2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

On solutions of bounded-real LMI for singularly bounded-real systems

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Abstract— **In this paper, we provide an algorithm to compute the solutions of the LMI arising from the bounded-real lemma for a special class of bounded-real systems that do not admit algebraic Riccati equations due to singularity condition in the feedthrough term. We call such systems *singularly bounded-real* systems. We show that unlike strictly bounded-real systems, the solutions of the bounded-real LMI for singularly bounded-real systems can be computed by a suitable arrangement of the controllability and observability matrices of the system. This is intrinsically linked to the Markov parameters and relative degree of the system. Further, we also show that the same algorithm can be used to compute the solutions of the bounded-real LMI corresponding to allpass systems, as well.**

Keywords: Bounded-real lemma, Linear matrix inequality, Storage functions, Allpass systems, Markov parameters.

1. Introduction

One of the most important tools used in the study and design of control systems is the linear matrix inequality (LMI) arising from the bounded-real lemma. Solutions of such an LMI is used in different fields of control namely, H_∞ synthesis problems, H_2 synthesis problems, optimal control, moving average parameter estimation, design of filters, equalization filters and filterbanks, etc. [4, Section 5.9], [6], [7]. The bounded-real lemma states that a system with a minimal input-state-output (i/s/o) representation $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + Du$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$, is bounded-real if and only if there exists a $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$\begin{bmatrix} A^TK + KA + C^TC & KB + C^TD \\ B^TK + D^TC & -(I - D^TD) \end{bmatrix} \leqslant 0. \tag{1}$$

We call this LMI the *bounded-real* LMI. One of the well-known methods to compute solutions of the bounded-real LMI involves finding solutions to a matrix equation of the form:

$$A^TK + KA + C^TC + (KB + C^TD)(I - D^TD)^{-1}(B^TK + D^TC) = 0. \tag{2}$$

Equation (2) is known as the algebraic Riccati equation (ARE). One of the methods to compute solutions of this ARE is by using a Hamiltonian matrix of the form:

$$\mathscr{H} = \begin{bmatrix} A + B(I - D^TD)^{-1}D^TC & B(I - D^TD)^{-1}B^T \\ -C^T(I - DD^T)^{-1}C & -(A + B(I - D^TD)^{-1}D^TC)^T \end{bmatrix} \tag{3}$$

Note that existence of the ARE and the Hamiltonian matrix crucially depends on the nonsingularity of $(I - D^TD)$. Since this arises repeatedly in this paper, we call the nonsingularity of $I - D^TD$ the *feedthrough regularity condition*. In this paper, we deal with a special class of bounded-real systems that do not satisfy the feedthrough regularity condition and therefore do not admit an ARE and its corresponding Hamiltonian matrix. We call this special class of systems the *singularly bounded-real systems*. In this paper, we focus on singularly bounded-real SISO systems, only. A typical example of such a system is the bounded-real counterpart of a parallel RC network (see [3, Section 4] for definition of bounded-real counterpart). The bounded-real LMI (1) customized to singularly bounded-real SISO systems takes the following form:

$$\begin{bmatrix} A^TK + KA + C^TC & KB + C^T \\ B^TK + C & 0 \end{bmatrix} \leqslant 0. \tag{4}$$

We call inequality (4) the *singular bounded-real LMI*. Note that computation of solutions K of LMI (4) is equivalent to solving the linear matrix equation $KB + C^T = 0$ combined with the LMI $A^TK + KA + C^TC \leqslant 0$. Hence solving LMI (4) is equivalent to finding the intersection between two convex sets; the convex sets being the solution set of $KB + C^T = 0$ and that of $A^TK + KA + C^TC \leqslant 0$. Therefore, iterative algorithms like alternating projection based methods can be used to compute solutions of LMI (4): see [9]. One of the known methods to compute solutions to the singular bounded-real LMI (4) for bounded-real systems with transfer function $G(s)$ involves the computation of spectral factors of $I - G(-s)^TG(s)$ [1, Chapter 7], [17]. Another method known in the literature for computation of solutions of the passive counterpart of the bounded-real LMI, i.e., the singular case of KYP LMI uses the notion of neutral deflating subspaces of a matrix pencil [12], [13]. However, the method, we propose in this paper, neither involves iterations nor spectral factorization. Further, our method is devoid of computation of deflating subspaces. We provide a simple closed form solution involving the system matrices (A, B, C, D). The method we propose has close parallel to the already existing method to compute solutions of the bounded-real LMI of a bounded-real system that admits an ARE, although the concepts involved are different.

The paper is structured as follows. The following section contains the notation and preliminaries required for this paper. Section 3 contains the main result of this paper, Theorem 3.1. In this main result, we propose a closed form solution to the singular bounded-real LMI (4) for singularly bounded-real SISO systems. In Section 4, we show that the proposed closed form solution to the bounded-real LMI for singularly bounded-real systems is applicable to allpass systems as well. We present the concluding remarks in Section 5.

2. Notation and Preliminaries

We use symbols $\mathbb{R}$ and $\mathbb{C}$ for the sets of real and complex numbers, respectively. The symbol $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of

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$\mathtt{n} \times \mathtt{p}$ matrices with elements from $\mathbb{R}$. We use $\mathbb{R}[s]$ and $\mathbb{R}(s)$, respectively, for denoting the sets of polynomials and rational functions in one-variable s with coefficients from $\mathbb{R}$. Likewise, we use $\mathbb{R}[s]^{\mathtt{n}\times\mathtt{p}}$ and $\mathbb{R}(s)^{\mathtt{n}\times\mathtt{p}}$ for the sets of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}[s]$ and $\mathbb{R}(s)$, respectively. Symbol $\mathtt{col}(B_1, B_2)$ represents a matrix of the form $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $\mathtt{det}(A)$ represents the determinant of the matrix A. Symbol $\mathtt{img}(A)$ is used to denote the subspace spanned by the columns of matrix A. The set of eigenvalues of a matrix A (counted with multiplicity) is denoted by $\sigma(A)$. Symbol $I_{\mathtt{n}}$ represents the $\mathtt{n}\times\mathtt{n}$ identity matrix. A block diagonal matrix G is represented as $\mathtt{diag}(G_1, \ldots, G_{\mathtt{m}})$, where each of $G_1, \ldots, G_{\mathtt{m}}$ are square matrices of possibly different sizes. Symbol $\mathbf{0}_{\mathtt{n}} \in \mathbb{R}^{\mathtt{n}}$ is used for the vector having all elements equal to zero. Next we give a brief review of various preliminary concepts required for this paper.

A. Bounded-real systems and their storage functions

In this section, we review the definition and properties of bounded-real systems.

Definition 2.1. *Consider a linear system Σ with a minimal i/s/o representation*

$$\frac{d}{dt}x = Ax + Bu \quad \text{and} \quad y = Cx + Du, \tag{5}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$. The system Σ is called bounded-real *if there exists a continuously differentiable*[1] *function $V(x)$ such that*

$$\frac{d}{dt}V(x) \leqslant \left(u^T u - y^T y\right) \tag{6}$$

for all (x, u, y) that satisfy equation (5)*.*

The function $V(x)$ is called a *storage function* of the system Σ. It is shown in [14] that a system Σ is bounded-real if and only if a storage function $V(x)$ exists that is of the form $x^T K x$, where $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Hence, inequality (6) takes the following form:

$$\frac{d}{dt}\left(x^T K x\right) \leqslant \left(u^T u - y^T y\right) \tag{7}$$

for all (x, u, y) that satisfy equation (5). A relevant question here is how to compute a matrix K that induces a storage function of a bounded-real system. Note that inequality (7) together with the i/s/o representation of a bounded-real system results in the bounded-real LMI (1). This means that solutions K of the bounded-real LMI (1) induce storage functions of the system under consideration. Therefore, henceforth, we use the term storage function of a bounded-real system and the term solution matrix of the corresponding bounded-real LMI, interchangeably. Recall that computation of solutions of the bounded-real LMI (1) is done using the corresponding ARE (2) and suitable $\mathtt{n}$-dimensional invariant subspaces of the Hamiltonian matrix (3), provided the systems under consideration satisfy the feedthrough regularity condition. An important class of bounded-real systems that satisfy the feedthrough regularity condition and hence admit ARE are the *strictly bounded-real systems*. We define such systems next.

Definition 2.2. *A bounded-real system Σ with a minimal i/s/o representation as given in equation* (5) *is called* strictly bounded-real *if for some positive definite*[2] *function $\psi(x)$ and $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ we have*

$$\frac{d}{dt}\left(x^T K x\right) + \psi(x) \leqslant \left(u^T u - y^T y\right)$$

for all (x, u, y) that satisfy equation (5)*.*

For easy reference, we present a proposition next that provides the algorithm to compute storage functions of strictly bounded-real systems using their corresponding ARE and Hamiltonian matrix. Before we review the proposition, we need the following definition.

Definition 2.3. *Let the characteristic polynomial of $\mathscr{H}$, as defined in equation* (3)*, be denoted as $\mathscr{X}(s)$. A Lambda-set $\Lambda \subset \sigma(\mathscr{H})$, if it exists, is the set of roots of a polynomial $p(s) \in \mathbb{R}[s]$ such that $\mathscr{X}(s) = p(s)p(-s)$ with $p(s)$ and $p(-s)$ coprime (sets are counted with multiplicity).*

The significance of the main result of this paper is the close parallel with the statements below (see [11], [16]), though the concepts involved in the next proposition are very different.

Proposition 2.4. *Consider a strictly bounded-real system Σ with a minimal i/s/o representation as given in equation* (5) *and let the corresponding Hamiltonian matrix $\mathscr{H}$ be as given in equation* (3)*. Assume $\mathscr{H}$ has no eigenvalues on the imaginary axis and let Λ be a Lambda-set of $\mathtt{det}(sI - \mathscr{H})$. Suppose the $\mathtt{n}$-dimensional $\mathscr{H}$-invariant subspace corresponding to Λ is given by*

$$\mathscr{V} := \mathtt{img}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad V_1, V_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}. \tag{8}$$

Then, the following statements hold.

1) V_1 is invertible.

2) $K := V_2 V_1^{-1}$ is symmetric.

3) K is a solution to the ARE:
$A^T K + KA + C^T C + (KB + C^T D)(I - D^T D)^{-1}(B^T K + D^T C) = 0.$

4) $x^T K x$ is a storage function of the system Σ, i.e., $\frac{d}{dt}\left(x^T K x\right) \leqslant \left(u^T u - y^T y\right)$ for all (x, u, y) that satisfy equations (5)*.*

Interestingly, a necessary and sufficient condition for a system with a transfer function $G(s) \in \mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$ to be bounded-real is

$$I - G(-i\omega)^T G(i\omega) \geqslant 0 \quad \text{for all } \omega \in \mathbb{R}. \tag{9}$$

This property of bounded-real systems is crucially used to derive the main result of this paper: see [1, Section 2.6] for

[1] In this paper, since we focus only on fast solutions, we do not dwell on stability and therefore, we relax non-negativity of $V(x)$: link with stability can be seen in [17, Theorem 6.3].

[2] A function $\psi : \mathbb{D} \mapsto \mathbb{R}$, where $\mathbb{D} \subseteq \mathbb{R}^{\mathtt{n}}$, is said to be a *positive definite function* if $\psi(0) = 0$ and $\psi(x) > 0$ for all nonzero $x \in \mathbb{D}$.

more on this condition.

Now that we have reviewed the properties of bounded-real systems, in the next section we define the systems of interest in this paper, i.e., *singularly* bounded-real SISO systems.

B. Singularly bounded-real SISO systems

In this section, we define singularly bounded-real SISO systems and discuss why Proposition 2.4 does not work for computing the storage functions of such systems.

Definition 2.5. *A bounded-real SISO system with transfer function $G(s) \in \mathbb{R}(s)$ is called* singularly bounded-real *if the numerator of $I - G(-s)G(s)$ is a nonzero constant.*

From Definition 2.5, it is clear that a necessary condition for a bounded-real SISO system Σ to be singularly bounded-real is $D = \pm 1$. Without loss of generality, we assume $D = 1$ throughout the paper. Note that $D = 1$ implies that singularly bounded-real SISO systems do not admit an ARE as they do not satisfy the feedthrough regularity condition. However they admit the singular bounded-real LMI (4). Nonexistence of the Hamiltonian matrix $\mathscr{H}$ for singularly bounded-real SISO systems mean that Proposition 2.4 can no longer be used to compute the storage functions of such systems. One can argue, to avoid using the Hamiltonian matrix $\mathscr{H}$ and instead find Lambda-set corresponding to the matrix pencil (E, H), where

$$E = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } H = \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix}.$$

This method of finding storage functions was introduced in [5] and does not involve inversion of $I - D^T D$. However, this method does not work for singularly bounded-real systems, since the Lambda-sets for such systems are empty, i.e., there is deficiency of suitable $\mathtt{n}$-dimensional invariant subspaces. In this paper, we show that the absence of suitable $\mathtt{n}$-dimensional invariant subspace of the Hamiltonian matrix $\mathscr{H}$ for singularly bounded-real SISO systems is compensated by a convenient arrangement of controllability and observability matrices. Further, the main result is in close parallel to Proposition 2.4. Next we list down some properties of singularly bounded-real SISO systems that distinguishes them from strictly bounded-real SISO systems.

For a singularly bounded-real SISO system Σ, the following statements hold.

1) The system Σ does not satisfy the feedthrough regularity condition and hence does not admit an ARE.

2) The system Σ has no spectral zeros[3].

3) Storage function is unique[4].

[3]The spectral zeros of a bounded-real SISO system with transfer function $G(s)$ are the roots of the numerator $I - G(-s)^T G(s)$.

[4]Since the numerator of $I - G(-s)^T G(s)$ for a singularly bounded-real SISO system is a nonzero constant, such a system admits unique spectral factorization (see [1, Section 5.2], [17, Proposition 5.6] for uniqueness of spectral factorization in this case). Hence, a singularly bounded-real SISO system admits a unique storage function.

On the contrary, for a strictly bounded-real SISO system Σ, the following statements hold.

1) The system Σ satisfies the feedthrough regularity condition and therefore admits an ARE.

2) The degree of the numerator of $I - G(-s)^T G(s)$ is $2\mathtt{n}$, i.e., the system Σ has $2\mathtt{n}$ spectral zeros.

3) Storage functions are in general non-unique.

3. Storage functions of singularly bounded-real SISO systems

We now state and prove our main result. The main result provides closed form solutions of the singular bounded-real LMI for singularly bounded-real SISO systems. Note the close parallel with Proposition 2.4. This is the main result in this paper.

Theorem 3.1. *Consider a singularly bounded-real SISO system Σ with a minimal i/s/o representation $\frac{d}{dt}x = Ax + Bu, y = Cx + u$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B, C^T \in \mathbb{R}^{\mathtt{n}}$. Define*

$$\widehat{A} := \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \text{ and } \widehat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}.$$

Suppose $W := \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \dots & \widehat{A}^{\mathtt{n}-1}\widehat{B} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$. Define

$$W =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \text{ where } X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$$

Then, the following statements hold.

1) *X_1 is invertible.*

2) *$K := X_2 X_1^{-1}$ is symmetric.*

3) *$KB + C^T = 0$ and $A^T K + KA + C^T C \leqslant 0$.*

4) *$x^T K x$ is the storage function of Σ.*

Proof of Statement 1 of Theorem 3.1:
Note that $X_1 = \begin{bmatrix} B & AB & A^2 B & \cdots & A^{\mathtt{n}-1}B \end{bmatrix}$ Since, the system is controllable, X_1 is invertible. □

Next we prove that $K := X_2 X_1^{-1}$ is a symmetric matrix. In other words, we need to show that $X_2 X_1^{-1} = (X_2 X_1^{-1})^T$, i.e., $X_1^T X_2 = X_2^T X_1$. The next few lemmata are crucially used for the proof of Statement 2 and 3 of Theorem 3.1.

Lemma 3.2. *Consider a singularly bounded-real SISO system Σ with a minimal i/s/o representation $\frac{d}{dt}x = Ax + Bu$, $y = Cx + u$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B, C^T \in \mathbb{R}^{\mathtt{n}}$. Define*

$$\widehat{A} := \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix}, \widehat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix} \text{ and } \widehat{C} := -\begin{bmatrix} C & B^T \end{bmatrix}.$$

Then, $$\widehat{C}\widehat{A}^k\widehat{B} = 0 \text{ for } k \in \{0, 1, 2, \dots, 2\mathtt{n} - 2\}. \quad (10)$$

Proof. The transfer function for the system Σ is $G(s) := I + C(sI - A)^{-1}B$. Therefore, $G(-s) = G(-s)^T = I - B^T(sI +$

$A^T)^{-1}C^T$. Note that

$$\widehat{C}\left(sI_{2\mathtt{n}}-\widehat{A}\right)^{-1}\widehat{B} = -\begin{bmatrix}C & B^T\end{bmatrix}\begin{bmatrix}(sI-A) & 0\\ C^TC & (sI+A^T)\end{bmatrix}^{-1}\begin{bmatrix}B\\ -C^T\end{bmatrix}$$
$$= -\begin{bmatrix}C^T\\ B\end{bmatrix}^T\begin{bmatrix}(sI-A)^{-1} & 0\\ -(sI+A^T)^{-1}C^TC(sI-A)^{-1} & (sI+A^T)^{-1}\end{bmatrix}\begin{bmatrix}B\\ -C^T\end{bmatrix}$$
$$= I-\left(I-B^T(sI+A^T)^{-1}C^T\right)\left(I+C(sI-A)^{-1}B\right)$$
$$= I-G(-s)^TG(s). \tag{11}$$

Define $H(s) := I-G(-s)^TG(s)$. Recall Σ being singularly bounded-real means that the relative degree[5] of $I-G(-s)^TG(s)$ is $2\mathtt{n}$. Therefore, using equation (11), we have

$$\lim_{s\to\infty} s^jH(s)=0 \implies \lim_{s\to\infty} s^j\widehat{C}\left(sI_{2\mathtt{n}}-\widehat{A}\right)^{-1}\widehat{B}=0$$

for $j\in\{1,2,\cdots,2\mathtt{n}-1\}$. Expanding $(sI_{2\mathtt{n}}-\widehat{A})^{-1}$ about $s=\infty$, we have

$$\lim_{s\to\infty} s^j\sum_{k=0}^{\infty}\left(\frac{1}{s^{k+1}}\widehat{C}\widehat{A}^k\widehat{B}\right)=0 \text{ for } j\in\{1,\cdots,2\mathtt{n}-1\}. \tag{12}$$

Therefore from equation (12), we have $\widehat{C}\widehat{A}^k\widehat{B}=0$ for $k\in\{0,1,2,\ldots,2\mathtt{n}-2\}$. $\square$

Note that $\widehat{C}\widehat{A}^k\widehat{B}$ are nothing but the Markov parameters of a system with system matrices $\left(\widehat{A},\widehat{B},\widehat{C}\right)$. Such a system can be formed by a suitable interconnection of the system (A,B,C) with transfer function $G(s)$ and its dual system (adjoint system): see [10], [18], [2, Section VI] for more on dual systems and its interconnection to primal systems.

Next we present a simple matrix result that is crucially used to prove Theorem 3.1.

Lemma 3.3. *Consider*

$$\widehat{A}:=\begin{bmatrix}A & 0\\ -C^TC & -A^T\end{bmatrix}, \widehat{B}:=\begin{bmatrix}B\\ -C^T\end{bmatrix} \text{ and } \widehat{C}:=-\begin{bmatrix}C & B^T\end{bmatrix},$$

where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B,C^T\in\mathbb{R}^{\mathtt{n}}$. Define

$$W:=\begin{bmatrix}\widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n}-1}\widehat{B}\end{bmatrix} \text{ and } J:=\begin{bmatrix}0 & -I_{\mathtt{n}}\\ I_{\mathtt{n}} & 0\end{bmatrix}\in\mathbb{R}^{2\mathtt{n}\times 2\mathtt{n}}.$$

Then, $$W^TJ=\mathtt{col}\left(\widehat{C},-\widehat{C}\widehat{A},\ldots,(-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}-1}\right). \tag{13}$$

Proof. We claim that $(\widehat{A}^k\widehat{B})^TJ=(-1)^k\widehat{C}\widehat{A}^k$ for $k\in\{0,1,2,\ldots,\mathtt{n}-1\}$. To prove this we use induction.

Base step: For $k=0$, $\widehat{B}^TJ=-\begin{bmatrix}C & B^T\end{bmatrix}=\widehat{C}$.
Induction step: Let $(\widehat{A}^k\widehat{B})^TJ=(-1)^k\widehat{C}\widehat{A}^k$.
We prove that $(\widehat{A}^{k+1}\widehat{B})^TJ=(-1)^{k+1}\widehat{C}\widehat{A}^{k+1}$.

$$\left(\widehat{A}^{k+1}\widehat{B}\right)^TJ=\left(\widehat{A}^k\widehat{B}\right)^T\widehat{A}^TJ=\left((-1)^k\widehat{C}\widehat{A}^kJ^{-1}\right)\widehat{A}^TJ. \tag{14}$$

Note that

$$J^{-1}\widehat{A}^TJ=\begin{bmatrix}0 & I_{\mathtt{n}}\\ -I_{\mathtt{n}} & 0\end{bmatrix}\begin{bmatrix}A^T & -C^TC\\ 0 & -A\end{bmatrix}\begin{bmatrix}0 & -I_{\mathtt{n}}\\ I_{\mathtt{n}} & 0\end{bmatrix}=-\widehat{A}. \tag{15}$$

Using equation (15) in equation (14), we have

$$\left(\widehat{A}^{k+1}\widehat{B}\right)^TJ=(-1)^k\left(\widehat{C}\widehat{A}^k\right)\left(-\widehat{A}\right)=(-1)^{k+1}\widehat{C}\widehat{A}^{k+1}.$$

[5]Relative degree of a rational polynomial $\frac{n(s)}{d(s)}$ is $\mathtt{deg}\,(d(s))-\mathtt{deg}\,(n(s))$.

Therefore, using induction we infer that

$$(\widehat{A}^k\widehat{B})^TJ=(-1)^k\widehat{C}\widehat{A}^k \text{ for } k\in\{0,1,2,\ldots,\mathtt{n}-1\}. \tag{16}$$

Writing equation (16) in matrix form we have $W^TJ=\mathtt{col}\left(\widehat{C},-\widehat{C}\widehat{A},\ldots,(-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}-1}\right)$. $\square$

Using Lemma 3.2 and Lemma 3.3, we prove Statement 2 of Theorem 3.1 next. Recall that proving K is symmetric is equivalent to proving $X_1^TX_2-X_2^TX_1=0$.

Proof of Statement 2 of Theorem 3.1:
Note that $X_1^TX_2-X_2^TX_1=\begin{bmatrix}X_1^T & X_2^T\end{bmatrix}\begin{bmatrix}0 & I_{\mathtt{n}}\\ -I_{\mathtt{n}} & 0\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}$. Recall from Lemma 3.3: $J=\begin{bmatrix}0 & -I_{\mathtt{n}}\\ I_{\mathtt{n}} & 0\end{bmatrix}$ and $\begin{bmatrix}X_1\\ X_2\end{bmatrix}=W$. Therefore, using Lemma 3.3, we have $X_1^TX_2-X_2^TX_1=$

$$-W^TJW=-\mathtt{col}\left(\widehat{C},-\widehat{C}\widehat{A},\ldots,(-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}-1}\right)W.$$

Note that using Lemma 3.2 we can infer that

$$\begin{bmatrix}\widehat{C}\\ -\widehat{C}\widehat{A}\\ \vdots\\ (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}-1}\end{bmatrix}W=\begin{bmatrix}\widehat{C}\\ -\widehat{C}\widehat{A}\\ \vdots\\ (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}-1}\end{bmatrix}\begin{bmatrix}\widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n}-1}\widehat{B}\end{bmatrix}.$$
$$=\begin{bmatrix}\widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \cdots & \widehat{C}\widehat{A}^{\mathtt{n}-1}\widehat{B}\\ -\widehat{C}\widehat{A}\widehat{B} & -\widehat{C}\widehat{A}^2\widehat{B} & \cdots & -\widehat{C}\widehat{A}^{\mathtt{n}}\widehat{B}\\ \vdots & \vdots & \ddots & \vdots\\ (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}-1}\widehat{B} & (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}}\widehat{B} & \cdots & (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{2\mathtt{n}-2}\widehat{B}\end{bmatrix}$$
$$=0.$$

Therefore, $W^TJW=0$. Thus, $X_1^TX_2-X_2^TX_1^T=0$, i.e., $X_2X_1^{-1}=(X_2X_1^{-1})^T$. Therefore, K is symmetric. $\square$

Next we prove Statement 3 of Theorem 3.1.

Proof of Statement 3 of Theorem 3.1:
First we prove that $KB+C^T=0$.

$$KB+C^T=X_2X_1^{-1}B+C^T=X_2\begin{bmatrix}B & AB & \cdots & A^{\mathtt{n}-1}B\end{bmatrix}^{-1}B+C^T$$
$$=X_2\mathtt{col}\,(1,\mathbf{0}_{\mathtt{n}-1})+C^T=-C^T+C^T=0.$$

Next we prove that $A^TK+KA+C^TC\leqslant 0$. To prove this we first prove that $X_1^T(A^TK+KA+C^TC)X_1\leqslant 0$. Note that

$$X_1^T(A^TK+KA+C^TC)X_1$$
$$=X_1^TA^T(X_2X_1^{-1})X_1+X_1^T(X_1^T)^{-1}X_2^TAX_1+X_1^TC^TCX_1$$
$$=X_1^TA^TX_2+X_2^TAX_1+X_1^TC^TCX_1$$
$$=\begin{bmatrix}X_1\\ X_2\end{bmatrix}^T\begin{bmatrix}0 & -I_{\mathtt{n}}\\ I_{\mathtt{n}} & 0\end{bmatrix}\begin{bmatrix}A & 0\\ -C^TC & -A^T\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}. \tag{17}$$

Recall $J=\begin{bmatrix}0 & -I_{\mathtt{n}}\\ I_{\mathtt{n}} & 0\end{bmatrix}$ from Lemma 3.3. Therefore, from equation (17) we have $X_1^T(A^TK+KA+C^TC)X_1=W^TJ\widehat{A}W$.
Using Lemma 3.3, we have

$$W^TJ\widehat{A}W=\begin{bmatrix}\widehat{C}^T & -(\widehat{C}\widehat{A})^T\cdots & (-1)^{\mathtt{n}-1}(\widehat{C}\widehat{A}^{\mathtt{n}-1})^T\end{bmatrix}\widehat{A}\begin{bmatrix}\widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n}-1}\widehat{B}\end{bmatrix}$$

$$=\begin{bmatrix}\widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{A}^2\widehat{B} & \cdots & \widehat{C}\widehat{A}^{\mathtt{n}}\widehat{B}\\ -\widehat{C}\widehat{A}^2\widehat{B} & -\widehat{C}\widehat{A}^3\widehat{B} & \cdots & -\widehat{C}\widehat{A}^{\mathtt{n}+1}\widehat{B}\\ \vdots & \vdots & \ddots & \vdots\\ (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}}\widehat{B} & (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{\mathtt{n}+1}\widehat{B} & \cdots & (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B}\end{bmatrix}. \tag{18}$$

Now using Lemma 3.2 in equation (18), we infer that

$$X_1^T(A^TK+KA+C^TC)X_1 = \begin{bmatrix}0 & 0\\ 0 & (-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B}\end{bmatrix}. \quad (19)$$

We claim that $(-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B} \leqslant 0$. Note that using equation (11), we rewrite $\omega^{2\mathtt{n}}\left(I-G(-i\omega)^TG(i\omega)\right)$ as

$$\omega^{2\mathtt{n}}\left(I-G(-i\omega)^TG(i\omega)\right) = \omega^{2\mathtt{n}}\widehat{C}(i\omega-\widehat{A})^{-1}\widehat{B}.$$

Expanding the $\widehat{C}(i\omega-\widehat{A})^{-1}\widehat{B}$ about $\omega=\infty$, we have

$$\omega^{2\mathtt{n}}\widehat{C}(i\omega-\widehat{A})^{-1}\widehat{B} = (-1)^{\mathtt{n}}(i\omega)^{2\mathtt{n}}\sum_{j=0}^{\infty}\frac{1}{(i\omega)^{j+1}}\widehat{C}\widehat{A}^j\widehat{B} \quad (20)$$

From Lemma 3.2, we know that the first $(2\mathtt{n}-1)$ summands in equation (20) are zero. Therefore, we rewrite the equation (20) as

$$\omega^{2\mathtt{n}}\left(I-G(-i\omega)^TG(i\omega)\right) = (-1)^{\mathtt{n}}(i\omega)^{2\mathtt{n}}\sum_{j=2\mathtt{n}-1}^{\infty}\frac{1}{(i\omega)^{j+1}}\widehat{C}\widehat{A}^j\widehat{B} \quad (21)$$

Note that the relative degree of each of the summands in equation (21) are nonnegative. Hence, we infer that

$$\lim_{\omega\to\infty}\omega^{2\mathtt{n}}\left(I-G(-i\omega)^TG(i\omega)\right) < \infty \quad (22)$$

Further, singularly bounded-real SISO systems being bounded-real satisfies $I-G(-i\omega)^TG(i\omega)\geqslant 0$ for all $\omega\in\mathbb{R}$ (see equation (9)). Therefore,

$$\lim_{\omega\to\infty}\omega^{2\mathtt{n}}\left(I-G(-i\omega)^TG(i\omega)\right) \geqslant 0. \quad (23)$$

for all $\omega\in\mathbb{R}$. Therefore, from equation (21) and (23), we have $\lim_{\omega\to\infty}(-1)^{\mathtt{n}}(i\omega)^{2\mathtt{n}}\sum_{j=2\mathtt{n}-1}^{\infty}\frac{1}{(i\omega)^{j+1}}\widehat{C}\widehat{A}^j\widehat{B}\geqslant 0$. Expanding the sum in this inequality, we have

$$(-1)^{\mathtt{n}}\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B} + (-1)^{\mathtt{n}}\lim_{\omega\to\infty}(i\omega)^{2\mathtt{n}}\sum_{j=2\mathtt{n}}^{\infty}\frac{1}{(i\omega)^{j+1}}\widehat{C}\widehat{A}^j\widehat{B}$$
$$= (-1)^{\mathtt{n}}\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B}\geqslant 0. \quad (24)$$

From equation (24) it is clear that $(-1)^{\mathtt{n}-1}\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B}\leqslant 0$. Therefore, from equation (19), we infer that $X_1^T(A^TK+KA+C^TC)X_1\leqslant 0$. Since X_1 is a nonsingular matrix, by Sylvester's law of intertia, we infer that signature of $(A^TK+KA+C^TC)$ and $X_1^T(A^TK+KA+C^TC)X_1$ is the same. Therefore, $A^TK+KA+C^TC\leqslant 0$. ☐

Now we prove Statement 4 of Theorem 3.1.

Proof of Statement 4 of Theorem 3.1:

Note that Σ being a singularly bounded-real SISO system must admit a storage function x^TKx, where $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a solution to the singular KYP LMI (4), i.e.,

$$\begin{bmatrix}A^TK+KA+C^TC & KB+C^T\\ B^TK+C & 0\end{bmatrix}\leqslant 0.$$

In other words, $A^TK+KA+C^TC\leqslant 0$ and $KB+C^T=0$. Therefore, from Statement 3 of Theorem 3.1 we conclude that x^TKx is a storage function of Σ. Further, from the properties of singularly bounded-real systems, we know that they admit unique storage functions (see Footnote 4). Therefore, x^TKx is *the* storage function of Σ. ☐

Thus, in Theorem 3.1, we have constructed the unique closed form solution to the singular bounded-real LMI (4) corresponding to a singularly bounded-real SISO system. Based on Theorem 3.1, we present the algorithm to compute the storage function of a singularly bounded-real SISO system next.

Algorithm 3.4 Algorithm to compute the storage function of a singularly bounded-real SISO system.

Input: $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T\in\mathbb{R}^{\mathtt{n}}$.
Output: $K=K^T\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.
1: Construct $\widehat{A} := \begin{bmatrix}A & 0\\ -C^TC & -A^T\end{bmatrix}$ and $\widehat{B} := \begin{bmatrix}B\\ -C^T\end{bmatrix}$.
2: Construct $W := \begin{bmatrix}\widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n}-1}\widehat{B}\end{bmatrix}\in\mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$
3: Partition W as $W =: \begin{bmatrix}X_1\\ X_2\end{bmatrix}$, where $X_1, X_2\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$
4: Compute the storage function: $K = X_2X_1^{-1}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

Next we present an example to demonstrate the working of Algorithm 3.4.

Example 3.5. Consider the singularly bounded-real SISO system Σ with transfer function $G(s)=\frac{s^3+s^2+2s+0.5}{s^3+3s^2+6s+5.5}$. An i/s/o representation of the system is

$$\frac{d}{dt}x = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -5.5 & -6 & -3\end{bmatrix}x + \begin{bmatrix}0\\0\\1\end{bmatrix}u,\ y = -\begin{bmatrix}5 & 4 & 2\end{bmatrix}x+u.$$

Using Theorem 3.1, we get

$$W = \begin{bmatrix}\widehat{B} & \widehat{A}\widehat{B} & \widehat{A}^2\widehat{B}\end{bmatrix} = \left[\begin{array}{ccc}0 & 0 & 1\\ 0 & 1 & -3\\ 1 & -3 & 3\\ \hline 5 & 1 & -1\\ 4 & -1 & -5\\ 2 & -2 & -1\end{array}\right] =: \left[\begin{array}{c}X_1\\ \hline X_2\end{array}\right].$$

Therefore, $K = X_2X_1^{-1} = \begin{bmatrix}32 & 16 & 5\\ 16 & 11 & 4\\ 5 & 4 & 2\end{bmatrix}$ induces the storage function of Σ. It is easy to verify that $KB+C^T=0$ and $A^TK+KA+C^TC=\mathtt{diag}(-30,0,0)\leqslant 0$.

Remark 3.6. Step 2 of Algorithm 3.4 involves $(\mathtt{n}-1)$ matrix-vector multiplication and hence, requires $\mathscr{O}(\mathtt{n}^3)$ floating point operations (flops). Step 3 involves inversion of matrix X_1 and matrix-matrix multiplication. Hence, Step 3 requires $\mathscr{O}(\mathtt{n}^3)$ operations. Thus, the total flop-count for Algorithm 3.4 is $\mathscr{O}(\mathtt{n}^3)$. On the other hand, solving the singular bounded-real LMI using SDP techniques require $\mathscr{O}(\mathtt{n}^6)$ flops which can be improved to $\mathscr{O}(\mathtt{n}^{4.5})$ exploiting the structure of the system matrices [15].

In the next section we show that Algorithm 3.4 can be used to compute storage functions of allpass systems, as well.

4. Storage functions of allpass SISO systems

In this section, we show that Theorem 3.1 is not only applicable to singularly bounded-real SISO systems but also to allpass SISO systems. An allpass SISO system $\Sigma_{\mathtt{all}}$, with a minimal i/s/o representation $\frac{d}{dt}x=Ax+Bu$ and $y=Cx+u$, where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T\in\mathbb{R}^{\mathtt{n}}$, is a special class of bounded-real systems that has the following properties:

1) There exists a unique $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$\frac{d}{dt}\left(x^T Kx\right) = u^T u - y^T y \tag{25}$$

for all (x,u,y) that satisfy the i/s/o representation of $\Sigma_{\mathtt{all}}$.

2) Let $G(s)$ be the transfer function of $\Sigma_{\mathtt{all}}$ then

$$I - G(-i\omega)G(i\omega) = 0 \text{ for all } \omega \in \mathbb{R}. \tag{26}$$

See [1], [8, Section 5] for properties of allpass systems. Next we present a corollary that establishes that Theorem 3.1 is applicable to allpass systems as well.

Corollary 4.1. *Consider an allpass SISO system $\Sigma_{\mathtt{all}}$ with a minimal i/s/o representation $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + u$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B, C^T \in \mathbb{R}^{\mathtt{n}}$. Define*

$$\widehat{A} = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \text{ and } \widehat{B} = \begin{bmatrix} B \\ -C^T \end{bmatrix}.$$

Suppose $W := \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \cdots \widehat{A}^{\mathtt{n}-1}\widehat{B} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$. Define

$$W =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \text{ where } X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$

Then, the following statements hold.

1) X_1 is invertible.

2) $K := X_2 X_1^{-1}$.

3) $KB + C^T = 0$ and $A^T K + KA + C^T C = 0$.

4) $x^T Kx$ is the storage function of $\Sigma_{\mathtt{all}}$.

Proof. The proof of Statement 1 and 2 is the same as that of Theorem 3.1. We prove Statement 3 and 4 of the corollary. **3:** As shown in the proof of Theorem 3.1, by construction $KB = -C^T$. Therefore, $KB + C^T = 0$.

Next we prove $A^T K + KA + C^T C = 0$. The proof follows the same line of reasoning as that for singularly bounded-real systems. From equation (19), we have

$$X_1^T (A^T K + KA + C^T C) X_1 = \begin{bmatrix} 0 & 0 \\ 0 & (-1)^{\mathtt{n}-1}\left(\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B}\right) \end{bmatrix} \tag{27}$$

We claim $\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B} = 0$. From equation (26) we have, for all $\omega \in \mathbb{R}$, $\lim_{\omega\to\infty} \omega^{2\mathtt{n}}(I - G(-i\omega)G(i\omega)) = 0$. Hence equation (24) customized to allpass systems become $\widehat{C}\widehat{A}^{2\mathtt{n}-1}\widehat{B} = 0$. Therefore from equation (27), $A^T K + KA + C^T C = 0$.
4: From equation (25) it is evident that an allpass SISO system $\Sigma_{\mathtt{all}}$ admits a singular bounded-real LMI (4) with equality, i.e.,

$$\begin{bmatrix} A^T K + KA + C^T C & KB + C^T \\ B^T K + C & 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} A^T K + KA + C^T C = 0. \\ KB + C^T = 0. \end{cases}$$

Hence, from Statement 3 of Corollary 4.1 and properties of allpass systems, $x^T Kx$ is the storage function of $\Sigma_{\mathtt{all}}$. □

5. Conclusion

In this paper, we proposed closed form solutions of the singular bounded-real LMI for singularly bounded-real SISO systems. When the feedthrough regularity condition is satisfied the Hamiltonian matrix provides suitable invariant subspaces needed for the computation of K. This provision is provided by a suitable arrangement of the controllability and observability matrices in case of singularly bounded-real SISO systems. We showed that the Markov parameters of a system with system matrices $(\widehat{A}, \widehat{B}, \widehat{C})$ play a crucial role in formulating the main result of this paper. Further, we also showed using Corollary 4.1 that Theorem 3.1 is not only applicable to singularly bounded-real SISO systems but also to allpass SISO systems.

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