathtt{n}$, the term $\zeta+\eta$ in equation (7.38) can be rewritten as

$$\zeta+\eta = \mathbf{X}^T J \mathbf{Y}, \text{ where } J := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{\mathtt{n}-1,\mathtt{n}-1} \end{bmatrix} \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}, \tag{7.48}$$

where $\mathbf{X} := \mathtt{col}(1,\zeta,\ldots,\zeta^{\mathtt{n}})$ and $\mathbf{Y} := \mathtt{col}(1,\eta,\ldots,\eta^{\mathtt{n}})$. Further, $\Psi(\zeta,\eta) = \mathbf{X}^T \begin{bmatrix} \tilde{\Psi} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Y}$, where $\tilde{\Psi}$ is as defined in equation (7.38). Then, using Theorem 7.28 we have the following corollary.

Relation between 2D-DFT and Bezoutian corresponding to a lossless system

**Corollary 7.29.** *Let* $\tilde{\Psi} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *and* $\tilde{\Phi} \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}$ *be as defined in equation* (7.38). *Define* $\widehat{\Psi} := \begin{bmatrix} \tilde{\Psi} & 0 \\ 0 & 0 \end{bmatrix}$. *Assume* $J \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}$ *to be as defined in equation* (7.48). *Then,*

$$\mathscr{F}(J) \odot \mathscr{F}(\widehat{\Psi}) = \mathscr{F}(\tilde{\Phi}).$$

*Proof:* Consider the polynomial ring $\mathbb{C}[\zeta,\eta]$ and define $\mathbb{A} := \langle \zeta^{\mathtt{n}+1}-1, \eta^{\mathtt{n}+1}-1\rangle$. Define the map $\Pi : \mathbb{C}[\zeta,\eta] \mapsto \mathbb{C}[\zeta,\eta]/\mathbb{A}$. Under the action of $\Pi$, an element $p \in \mathbb{C}[\zeta,\eta]$ goes to $[p]$. Note that the Gröbner basis of the ideal $\mathbb{A}$ is the set $\{\zeta^{\mathtt{n}+1}-1, \eta^{\mathtt{n}+1}-1\}$. Therefore, any element in $\mathbb{C}[x,y]/\mathbb{A}$, e.g., $[p]$ is uniquely represented by the remainder obtained on dividing $p$ by $\zeta^{\mathtt{n}+1}-1$ and $\eta^{\mathtt{n}+1}-1$. From equation (7.38) it is clear that $(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) \Rightarrow 0\times(\zeta^{\mathtt{n}+1}-1) + 0\times(\eta^{\mathtt{n}+1}-1) + \Phi(\zeta,\eta)$. Therefore, the unique remainder obtained on dividing $(\zeta+\eta)\Psi(\zeta,\eta)$ with $(\zeta^{\mathtt{n}+1}-1)$ and $(\eta^{\mathtt{n}+1}-1)$ is $\Phi(\zeta,\eta)$, i.e.,

$$\Pi((\zeta+\eta)\Psi(\zeta,\eta)) = \Phi(\zeta,\eta). \tag{7.49}$$

Since $(\zeta+\eta) = \mathbf{X}^T J \mathbf{Y}$, $\Psi(\zeta,\eta) = \mathbf{X}^T \widehat{\Psi}\mathbf{Y}$ and $\Phi(\zeta,\eta) = \mathbf{X}^T \tilde{\Phi}\mathbf{Y}$, from equation (7.49) and Theorem 7.28, we infer that $\mathscr{F}(J) \odot \mathscr{F}(\widehat{\Psi}) = \mathscr{F}(\tilde{\Phi})$. ■

From Corollary 7.29, it is evident that the matrix $\widehat{\Psi}$ can be computed using the formula:

$$\tilde{\Psi} = \widehat{\Psi}(1:\mathtt{n}, 1:\mathtt{n}), \text{ where } \widehat{\Psi} = \mathscr{F}^{-1}\left(\mathscr{F}(\tilde{\Phi}) \oslash \mathscr{F}(J)\right) \tag{7.50}$$

Note that for the operation $\oslash$, i.e., element-wise division to be well-defined, *every* element of $\mathscr{F}(J)$ needs to be nonzero. Therefore, a relevant question to ask here is: when will every element of $\mathscr{F}(J)$ be nonzero. We answer this in the next lemma.

Condition for the operation $\oslash$, in equation (7.50), to be well-defined.

**Lemma 7.30.** *Let* $J \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}$ *be as defined in equation* (7.48). *Then,* $\mathscr{F}(J)$ *has every element nonzero if and only if* $\mathtt{n}$ *is even.*

*Proof:* Let $J = [J_{p,q}]_{p,q=0,1,\ldots,\mathtt{n}}$ and $\mathscr{F}(J) = [\mathscr{F}(J)_{p,q}]_{p,q=0,1,\ldots,\mathtt{n}}$. Define $\mathtt{N} := \mathtt{n}+1$. Then, by equation (7.6) in Section 7.2.4, we get

$$\begin{aligned}\mathscr{F}(J)_{p,q} &= \sum_{i=0}^{\mathtt{n}}\left(\sum_{k=0}^{\mathtt{n}} J_{ik}\omega^{kp}\right)\omega^{iq} = \omega^{p}+\omega^{q} = \mathrm{e}^{-j\frac{2\pi}{\mathtt{N}}p}+\mathrm{e}^{-j\frac{2\pi}{\mathtt{N}}q} \\ &= \mathrm{e}^{-j\frac{2\pi}{\mathtt{N}}\left(\frac{p+q}{2}\right)}\left(\mathrm{e}^{-j\frac{2\pi}{\mathtt{N}}\left(\frac{p-q}{2}\right)}+\mathrm{e}^{j\frac{2\pi}{\mathtt{N}}\left(\frac{p-q}{2}\right)}\right) = 2\cos\left(\frac{2\pi}{\mathtt{N}}\left(\frac{p-q}{2}\right)\right)\mathrm{e}^{-j\frac{2\pi}{\mathtt{N}}\left(\frac{p+q}{2}\right)}.\end{aligned}$$

It is evident that $\mathscr{F}(J)_{p,q} = 0$ if and only if $\cos\left(\frac{2\pi}{\mathtt{N}}\left(\frac{p-q}{2}\right)\right) = 0$. Now, $\cos\left(\frac{2\pi}{\mathtt{N}}\left(\frac{p-q}{2}\right)\right) = 0$ for any $p,q = 0,1,\ldots,\mathtt{N}-1$ if and only if for any $\ell \in \mathbb{Z}$,

$$\frac{2\pi(p-q)}{2\mathtt{N}} = (2\ell+1)\frac{\pi}{2} \implies p-q = \frac{(2\ell+1)\mathtt{N}}{2}. \tag{7.51}$$

Note that $p,q \in \mathbb{Z}$ and therefore from equation (7.51), $p-q \in \mathbb{Z}$ if and only if $\mathtt{N}$ is even, i.e., $\mathtt{n}$ is odd. Thus $\mathscr{F}(J)$ has every element nonzero if and only if $\mathtt{n}$ is even. ∎

From Lemma 7.30 it is evident that the 2D-DFT method for computation of the storage function of a lossless system is applicable only for systems with even McMillan degree.

Now that we have presented three methods to compute the Bezoutian corresponding to a lossless system, we next compare these methods for accuracy and time.

### 7.7.4 Experimental setup and procedure

**Experimental setup:** The experiments were carried out on an Intel Core i3 computer at 3.30 GHz with 4 GB RAM using Ubuntu 14.04 LTS operating system. The relative machine precision is $\varepsilon \approx 2.22\times 10^{-16}$. Numerical computational package SCILAB 5.5 (which, like MATLAB and Python-SciPy, NumPy, relies on LAPACK for core numerical computations) has been used to implement the algorithms.

**Experimental procedure:** Randomly generated transfer functions of lossless systems are used to test the algorithms. Computation time and error for each transfer function order has been averaged over three different randomly generated transfer functions. To nullify the effect of CPU delays the computation time to calculate $K$ for each transfer function is further averaged over hundred iterations.

### 7.7.5 Experimental results

#### Computation Time

Figure 7.5 shows a comparison of the time taken by each of the Bezoutian based methods, viz., Euclidean long division, Pseudo-inverse and 2D-DFT. From the figure it is evident that the time taken to compute the storage function of a lossless transfer function using Euclidean long division based method and pseudo-inverse methods are comparable to each other and are much faster than 2D-DFT.

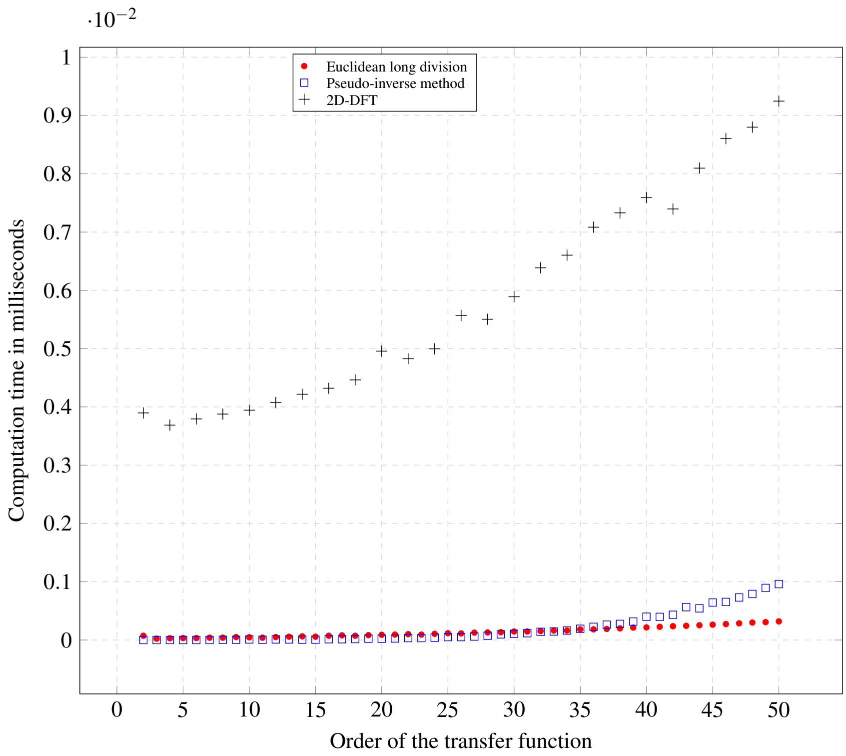



Figure 7.5: Plot of computation time versus system's order.

### Computation error

As discussed in Section 7.1, lossless systems satisfy the LME (7.3). In view of this, we define the error associated with the computation of $K$ as

$$\operatorname{Err}(K) := \left\| \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} \right\|_2 . \tag{7.52}$$

The matrix $K$ obtained from the above procedures must ideally yield $\operatorname{Err}(K) = 0$. Figure 7.6 shows the error in computation of storage function using the three Bezoutian based methods discussed above. All the three methods have comparable errors. From the comparison it is clear that both the Euclidean long division and Pseudo-inverse method are comparable in computational time and error. Next we present an algorithm to compute the storage function of a lossless system using the Euclidean long division method. In the algorithm, we use the symbol $F(i,:)$ to represent the $i$-th row of a matrix $F$. The symbol $F(1:\mathtt{n}, 1:\mathtt{m})$ is used to denote a $\mathtt{n} \times \mathtt{m}$ submatrix of a matrix $F$ with the $1^{st}$ to $\mathtt{n}^{th}$ rows and $1^{st}$ to $\mathtt{m}^{th}$ columns of $F$.

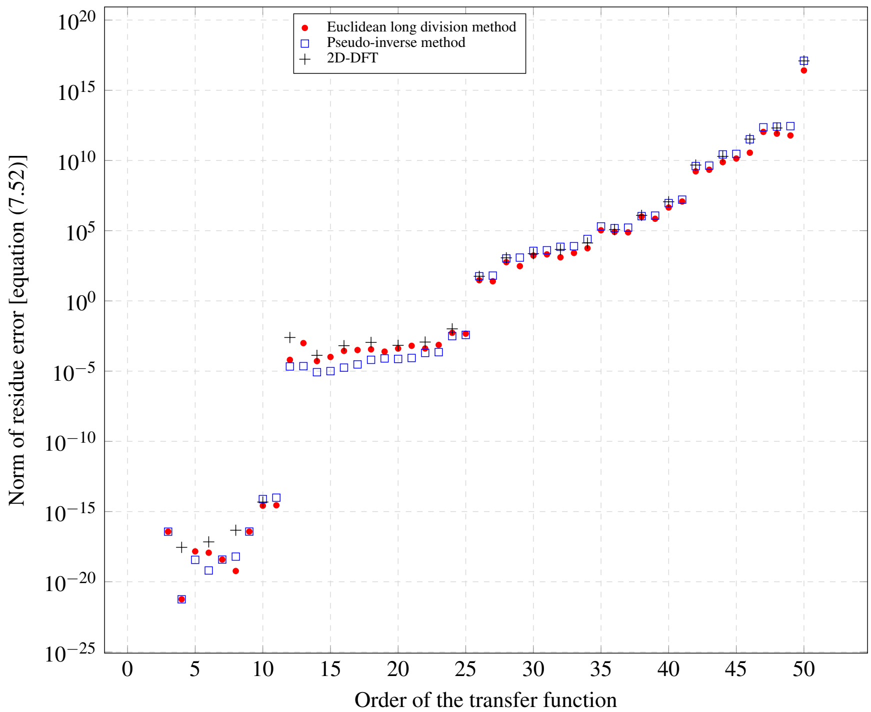



Figure 7.6: Plot of error versus system's order.

**Algorithm 7.31** Bezoutian algorithm - SISO.

**Input:** Transfer function of a lossless system $G(s) = \frac{n(s)}{d(s)}$ of order $\mathtt{n}$ where $d(s)$ is monic and $G(s)$ proper.

**Output:** $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with $x^T Kx$ the storage function.

1: Extract coefficients of the polynomials $n(s)$ and $d(s)$ into arrays $N \in \mathbb{R}^{1\times\mathtt{n}}$ and $D \in \mathbb{R}^{1\times(\mathtt{n}+1)}$ with constant term coefficient first.

2: Equate length of $N$ and $D$ by appending a zero after the last element of $N$.

3: Compute Bezoutian coefficient matrix using equation (7.38)

$$K_b := N^T D + D^T N \ \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}.$$

4: Implement the division in first equation of (7.39) by constructing a row vector from the first row of $K_b$

$$F_{\text{old}} := \begin{bmatrix} K_b(1:1, 2:\mathtt{n}+1) & 0 \end{bmatrix} \in \mathbb{R}^{1\times(\mathtt{n}+1)}.$$

5: Set $F_{\text{new}} := F_{\text{old}}$.

6: Append new rows to get $F_{\text{new}} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n}+1)}$ by implementing the division in (7.39) by the following iteration:

7: **for** i=2,..., n **do**

8: $\quad r := K_b(i,:) - F_{\text{new}}(i-1,:)$

9: $$\quad F_{\text{new}} := \begin{bmatrix} & F_{\text{old}} & \\ r(1:1, 2:\mathtt{n}+1) & & 0 \end{bmatrix}$$

```
10:     F_old := F_new
11: end for
12: Define K := F_new(1 : n, 1 : n).
```

Now, we revisit Example 7.9 and compute the storage function of the system using Algorithm 7.31.

**Example 7.32.** *Recall that the lossless system in Example 7.9 has the transfer function*

$$G(s) = \frac{8s^2+1}{6s^3+s} = \frac{\frac{8}{6}s^2+\frac{1}{6}}{s^3+\frac{1}{6}s}.$$

*Here* $\mathtt{n} = 3$ *and therefore using Step (1) and Step (2) of the Algorithm 7.31, we have*

$$N = \begin{bmatrix} \frac{1}{6} & 0 & \frac{8}{6} & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{6} \end{bmatrix}.$$

*The i/s/o representation of the system is*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{6} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \frac{1}{6}\begin{bmatrix} 1 & 0 & 8 \end{bmatrix} x.$$

*Hence, we have* $\Phi(\zeta,\eta) = n(\zeta)d(\eta) + n(\eta)d(\zeta) =$

$$\frac{1}{36}\left\{ \underbrace{(\eta+6\eta^3)}_{\phi_0(\eta)} + \underbrace{(1+8\eta^2)}_{\phi_1(\eta)}\zeta + \underbrace{(8\eta+48\eta^3)}_{\phi_2(\eta)}\zeta^2 + \underbrace{(6+48\eta^2)}_{\phi_3(\eta)}\zeta^3 \right\}$$

$$= \frac{1}{36}\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 1 & 0 & 6 \\ 1 & 0 & 8 & 0 \\ 0 & 8 & 0 & 48 \\ 6 & 0 & 48 & 0 \end{bmatrix}}_{N^T D + D^T N = K_b} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \\ \eta^3 \end{bmatrix}$$

*This corresponds to step (3) of Algorithm 7.31. Using the equations* (7.39)*, we have*

$$\psi_0(\xi) = \frac{\phi_0(\xi)}{\xi} = \frac{1+6\xi^2}{36}, \; \psi_1(\xi) = \frac{\phi_1(\xi)-\psi_0(\xi)}{\xi} = \frac{2\xi}{36}$$

$$\psi_2(\xi) = \frac{\phi_2(\xi)-\psi_1(\xi)}{\xi} = \frac{6+48\xi^2}{36}$$

*Note that the polynomial subtraction and division shown in these steps can also be done using corresponding vector shift and subtraction operations. This is implemented with Step (4) to*

*Step (10) of Algorithm 7.31. Hence the storage function is*

$$\Psi(\zeta,\eta) = \frac{1}{36}\{(1+6\eta^2)+2\eta\zeta+(6+48\eta^2)\zeta^2\} = \frac{1}{36}\begin{bmatrix}1\\ \zeta\\ \zeta^2\end{bmatrix}\underbrace{\begin{bmatrix}1 & 0 & 6\\ 0 & 2 & 0\\ 6 & 0 & 48\end{bmatrix}}_{K}\begin{bmatrix}1\\ \eta\\ \eta^2\end{bmatrix}$$

## 7.8 Bezoutian method: MIMO case

In this section we propose an extension of the Bezoutian method for the SISO case to MIMO case when each of the elements in $G(s)$ are considered to be lossless, i.e., we consider each element of $G(s)$ to have poles on the imaginary axis. In general the elements of $G(s)$ have the form given in equation (7.33). However, since we consider each element of $G(s)$ to be lossless therefore $\beta_{iv} = 0$. For such systems, we consider each of the elements in $G(s)$ as lossless systems and use the procedure described in Section 7.7 to compute the storage function of the system. We present a theorem next to compute the storage function of MIMO systems with the Bezoutian method.

Bezoutian method to compute the storage function of a MIMO lossless system

**Theorem 7.33.** *Consider a lossless transfer matrix $G(s)$ with the $(i,k)$-th element represented as $g_{ik}$. Recall again that $g_{ik}$ has the form of equation* (7.33) *with $\beta_{ik} = 0$. The controller canonical form of each element $g_{ik}$ is represented by the triplet $(A_{ik}, b_{ik}, c_{ik})$. Construct matrices $B_{ik} \in \mathbb{R}^{2\times \mathtt{p}}$ such that k-th column of $B_{ik} := b_{ik}$ and rest entries zero. Let $C_{ik} \in \mathbb{R}^{\mathtt{p}\times 2}$ be matrices with i-th row of $C_{ik} := c_{ik}$ and rest entries zero. Suppose $K_{ik}$ represents the storage function corresponding to each $g_{ik}$ given by Theorem 7.26. Then (possibly nonminimal) state-space representation of the system $G(s)$ is given by the following $(A,B,C)$ matrices.*

1. $A = \mathtt{diag}\left(A_{11}, A_{12}, \ldots, A_{1\mathtt{p}}, A_{21}, \ldots, A_{\mathtt{pp}}\right) \in \mathbb{R}^{2\mathtt{p}^2\times 2\mathtt{p}^2}$.
2. $B = \mathrm{col}\left(B_{11}, B_{12}, \ldots, B_{1\mathtt{p}}, B_{21}, \ldots, B_{\mathtt{pp}}\right) \in \mathbb{R}^{2\mathtt{p}^2\times \mathtt{p}}$.
3. $C = \begin{bmatrix}C_{11} & C_{12} & \ldots & C_{1\mathtt{p}} & C_{21} & \ldots & C_{\mathtt{pp}}\end{bmatrix} \in \mathbb{R}^{\mathtt{p}\times 2\mathtt{p}^2}$.

*The K matrix that induces the storage function of the lossless system $G(s)$ with respect to the triplet $(A,B,C)$ is given by $K = \mathtt{diag}\left(K_{11}, K_{12}, \cdots, K_{1\mathtt{p}}, K_{21}, \cdots, K_{\mathtt{pp}}\right) \in \mathbb{R}^{2\mathtt{p}^2\times 2\mathtt{p}^2}$.*

*Proof:* We present the proof for a specific case. The general proof is essentially a book-keeping version of this simplified case. Consider the transfer matrix of the form

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s)\\ g_{21}(s) & g_{22}(s)\end{bmatrix} := \begin{bmatrix}\dfrac{\alpha_{11}s}{s^2+\omega^2} & \dfrac{\alpha_{12}s}{s^2+\omega^2}\\ \dfrac{\alpha_{21}s}{s^2+\omega^2} & \dfrac{\alpha_{22}s}{s^2+\omega^2}\end{bmatrix}.$$

Consider $c_{ik}(sI-A_{ik})^{-1}b_{ik}=g_{ik}$ where $i,k=1,2$. Construct

$$A=\begin{bmatrix} A_{11} & & & \\ & A_{12} & & \\ & & A_{21} & \\ & & & A_{22} \end{bmatrix}, B=\begin{bmatrix} b_{11} & 0 \\ 0 & b_{12} \\ b_{21} & 0 \\ 0 & b_{22} \end{bmatrix}, \text{ and } C=\begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} \end{bmatrix}.$$

It can be verified that $C(sI-A)^{-1}B=G(s)$.
The storage function corresponding to each $g_{ik}$ is $K_{ik}$. Hence $A_{ik}^TK_{ik}+K_{ik}A_{ik}=0$ is satisfied. Further, $b_{ik}^TK_{ik}-c_{ik}=0$. Construct $K=\texttt{diag}(K_{11},K_{12},K_{21},K_{22})$. From the construction of $A$ and $K$ it follows that $A^TK+KA=0$. Further, it can also be verified that $B^TK-C=0$. This proves that $K$ gives the required storage function. ■

Note that the $K$ matrix obtained by the method described in Theorem 7.33 is not minimal in general. This is due the non-minimal state-space representation obtained for the lossless transfer matrix $G(s)$. It is not clear whether a minimal state-space realization is always possible for a lossless MIMO system with the states of the form given in Section 5.2.1: for more on nonminimality of RLC circuits in general see [BD49, HS14]. This requires further investigation and is not dealt here.

## 7.9 Gramian method

In this section we present a method to compute the storage function of a lossless system using controllability and observability Gramian. As reviewed in Section 7.2.6, the observability Gramian of a system with an i/s/o representation as given in equation (7.1) is the solution of the Lyapunov equation $A^TQ+QA+C^TC=0$. For a lossless system since all the eigenvalues of the system are on the imaginary axis, the solution to the Lyapunov equation $A^TQ+QA=-C^TC$ is not unique. Further, for a lossless system the storage function of the system is unique and must satisfy the equation $A^TK+KA=0$. Thus, the observability Gramian and the storage function are not the same for a lossless system. However, in this section we establish that the observability Gramian of a suitable allpass system reveals the storage function of a lossless system.

At the every outset of this section, we present a theorem that establishes the link between allpass systems, and Gramians.

Observability Gramian of an allpass system is its storage function

**Theorem 7.34.** *Consider a stable, allpass system with a minimal i/s/o representation as given in equation* (7.1)*. Assume $Q\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ to be the observability Gramian of the system. Then, $x^TQx$ is the unique storage function of the system.*

*Proof:* We know that for an allpass system there exists a $K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ that satisfies equation (7.13). Since the system is stable, the equation $A^TK+KA+C^TC=0$ must have a unique solution. Note that $A^TK + KA + C^TC = 0$ is the observability Gramian equation. Therefore, $K = Q$. Hence, we conclude using equation (7.13) that $x^TQx$ is a unique storage function of the system. ■

For an allpass system, the observability Gramian $Q$ and controllability Gramian $P$ are related as $PQ = I_{\mathtt{n}}$: see [Glo84, Theorem 5.1]. Hence, the matrix $P^{-1}$ also induces the storage function associated with an allpass system. The next result uses the concept of balanced basis to infer a noteworthy property of the storage function of an allpass system.

The storage function of an allpass system is induced by an identity matrix

**Theorem 7.35.** *Consider a stable, allpass system with a transfer function* $G(s) \in \mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$. *Let the minimal state-space realization of the system be in a* balanced *state-space basis. Let the storage function in the balanced state-space basis be* $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. *Then* $K = I_{\mathtt{n}}$.

*Proof:* Let the observability and controllability Gramian in the balanced state-space basis be $W_o$ and $W_r$, respectively. By the definition of balanced state-space basis, we have $W_o = W_r = W$. Since $G(s)$ is allpass, using Proposition 7.5, we have $W_oW_r = I_{\mathtt{n}} \implies W^2 = I_{\mathtt{n}}$. Further, since $G(s)$ is stable, we have $W > 0$. Therefore, $W = I_{\mathtt{n}}$. Thus, using Theorem 7.34, we conclude that the storage function of the allpass system is $I_{\mathtt{n}}$. ■

Now that we have established a few of the properties of the storage function of an allpass system, we establish the link between the storage function of a lossless system and an allpass system. In order to do so, we first show the link between storage functions of a bounded-real system and its passive counterpart.

Storage function of a bounded-real system and its passive counterpart is the same

**Theorem 7.36.** *Consider a controllable, bounded-real system* $\Sigma_{\mathtt{br}}$ *with input-output variables* $(u,y)$. *Suppose* $\Sigma_{\mathtt{pas}}$ *is the passive counterpart of* $\Sigma_{\mathtt{br}}$. *Then, the set of storage functions of* $\Sigma_{\mathtt{br}}$ *and* $\Sigma_{\mathtt{pas}}$ *remains invariant.*

*Proof:* Since $\Sigma_{\mathtt{pas}}$ is the passive counterpart of $\Sigma_{\mathtt{br}}$, the input and output variables of $\Sigma_{\mathtt{pas}}$ are $v := \frac{u+y}{\sqrt{2}}$ and $r := \frac{u-y}{\sqrt{2}}$, respectively. Note that for controllable systems the storage function is $x^TKx$, where $x$ is the state-vector of the system $\Sigma_{\mathtt{br}}$ and $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Hence, we have

$$\frac{d}{dt}(x^TKx) \leqslant \begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u \\ y \end{bmatrix}^T J^T \begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{p}} & 0 \end{bmatrix} J \begin{bmatrix} u \\ y \end{bmatrix}, \text{ where } J := \begin{bmatrix} I_{\mathtt{p}} & I_{\mathtt{p}} \\ I_{\mathtt{p}} & -I_{\mathtt{p}} \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}(x^TKx) \leqslant \begin{bmatrix} \frac{u+y}{\sqrt{2}} \\ \frac{u-y}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{p}} & 0 \end{bmatrix} \begin{bmatrix} \frac{u+y}{\sqrt{2}} \\ \frac{u-y}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} v \\ r \end{bmatrix}^T \begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{p}} & 0 \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}. \tag{7.53}$$

Inequality (7.53) is the dissipation inequality of the system $\Sigma_{\mathtt{pas}}$. Hence, $x^TKx$ is a storage function of the system $\Sigma_{\mathtt{pas}}$. Therefore, $K$ is a storage function of $\Sigma_{\mathtt{pas}}$. Along similar lines, it can be shown that any $K$ that induces a storage function for $\Sigma_{\mathtt{pas}}$ is a storage function of $\Sigma_{\mathtt{br}}$, as well. This proves that the set of storage function for $\Sigma_{\mathtt{br}}$ and $\Sigma_{\mathtt{pas}}$ remains invariant. ■

Note that the passive counterpart of an allpass system $\Sigma_{\texttt{all}}$ with transfer function $G(s)$ is a lossless system $\Sigma_{\texttt{loss}}$ with transfer function $[1 - G(s)][1 + G(s)]^{-1}$. Hence, we call $\Sigma_{\texttt{loss}}$ the *lossless counterpart* of $\Sigma_{\texttt{all}}$ and $\Sigma_{\texttt{all}}$ the *allpass counterpart* of $\Sigma_{\texttt{loss}}$. The next corollary relates the storage function of an allpass system and its lossless counterpart.

Observability Gramian induces the storage function of an allpass/lossless system

**Corollary 7.37.** *Consider a lossless system $\Sigma_{\texttt{loss}}$ with a minimal i/s/o representation as in equation* (7.1). *Let the bounded-real counterpart of $\Sigma_{\texttt{loss}}$ be $\Sigma_{\texttt{all}}$. Assume $Q$ to be the observability Gramian of $\Sigma_{\texttt{all}}$ corresponding to the minimal i/s/o representation* (7.12). *Then, $x^T Qx$ is the unique storage function of $\Sigma_{\texttt{loss}}$.*

*Proof:* From Theorem 7.34, we know that the observability Gramian $Q$ induces the storage function of $\Sigma_{\texttt{all}}$. Since $\Sigma_{\texttt{loss}}$ is the lossless counterpart of $\Sigma_{\texttt{all}}$, from Theorem 7.36 we know that the storage function of $\Sigma_{\texttt{all}}$ and $\Sigma_{\texttt{loss}}$ are the same. Therefore, $x^T Qx$ is the unique storage function of $\Sigma_{\texttt{loss}}$. ■

Thus, we have a method to compute storage function of a lossless system using Gramian. We revisit the example in Example 7.9 to illustrate this method.

**Example 7.38.** *For the transfer function $G(s)$ in Example 7.9, the allpass counterpart has the following transfer function $\frac{1-G(s)}{1+G(s)} = \frac{6s^3-8s^2+s-1}{6s^3+8s^2+s+1}$. Corresponding to the i/s/o representation of the lossless system in equation* (7.16)*, the minimal i/s/o representation of its allpass counterpart computed using equation* (7.12) *and the fact that $D = 0$ is given by the following equation:*

$$\frac{d}{dt}x = (A - BC)x + \sqrt{2}B\left(\frac{u+y}{\sqrt{2}}\right), \text{ and } \left(\frac{u-y}{\sqrt{2}}\right) = -\sqrt{2}Cx, \text{ i.e. },$$

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{8}{6} \end{bmatrix} x + \sqrt{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\left(\frac{u+y}{\sqrt{2}}\right), \left(\frac{u-y}{\sqrt{2}}\right) = -\sqrt{2}\begin{bmatrix} -\frac{1}{6} & 0 & -\frac{8}{6} \end{bmatrix} x + \left(\frac{u+y}{\sqrt{2}}\right).$$

*The observability Gramian for the system in the equation above is*

$$Q = \frac{1}{18}\begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & 0 \\ 6 & 0 & 48 \end{bmatrix}$$

*It can be easily verified that $Q$ is indeed the storage function of the lossless system, since it satisfies the KYP LME* (7.41).

Therefore, the algorithm to compute storage function of a lossless system using Gramian is as follows:

**Algorithm 7.39** Gramian balancing method based algorithm.

**Input:** A minimal i/s/o representation $(A, B, C)$ of the lossless system.

**Output:** $K \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ with $x^T K x$ the storage function.

1: Compute $A_{\mathtt{br}} := A - BC$, $B_{\mathtt{br}} = \sqrt{2}B$ and $C_{\mathtt{br}} = -\sqrt{2}C$.

2: Solve $A_{\mathtt{br}}^T Q + Q A_{\mathtt{br}} + C_{\mathtt{br}}^T C_{\mathtt{br}} = 0$ to find the observability Gramian $Q \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$.

3: The storage function of the system is $K = Q$.

## 7.10 Comparison of the methods for computational time and numerical error

Using the experimental setup and procedure described in Section 7.7.4, we compare the computational performance of the Algorithms 7.10, 7.16, 7.19, 7.31, and 7.39 in this section.

**Computation time:** The plot in Figure 7.7 shows the time taken by each algorithm to compute the matrix $K$ for lossless systems of different orders. The controllability matrix method, the Bezoutian method, and the Gramian method take relatively less computation time compared to MPB method.

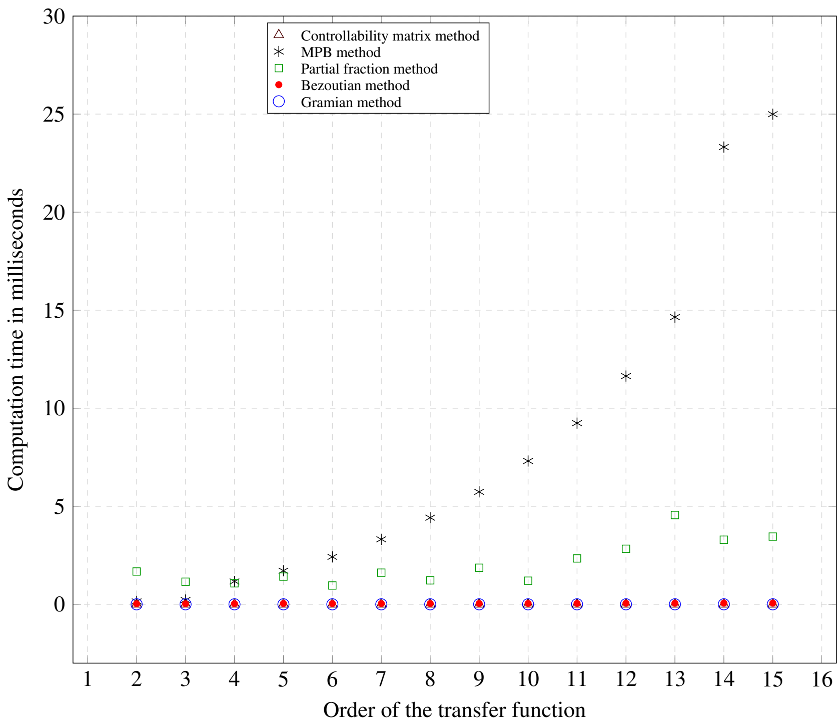



Figure 7.7: Plot of computation time versus system's order.

**Computation error:** Error in $K$ is computed using equation (7.52) and is plotted for comparison. We calculate $\mathrm{Err}(K)$ for test cases used above for computation time. Figure 7.8 shows a comparison of the error associated in the computation of $K$ using the five algorithms presented

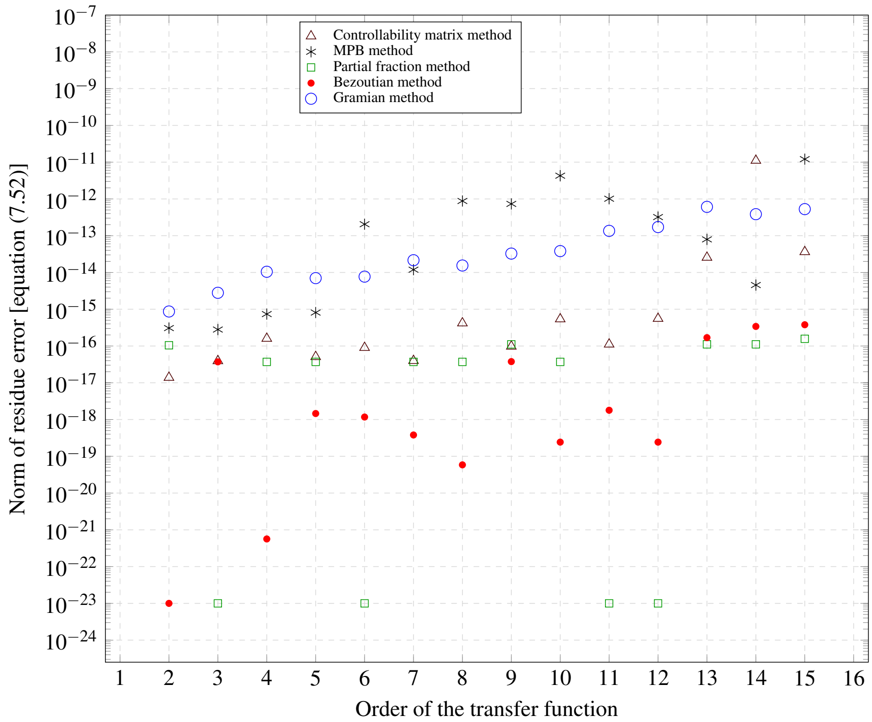



Figure 7.8: Plot of error residue versus system's order.

in this chapter. The error has been plotted in the logarithmic scale for better comparison of data. From the plot we infer that Bezoutian method and partial fraction method is marginally better than the other three methods.

From the plots it is evident that the MPB method is not efficient in spite of using efficient and stable algorithm for the computation of nullspace basis of a polynomial matrix. A probable reason for this is as follows. The MPB method is based on finding minimal polynomial basis of the polynomial matrix $R(s)$. The algorithm of finding the minimal polynomial basis, as reported in [KPB10], is an iterative algorithm and is based on writing the matrix $R(s)$ as $\sum_{i=0}^{\mathtt{d}} R_i s^i$ and then using the co-efficient matrices $R_i$ to form Toeplitz matrices at each iteration. Consider the matrices $R_i$ have size $\mathtt{N} \times \mathtt{N}$, rank col$\,(R_0, R_1, \ldots, R_{\mathtt{d}}) =: \mathtt{r}_0$ and the iteration step is $t$ then the Toeplitz matrix will have a size $(\mathtt{d}+1+t)\mathtt{r}_0 \times (\mathtt{r}_0 + \mathtt{r}_{t-1})$. At each iterations, singular value decomposition (SVD) of such augmented matrices needs to be computed to find the minimal polynomial basis of $R(s)$. Hence, the algorithm being iterative and the large size of the augmented matrix results in more error and computation time. Further, the operation of finding minimal basis is done twice in Algorithm 7.4 and this also adds to the error and computation time.

## 7.11 Summary

In this chapter we dealt with the computation of the stored energy in a lossless system. We presented five different conceptual methods to compute the unique storage function for a lossless system.

1. *Controllability matrix method*: In this method, we showed that the algorithm to compute storage function of a singularly passive SISO system (Algorithm 5.16) can be used to compute the storage function of a lossless system as well. This method is based on the inversion of the controllability matrix of the lossless system.

2. *Minimal polynomial basis (MPB) method*: In this method, we showed that for a lossless system the states and costates of a system satisfies the algebraic relation $z = Kx$, where $K$ is the storage function of the lossless system (Theorem 7.11). Using these algebraic relations, we developed an algorithm to compute the storage function of a lossless system (Theorem 7.14 and Algorithm 7.16).

3. *Partial fraction method*: This uses Foster/Cauer method (Theorem 7.17 and Theorem 7.23) and capacitor voltages & inductor currents as states.

4. *Bezoutian method*: (Theorem 7.26) States corresponding to controller canonical form are used in this method. Three different techniques are presented to compute the Bezoutian of such systems, viz., Euclidean long division, Pseudo-inverse, and 2D discrete Fourier transform.

5. *Controllable/Observable Gramians balancing method*: (Theorem 7.35). In this method we showed that the observability Gramian of the allpass counterpart of a lossless system is the storage function of the lossless system (Theorem 7.35).

From the comparisons among the different methods, based on computational time and computational error, it is evident that the Bezoutian, partial fraction and controllability matrix method are better. Note that in this chapter we did not perform a thorough investigation of the algorithmic aspects of the methods presented. This is a future direction of work that arises out of this chapter. However, we indicate our preliminary observations in this regards next.

The choice of the storage function computation method for a lossless system depends on the system description: for example transfer function or state-space. Loosely speaking, a few of the key factors that help in the choice of the algorithm are

1. Ability to diagonalize the system matrix $A$ using a well-conditioned matrix (i.e., the so-called 'departure from normality')[7].

2. Extents of uncontrollability/unobservability.

[7] A matrix $A \in \mathbb{R}^{n \times n}$ is called normal if $AA^T = A^TA$.

3. McMillan degrees of the elements in the transfer matrix.

*Partial fraction expansion algorithm* is about 'summing' over terms, this algorithm is favourable for a system whose transfer function is obtained as a 'sum'-of-parallel blocks (see [Kai80, Section 2.1.3]). Further, for a system whose system matrix $A$ is normal the similarity transform matrix $S$ that diagonalizes $A$ is well-conditioned (see [Loi69] and also [GL12, P7.2.3]). Hence for such systems, use of partial fraction based method is suitable. *Bezoutian algorithm* is best suited for systems whose matrix $A$ is non-diagonalizable: this is due to non-diagonalizability being linked to a *chain-of-'integrators'* type of interpretation. Hence systems with controller canonical forms and with $A$ matrices not diagonalizable are candidates for this algorithm. *Controllability matrix method* uses inversion of the controllability matrix for computation of the storage function. Hence this method is not favourable for systems which are "nearly uncontrollable" as this will result in inversion of an ill-conditioned matrix. The *MPB algorithm* is favourable for systems where the McMillan degree of the system is much higher than the degrees of the denominators of the transfer matrix of the system (this is especially relevant, in general, for MIMO systems). For such systems, the nullspace of the matrix $R(s)$ in Algorithm 7.4 will have a smaller degree and this will result in less computational effort and less error in the computation of $K$. Hence given a MIMO system realization which has neither the sum-of-parallel blocks form nor the controller canonical form, the MPB method is favourable.

# Chapter 8

# Conclusion and future work

In this chapter we summarize the contributions of this thesis and present a few future directions of the results arising out of the thesis.

## 8.1 Contributions of the thesis

In this thesis we developed a generalized Riccati theory using the notion of Hamiltonian systems. This led to interesting system-theoretic interpretations in terms of the LQR LMI and KYP LMI solutions. The following is a list of our findings in this context.

### 8.1.1 Singular LQR problems

The contributions of this thesis in the area of singular LQR problems are as follows:
**Algorithm to compute the maximal solution of an LQR LMI:** We provided a method to compute the maximal solution of the LQR LMI corresponding to a singular LQR problem with the underlying system admitting a single-input (Theorem 2.30). We showed that the maximal solution minimizes the rank of the LQR LMI, as well. To this end, we characterized the fast (strongly reachable) and slow (weakly unobservable) subspaces of a single-input system in terms of its Rosenbrock matrix (Theorems 2.24 and 2.25). We showed that in order to compute the maximal solution of an LQR LMI we need to use the basis of the good subspace of the weakly unobservable space corresponding to a Hamiltonian system and the basis of a suitably chosen subspace of the strongly reachable subspace of the Hamiltonian system.

**Characterization of the optimal trajectories of an LQR problem:** Using the maximal solution of an LQR LMI, we characterized the optimal trajectories of a single-input system corresponding to a singular LQR problem (Lemma 3.6, Lemma 3.7 and Theorem 3.9). Such a characterization revealed that a singular LQR problem not only admits exponential optimal trajectories but also impulsive ones.

**Static state-feedback controllers generically cannot solve a singular LQR problem:** It is known in the literature that a singular LQR problem admits a static state-feedback controller if and only if it admits a solution to the corresponding CGCARE. Hence, we formulated a few necessary and sufficient conditions for the solvability of CGCARE (Theorem 4.8). Using these conditions, we inferred that a CGCARE corresponding to a singular LQR problem is generically unsolvable (Theorem 4.22). Hence, we showed that a singular LQR problem generically cannot be solved using a static state-feedback. This led to the natural question: Is there a feedback policy that solves a singular LQR problem?

**Almost all singular LQR problems admit PD state-feedback controllers:** We showed that almost all singular LQR problems corresponding to single-input systems can indeed be solved using proportional-derivative (PD) state-feedback controllers (Theorem 3.12). We presented a method to design these state-feedback controllers using the method that we developed to compute the maximal solution of an LQR LMI corresponding to a singular LQR problem. The only assumption that we made here is that the Hamiltonian matrix pair does not have any finite eigenvalue on the imaginary axis. This assumption is true for almost all $A, B, C$ matrices.

### 8.1.2 Passive systems

The contributions of this thesis in the area of passivity theory are as follows:
**Algorithm to compute rank-minimizing solutions of a KYP LMI:** We provided an algorithm to compute the rank-minimizing solutions (also known as storage functions) of the KYP LMI corresponding to a singularly passive SISO system (Theorem 5.7). This method is analogous to the one we proposed to compute the maximal solution of the LQR LMI corresponding to a singular LQR problem. Further, we also showed that the method we proposed to compute the rank-minimizing solutions of a KYP LMI corresponding to a singularly passive SISO system can also be used to compute the maximal and minimal solutions of such an LMI (Theorem 6.7).

**Characterization of the lossless trajectories of a singularly passive SISO system:** We characterized the lossless trajectories of a singularly passive SISO system using the method to compute the rank-minimizing solutions of the KYP LMI corresponding to a singularly passive SISO system (Theorem 6.6 and Table 6.1). We showed that if the initial conditions of the system are confined to the space of regular initial conditions, then the lossless trajectories are exponential in nature (Lemma 6.4). On the other hand, if the initial conditions are from the space of irregular initial conditions, then the lossless trajectories are impulsive in nature (Lemma 6.5). It is important to note that the physical interpretation of a fast lossless trajectory is not known. We introduced such trajectories in this thesis in a formal setting only. Nevertheless, similar to the singular LQR case, we presented a method to design state-feedback controllers that confine the system-trajectories to its lossless trajectories (Theorem 6.16).

**Algorithms to compute the storage function of a lossless system:** Apart from singularly passive systems, we also presented algorithms to compute the storage functions of a special class of passive systems called the lossless systems. We presented five different conceptual methods to compute the unique storage function of a lossless system, viz.,

**Controllability matrix method:** This method is based on Algorithm 5.16 used to compute rank-minimizing solutions of the KYP LMI corresponding to a singularly passive SISO system (Theorem 7.8 and Algorithm 7.10).

**MPB method:** In order to develop this method, we showed that a lossless system admit algebraic relations between its states and the corresponding costates (Theorem 7.11). We used these relations to develop an minimal polynomial based (MPB) algorithm to compute the storage function of a lossless system (Theorem 7.14 and Algorithm 7.16).

**Partial fraction method:** We also developed a method to compute the storage function of a lossless system using partial fractions of the transfer function matrix (Theorem 7.17, Algorithm 7.19, Theorem 7.23, and Algorithm 7.24).

**Bezoutian method:** It is known in the literature that the storage function of a lossless SISO system is related to the Bezoutian obtained using the numerator and denominator of the corresponding transfer function. We presented three methods to compute the storage function of lossless systems using the Bezoutian. The first of these methods used Euclidean long division algorithm (Section 7.7.1 and Algorithm 7.31) to compute the Bezoutian and hence the storage function of a lossless system. In the second method, we used the notion of pseudo-inverse to compute the solution to a set of linear equation that eventually led to the storage function of a lossless system (Section 7.7.2). The last method used to compute the storage function of a lossless system using Bezoutian is that of 2D-DFT (Section 7.7.3), albeit for even order system only. For the MIMO case, however we presented a method that is applicable only for the case when all the elements of the transfer matrix are lossless.

**Gramian method:** We showed that the storage function of a lossless system is the same as that of its allpass counterpart. Further, using the fact that the storage function of an allpass system is given by its observability Gramian (Theorem 7.34), we developed a method to compute the storage function of a lossless system using Gramians (Corollary 7.37). Interestingly, the storage function of an allpass system if its i/s/o representation is in a balanced basis is induced by an identity matrix (Theorem 7.35).

From the findings in Part-I and Part-II of the thesis it is evident that there are parallels between the results on singular LQR problems and singularly passive SISO systems; we present a table (Table 8.1) next that highlights these parallels.

| Results | Part-I<br>Singular LQR problems | Part-II<br>Singularly passive system |
|---|---|---|
| Rank-minimizing solution of dissipation LMI (Maximal solution in case of LQR LMI) | Theorem 2.30 | Theorem 5.7 |
| Algebraic relations satisfied by solutions of the dissipation LMI | Lemma 2.37 | Lemma 6.8 |
| Maximal gap of the solutions of the dissipation LMI satisfies a Lyapunov equation | Lemma 2.38 | Lemma 6.9 |
| Slow optimal/lossless trajectories of the system | Lemma 3.7 | Lemma 6.4 |
| Fast optimal/lossless trajectories of the system | Lemma 3.6 | Lemma 6.5 |
| Optimal/lossless trajectories of the system | Theorem 3.9 | Theorem 6.6 |
| PD-controllers for confinement of system trajectories to the optimal/lossless ones | Theorem 3.12 | Theorem 6.16 |

Table 8.1: A table to demonstrate the analogous results in Part-I and Part-II of the thesis.

### 8.1.3 The generalized Riccati theory

From Table 8.1 it is evident that the main results in Part-I and Part-II of the thesis are analogous. All the results in Part-I and Part-II applied to the special case when the feed-through regularity term is satisfied, i.e., when ARE exists, corroborate the existing results on AREs. The results in Part-I and Part-II not only provide a method to compute solutions of the respective dissipation LMIs but also lead to a generalization of certain system-theoretic notions already present in the literature for the regular case. Hence, we have a generalized Riccati theory developed using Hamiltonian systems and its properties. Another class of systems that admit an ARE is the class of bounded-real systems (see Section 7.2.5 for definition). These systems admit a dissipation LMI, called the bounded-real LMI, of the form given in equation (7.9). The feed-through term corresponding to such an LMI is $I-D^TD$. In this thesis we did not explicitly develop the theory for the singular case of such systems, i.e., bounded-real SISO systems with $I-D^TD=0$. However, from Theorem 7.36 it is evident that a bounded-real SISO system with $I-D^TD=0$ can always be transformed to its passive counterpart with $D+D^T=0$. Further, the storage functions of the bounded-real system and the corresponding passive system remains the same during such a transformation (Theorem 7.36). Hence, the theory of singularly passive SISO systems developed in the thesis is directly applicable to their bounded-real SISO counterparts, as well. Further, using the same line of reasoning as presented in Part-II of the thesis and tweaking the matrices $\widehat{A}$, $\widehat{b}$ and $\widehat{c}$ used in Part-II, the theory developed in Part-II can be directly extended to the bounded-real case, as well. We did not explicitly develop this theory in the thesis to avoid repetition. We present a table next that shows the structure of the matrices $\widehat{A}$, $\widehat{b}$ and $\widehat{c}$ corresponding to singular LQR problems, singularly passive system and the singular case of bounded-real systems.

| Problem/System | $\widehat{A}$ | $\widehat{b}$ | $\widehat{c}$ |
|---|---|---|---|
| Singular LQR problem for single-input case | $\begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}$ | $\begin{bmatrix} b \\ 0_{\mathtt{n},1} \end{bmatrix}$ | $\begin{bmatrix} 0 & b^T \end{bmatrix}$ |
| Singularly passive SISO systems | $\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$ | $\begin{bmatrix} b \\ c^T \end{bmatrix}$ | $\begin{bmatrix} c & -b^T \end{bmatrix}$ |
| Bounded-real SISO systems with $I - D^T D = 0$ | $\begin{bmatrix} A & 0 \\ -c^T c & -A^T \end{bmatrix}$ | $\begin{bmatrix} b \\ -c^T \end{bmatrix}$ | $\begin{bmatrix} -c^T & -b^T \end{bmatrix}$ |

## 8.2 Future work

The results presented in this thesis can be extended in different directions. We list a few of these directions next.

**Extending the theory to MIMO systems:** Most of the main results presented in this thesis are for single-input or single-input single-output systems as the case may be. A natural extension of this work would be to extend the results in this thesis to the multi-input or multi-input multi-output case.

**Relaxation of the condition $\sigma(E,H) \cap j\mathbb{R}$:** Most of the results in this thesis assumes that the eigenvalues of Hamiltonian matrix pair are not on the imaginary axis. For an LQR problem such a condition needs to be assumed to guarantee convergence of the performance index. However, for singularly passive systems no such guarantee on convergence is required. Although the condition $\sigma(E,H) \cap j\mathbb{R}$ is generically true for passive systems, relaxing this condition would result in a complete Riccati theory for passive systems.

**Fast lossless trajectories:** The notion of fast lossless trajectories have been introduced in Chapter 6 in a formal setting only. The physical interpretation of such trajectories is a matter of future research.

**Generalized theory of model-order reduction:** In large scale systems, the system dimension makes computational analysis infeasible due to memory and time limitations. One of the approaches present in the literature to mitigate this problem is the notion of model reduction. The goal is to produce a low dimensional system with similar characteristics to the original one such that the computational analysis becomes less expensive. Some of the widely known methods of model order reduction like the stochastic balancing method, bounded real balancing method, etc. uses ARE solutions to obtain a lower order model [Gre88], [Sor05]. Hence, a natural shortcoming of these methods is that they can not be used to reduce the order of a system that does not admit an ARE. Since the theory developed in this thesis deals with such systems, the existing ARE based model order reduction methods can be generalized to systems that do not admit AREs.

Thus, in this thesis, we have presented a complete generalized Riccati theory for single-input LQR problems. Similarly, we have provided a complete generalized Riccati theory for passive systems, with the exception of systems that admit Hamiltonian systems with imaginary axis eigenvalues. We present a figure next that demonstrates the work we completed in this thesis and the knowledge gap still left in the literature after our contribution.

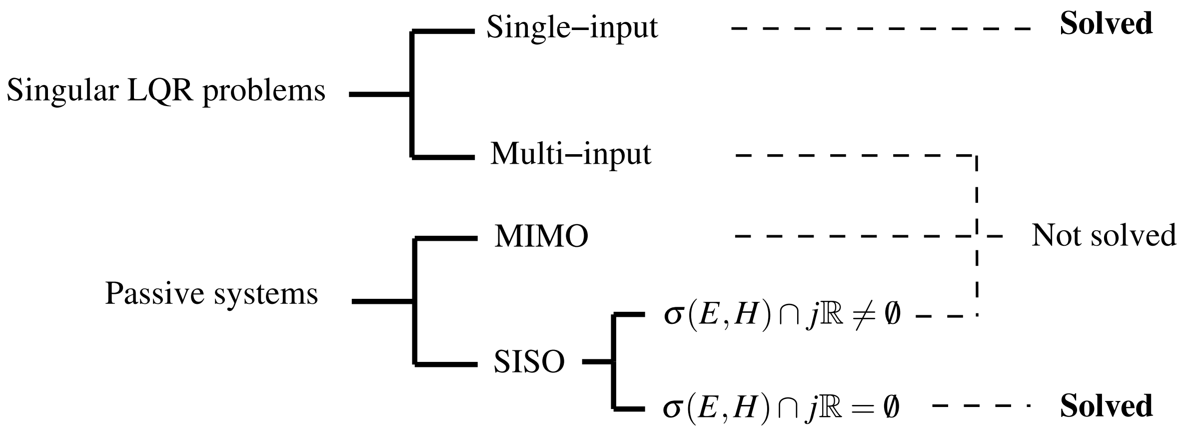



Figure 8.1: Works completed in this thesis and future directions.

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